# Equational Bounds: Determining Finite Algebras in Finitely Generated Varieties

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Suppose that $\mathcal{V}$ is a variety and that $\mathbf{B}$ is an algebra.

How can we determine whether $\mathbf{B} \in \mathcal{V}$?

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Describe the subdirectly irreducible algebras in $\mathcal{V}$, decompose $\mathbf{B}$ into subdirectly irreducible factors, and check whether each factor belongs to $\mathcal{V}$.

Describe the equations true in $\mathcal{V}$ and check whether they are true in $\mathbf{B}$.

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Here we take the third approach—the equational approach. But how much of the equational theory of $\mathcal{V}$ must we check to determine if $\mathbf{B}$ belongs to $\mathcal{V}$?

# The Equational Bound of $\mathcal{V}$

Let $\beta(n)$ be the least natural number $b$ such that for every algebra $\mathbf{B}$ with fewer than $n$ elements

$$\mathbf{B} \in \mathcal{V}$$

if and only if

every equation of length no more than $b$
which is true in $\mathcal{V}$ is also true in $\mathbf{B}$.

The function $\beta$ is called an **equational bound** of $\mathcal{V}$. When $\mathcal{V}$ is generated by the algebra $\mathbf{A}$ we also say that $\beta$ is an equational bound of $\mathbf{A}$.

# Easy Results

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Every variety of finite signature has an equational bound.

# Finite Algebras with Constant Bounds

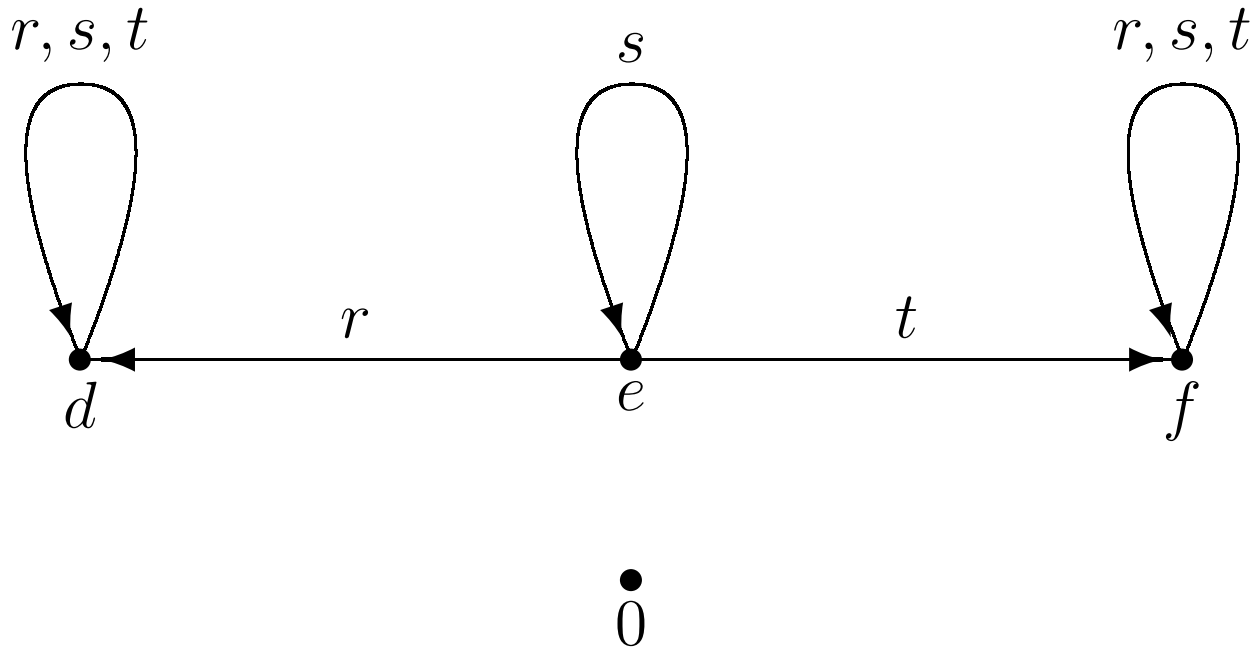
## Theorem

Suppose $\mathbf{A}$ is a finite algebra with a constant equational bound. Then either $\mathbf{A}$ is finitely based or $\mathbf{A}$ is inherently nonfinitely based.

## Corollary

Lyndon's seven element algebra is not constantly bounded.

# Lyndon's Algebra



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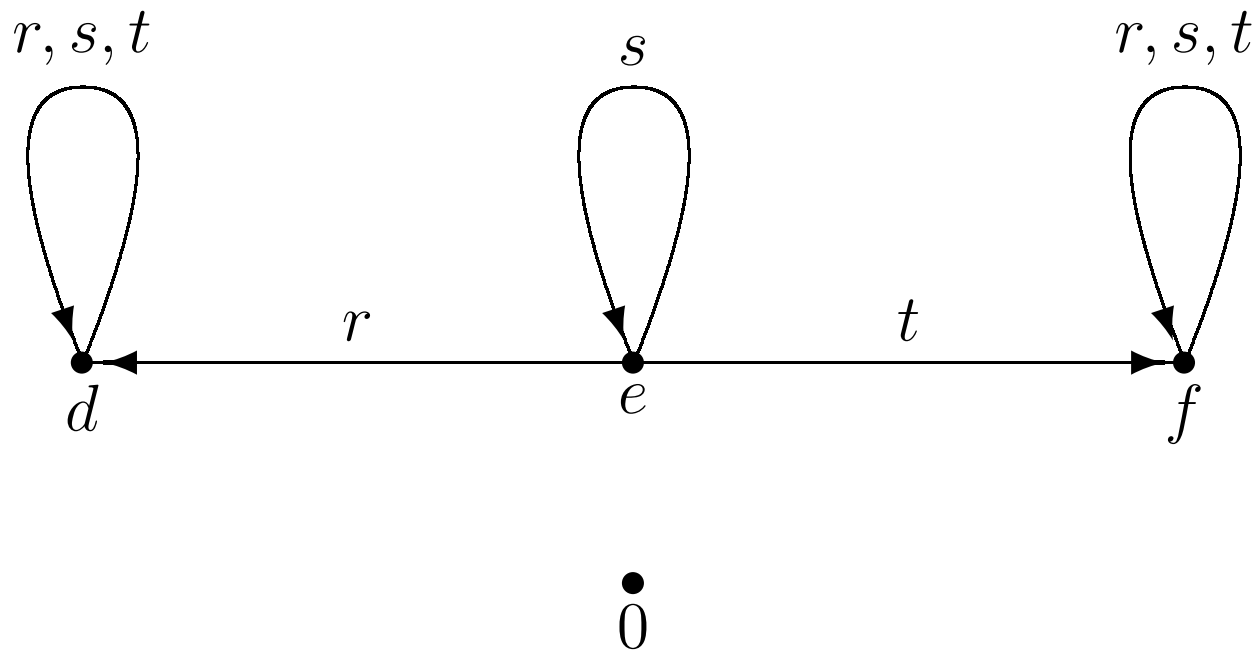



The labels are the elements of the algebra. There is only one operation and it is binary. Mostly it produces the default value $0$. The exceptions are given by the directed edges: for example $re = d$.

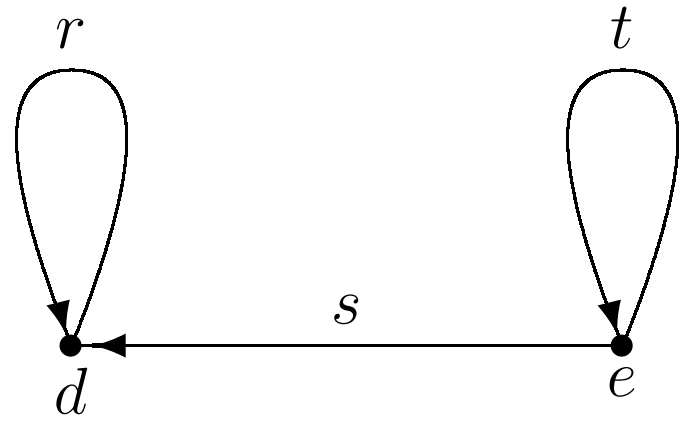
# An Open Problem

Is every constantly bounded finite algebra with only finitely many basic operations finitely based?

# Equational Bounds for Graph Algebras

Every graph algebra is $3n(n-1)+4$ bounded.

# McKenzie's Algebra



# Linear Bounds for the Algebras of Lyndon and McKenzie

$12n - 20$ is an equational bound for both Lyndon's algebra and McKenzie's algebra.

# The Twisted Torus