How Undecidable is the Elementary Theory of the Lattice of Equational Theories?

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Outline

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The Lattice of Equational Theories

The set of all equational theories of signature Δ is lattice-ordered by set-inclusion. We denote the resulting lattice by $\mathcal{L}_\Delta$.

A signature is a function that associates with operation symbols their (finite!) ranks.

An equational theory is a set of equations of some one signature that is closed with respect to logical consequence.

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A signature is large exactly when it is not small.

A signature is strictly large if and only if it provides at least one operation symbol of rank at least two.

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What We Knew Before Time Began

Birkhoff Told Us in 1935

The lattice $\mathcal{L}_\Delta$ is an algebraic lattice whose top element is compact and whose compact elements are exactly the finitely axiomatizable equational theories.

Folklore

If the signature provides only operation symbols of rank 0 (i.e. only constant symbols), then $\mathcal{L}_\Delta$ is isomorphic to the lattice of all equivalence relations on the set of these constant symbols.

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The lattice $\mathcal{L}_\Delta$ has at least $2^{\aleph_0}$ maximal elements, if the signature is strictly large.

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More Prehistorical Stuff

Jacobs and Schwabauer Gave us in 1964

a complete description of $\mathcal{L}_\Delta$ in the case when the signature has just one operation symbol and that one is unary. This lattice is distributive and has exactly 2 maximal elements.

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Results from the Beginning of History

Jarda in 1968

In a large signature every nontrivial equational theory lies above at least $2^{\aleph_0}$ other equational theories.

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Jarda supplies descriptions of all the lattices $\mathcal{L}_\Delta$ in the case when Δ is a finite small signature.

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That the lattice of all equivalence relations on a countably infinite set is embeddable, as an interval, into $\mathcal{L}_{\Delta}$ provided Δ is a large signature. The same applies to the lattice of equivalence relations on any finite set.

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Elementary Properties of $\mathcal{L}_\Delta$

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That certain sets of the form $\{T\}$ where definable by elementary formulas in the appropriate lattices $\mathcal{L}_\Delta$.

The results of Burris and of Jarda about embedding lattices in $\mathcal{L}_\Delta$ as intervals can be construed has definability results using parameters. McKenzie's result about essentially one-based equational theories can also be regarded as an elementary definability result.

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Suppose Δ is a large signature. As Stan Burris observed in 1971, it follows from his results by a theorem of Sachs that the equational theory of $\mathcal{L}_{\Delta}$ is the same as the equational theory of the class of all lattices. Thoralf Skolem proved that the equational theory of lattices is decidable in 1920.

How Undecidable in the Elementary Theory of $\mathcal{L}_\Delta$?

A Question

Let Δ be a large signature. Where, along the spectrum from the decidable *equational theory* of $\mathcal{L}_\Delta$ to the hereditarily undecidable *elementary theory* of $\mathcal{L}_\Delta$, does undecidability enter?

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Some Folk Wisdom

An infinite structure **A** that has a rich assortment of subsets that can be defined by elementary formulas is likely to have an undecidable elementary theory.

An infinite structure **A** that has a wide assortment of automorphisms is not likely to have rich assortment of definable subsets.

Tackling our Question

First show that the infinite lattice $\mathcal{L}_\Delta$ has a skimpy supply of automorphisms and a rich supply of elementarily definable subsets.

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Jarda on $\mathcal{L}_\Delta$: Automorphisms and Definable Subsets

Between 1981 and 1986, Jarda, in a tour de force, produced a four part paper which included the following theorems:

Theorem 0
The set of one-based equational theories is definable in $\mathcal{L}_\Delta$, for any signature Δ.

Theorem 1
The set of finitely based equational theories is definable in $\mathcal{L}_\Delta$, for any signature Δ.

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Theorem 2

The lattice $\mathcal{L}_\Delta$ has no nonobvious automorphisms, for any signature Δ with two exceptions. The exceptions are the signature just supplying one unary operation symbol and the signature supplying one unary operation symbol and one constant symbol.

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The orbit of any finitely based equational theory is definable in $\mathcal{L}_\Delta$, for any signature Δ.

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Proving $\mathcal{L}_\Delta$ has an Undecidable Elementary Theory

Jarda's Theorem 3, Redux

Suppose Δ is a finite signature. There is an effective procedure which associates with each finite set Σ of equations an elementary formula $\Theta_\Sigma(x)$ in one free variable x so that $\Theta_\Sigma(x)$ defines in $\mathcal{L}_\Delta$ the orbit of the equational theory based on Σ.

Strategy 0: Use a Finitely Based Undecidable Equational Theory

Assume each of the following about the finite set Σ of equations:

- The equational theory based on Σ is fixed by each automorphism of $\mathcal{L}_\Delta$.
- The equational theory based on Σ is undecidable.

Then

$\Sigma \vdash s \approx t$ if and only if $\mathcal{L}_\Delta \models \forall x \forall y \, [\Theta_{s \approx t}(x) \& \Theta_\Sigma(y) \implies x \leq y]$,

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Let Σ be the set of equational axioms for the variety of modular lattices. In 1980, Ralph Freese showed that even the set of equations in no more than five variables that are true in all modular lattices is undecidable. By 1983, Christian Herrmann even reduced the number of variables to 4. Meanwhile, Jarda, has told us what all the automorphisms of $\mathcal{L}_\Delta$ are and it is easy to see that the equational theory of modular lattices is fixed by each automorphism.

Surely in every finite large signature there is a finitely based undecidable equatonal theory which is fixed by each automorphism of $\mathcal{L}_\Delta$. Some one prove this please!

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Strategy 1: Use a Finitely Based Base Undecidable Equational Theory

In 1966 Peter Perkins showed that in every finite large signature the largest equational theory is base undecidable. This is theory is evidently finitely based and is fixed by every automorphism of $\mathcal{L}_\Delta$. In this case, the formulas we need can be simplified to

$$\forall x \forall y \, [\Theta_\Gamma(x) \implies y \leq x]$$

Question

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Due Diligence on the Theorem of Burris and Sankappanavar

The proof of Burris and Sankappanavar relied on a 1963 result of Lavrov that held that the $\exists^*\forall^*$ theory of two equivalence relations is strongly undecidable. Using a clever construction and following the Rabin-Scott Method, Burris and Sankappanavar actually proved that the $\forall^*\exists^*\forall^*\exists^*\forall^*$ theory of $\mathcal{L}_{\mathbf{\Delta}}$ is hereditarily undecidable, if Δ is a large signature.

We will say that a set Λ of elementary sentences (of the signature of lattice theory) is strongly undecidable provided U is undecidable whenever $U \subseteq \Lambda$ and U includes all the validities that belong to Λ.

Our Best Result to Date

Theorem

The $\forall^*\exists^*\forall^*$ theory of $\mathcal{L}_\Delta$ is strongly undecidable, whenever Δ is a large signature.

The proof of this theorem is very easy. We put together the following ingredients:

1. In 1963 Ershov and Taitslin proved the $\exists^*\forall^*$ theory of finite graphs in strongly undecidable.
2. In 1983, Jim Schmerl used the result of Ershov and Taitslin to prove that the $\exists^*\forall^*$ theory of finite lattices is strongly undecidable.
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A Barrier to Doing Better

The Use of Relational Signatures

The route to the undecidability results of Burris and Sankappanavar, and the route I just described involve elementary languages with relation symbols. It has long been known that the $\forall^*\exists^*$ validities in such signatures are decidable. So it is difficult to see how such arguments can produce strongly undecidable theories of this low a complexity.

The two routes through Jarda's work on definability in $\mathcal{L}_\Delta$ hit a similar barrier. Jarda's treats $\mathcal{L}_\Delta$ as a lattice-ordered set rather than a lattice.

There is some hope that by using operation symbols rather than relation symbols that lower quantifier complexity can be achieved.

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The route to the undecidability results of Burris and Sankappanavar, and the route I just described involve elementary languages with relation symbols. It has long been known that the $\forall^*\exists^*$ validities in such signatures are decidable. So it is difficult to see how such arguments can produce strongly undecidable theories of this low a complexity.

The two routes through Jarda's work on definability in $\mathcal{L}_\Delta$ hit a similar barrier. Jarda's treats $\mathcal{L}_\Delta$ as a lattice-ordered set rather than a lattice.

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Two Open Problems

Hilbert's Tenth Problem for $\mathcal{L}_\Delta$

Suppose that Δ is a finite large signature. Is there an algorithm that, upon input of a finite set Σ of equations in the signature of lattices with 1 and 0, will determine whether Σ has a solution in $\mathcal{L}_\Delta$?

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