Problems From Ring Theory

In the problems below, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote respectively the rings of integers, rational numbers, real numbers, and complex numbers. R generally denotes a ring, and I and J usually denote ideals. $R[x], R[x,y], \ldots$ denote rings of polynomials. $\operatorname{rad} R$ is defined to be $\{r \in R : r^n = 0 \text{ for some positive integer } n\}$, where R is a ring; $\operatorname{rad} R$ is called the *nil radical* or just the *radical* of R.

Problem 0.
Let b be a nilpotent element of the ring R. Prove that $1+b$ is an invertible element of R.

Problem 1.
Let R be a ring with more than one element such that $aR = R$ for every nonzero element $a \in R$. Prove that R is a division ring.

Problem 2.
If $(m,n) = 1$, show that the ring $\mathbb{Z}/(mn)$ contains at least two idempotents other than the zero and the unit.

Problem 3.
If a and b are elements of a commutative ring with identity such that a is invertible and b is nilpotent, then $a+b$ is invertible.

Problem 4.
Let R be a ring which has no nonzero nilpotent elements. Prove that every idempotent element of R commutes with every element of R.

PROBLEM 5.
Let A be a division ring, B be a proper subring of A such that $a^{-1}Ba \subseteq B$ for all $a \neq 0$. Prove that B is contained in the center of A.

PROBLEM 6.
Let R denote a ring. Prove that, if $x, y \in R$ and $x - y$ is invertible, then $x(x-y)^{-1}y = y(x-y)^{-1}x$.

PROBLEM 7.
a. If I and J are ideals of a commutative ring R with $I + J = R$, then prove that $I \cap J = IJ$.
b. If $I, J,$ and K are ideals in a principal ideal domain R, then prove that $I \cap (J + K) = (I \cap J) + (I \cap K)$.

PROBLEM 8.
Let R be a principal ideal domain, and let I and J be ideals of R. IJ denotes the ideal of R generated by the set of all elements of the form ab where $a \in I$ and $b \in J$. Prove that if $I + J = R$, then $I \cap J = IJ$.

PROBLEM 9.
Let R be a commutative ring with identity, and let I and J be ideals of R. Define IJ to be the ideal generated by all the products xy with $x \in I$ and $y \in J$; that is IJ is the set of all finite sums of such products.
a. Prove that $IJ \subseteq I \cap J$.
b. Prove that $IJ = I \cap J$ if R is a principal ideal domain and $I + J = R$.
c. We say that R has the *descending chain condition* if given any chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, there is an integer k such that $I_k = I_{k+1} = I_{k+2} = \ldots$. Prove that if R has the descending chain condition, then R has only finitely many maximal ideals.

Problem 10.
Let R be a commutative ring. Suppose that I is an ideal of R which is contained in a prime ideal P. Prove that the collection of prime ideals containing I and contained in P has a minimal member.

Problem 11.
Let R be a commutative ring with unit. Let I be a prime ideal of R such that R/I satisfies the descending chain condition on ideals. Prove that R/I is a field.

Problem 12.
Let R be a commutative ring with 1 and let K be a maximal ideal in R. Show that R/K is a field.

Problem 13.
Let R be a commutative ring with one. Prove that every maximal ideal of R is also a prime ideal of R.

Problem 14.
Let R be a commutative ring with unit and let n be a positive integer. Let $J, I_0, \ldots, I_{n-1}$ be ideals of R such that I_k is a prime ideal for all $k < n$ and that $J \subseteq I_0 \cup \cdots \cup I_{n-1}$. Prove that $J \subseteq I_k$ for some $k < n$.

Problem 15.
Let X be a finite set and R be the ring of all functions from X into the field $\mathbb{R}$ of real numbers. Prove that an ideal M of R is maximal if and only if there is an element $a \in X$ such that

$$M = \{f : f \in R \text{ and } f(a) = 0\}$$

Problem 16.
Let I be an ideal in a commutative ring R and let $\mathfrak{S}$ be a set of ideals of R defined by the property that $J \in \mathfrak{S}$ if and only if there is an element $a \in R$ such that $a \notin I$ and $J = \{r \in R : ra \in I\}$. Prove that every maximal element of $\mathfrak{S}$ is a prime ideal in R.

Problem 17.
Let R be the following subring of the field of rational functions in 3 variables with complex coefficients:

$$R = \{\frac{f}{g} : f, g \in \mathbb{C}[x, y, z] \text{ and } g(1, 2, 3) \neq 0\}$$

Find 3 prime ideals $P_1, P_2,$ and P_3 in R with

$$0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq P_3 \subsetneqq R.$$

Problem 18.
Let $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. (Of course, R is a subring of the reals.) Let $M = \{a + b\sqrt{2} \in R : 5|a \text{ and } 5|b\}$.

a. Show that M is a maximal ideal of R.

b. What is the order of the field R/M? Verify your answer.

Problem 19.
Let R be commutative ring with 1. Let $p \in R$ and suppose that the principal ideal (p) is prime. If Q is a prime ideal and $Q \subsetneqq (p)$, show that $Q \subseteq \bigcap_n (p^n)$.

Problem 20.
Let $R = \mathbb{Z}[x]$. Give three prime ideals of R that contain the ideal $(6, 2x)$, and prove that your ideals are prime.

PROBLEM 21.
Let R be a commutative ring with 1, and let J be the intersection of all the maximal proper ideals of R. Prove that $1+a$ is a unit of R for every $a \in J$.

PROBLEM 22.
Let I be an ideal of the commutative ring R. Prove that R/I is a field if and only if I is a maximal ideal of R.

PROBLEM 23.
Let R be a commutative ring with identity element 1, and let I be an ideal of R. Prove each of the following:

a. R is a field if and only if R has exactly two ideals.
b. R/I is a field if and only if I is a maximal proper ideal of R.

PROBLEM 24.
Let R be a ring with identity element 1. Prove each of the following:

a. Every proper ideal of R is included in a maximal proper ideal of R.
b. R has exactly one maximal proper ideal if and only if the set of nonunits of R is an ideal of R.

PROBLEM 25.
Let R be a principal ideal domain and $0 \neq r \in R$. Let I be the ideal generated by r. Prove:

a. If r is prime, then R/I is a field.
b. If r is not prime, then R/I is not an integral domain.

PROBLEM 26.
Show that a nonzero ideal in a principal ideal domain is maximal if and only if it is prime.

Problem 27.
Let R be a commutative ring with identity. For $x \in R$, let $A(x) = \{r \in R : xr = 0\}$. Suppose $\theta \in R$ has the property that $A(\theta)$ is not properly contained in $A(x)$ for any $x \in R$. Prove the $A(\theta)$ is a prime ideal of R.

Problem 28.
Show that any integral domain satisfying the descending chain condition on ideals is a field.

Problem 29.
Let R be a commutative ring with identity, and let I be a prime ideal of R. If R/I is finite, prove that I is maximal.

Problem 30.
Give an example of a commutative ring R with two maximal nonzero ideals M and N such that $M \cap N = \{0\}$.

Problem 31.
Is $y^3 - x^2y^2 + x^3y + x + x^4$ irreducible in $\mathbb{Z}[x,y]$?

Problem 32.
Prove that $y^4 + x^2y + 4xy + x_4y + 2$ is irreducible in $\mathbb{Q}[x,y]$.

Problem 33.
Prove that $x^4 + xy^2 + y$ is irreducible in $\mathbb{Q}[x,y]$.

Problem 34.
Let R be a unique factorization domain. Prove that $f(x,y,z) = x^5y^3 + x^4z^3 + x^3yz^2 + y^2z$ is irreducible in $R[x,y,z]$.

PROBLEM 35.
Give the prime factorization of $x^5 + 5x + 5$ in each of $\mathbb{Q}[x]$ and $\mathbb{Z}_2[x]$.

PROBLEM 36.
Prove that the polynomial $x^3y + x^2y_xy^2 + x^3 + y$ is irreducible in $\mathbb{Z}[x, y]$.

PROBLEM 37.
For each field F given below, factor $x^{31} - 1 \in F[x]$ into a product of irreducible polynomials and justify your answer.

a. F is the field of complex numbers.
b. F is the field of rational numbers.
c. F is the field with 31 elements.
d. F is the field with 32 elements.

PROBLEM 38.
Prove that the polynomial $3x^4 + 2x^2 - x + 15$ is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

PROBLEM 39.
Prove that $y^3 + x^2y^2 + x^3y + x$ is irreducible in $R[x, y]$, if R is a unique factorization domain.

PROBLEM 40.
Prove that $x^3 + 3x + 6$ is irreducible in $\mathbb{Z}[x]$.

PROBLEM 41.
Prove the following form of the Chinese Remainder Theorem: Let R be a commutative ring with unit 1 and suppose that I and J are ideals of R such that $I + J = R$. Then

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}.$$

PROBLEM 42.
Prove that there exists a polynomial $f \in \mathbb{R}[x]$ such that

- $f - 1$ is in the ideal $(x^2 - 2x + 1))$, and
- $f - 2$ is in the ideal $(x + 1)$, and
- $f - 3$ is in the ideal $(x^2 - 9)$.

PROBLEM 43.
Does their exist a polynomial $f(x) \in \mathbb{R}[X]$ such that

- $f(x) - 1$ is in the ideal $(x^2 + 2x + 1)$, and
- $f(x) - 2$ is in the ideal $(x - 1)$, and
- $f(x) - 3$ is in the ideal $(x^2 - 25)$?

PROBLEM 44.
Let F be a field. Let $f_1, \ldots, f_r$ be polynomials in the polynomial ring $F[x]$. Fill in the blank and prove the resulting statement: The natural map

$$F[x] \to \frac{F[x]}{(f_1)} \oplus \cdots \oplus \frac{F[x]}{(f_r)}$$

is onto if and only if ______________.

PROBLEM 45.
Fill in the blank and prove the resulting statement. If D is an integral domain, the $D[x]$ is a principal ideal domain if and only if D is ______________.

PROBLEM 46.
Explain why each of the following represents or does not represent a maximal ideal in the ring $\mathbb{C}[x, y]/(y^2 - x^3 - x^2 - 4)$:

a. $(x - 1, y + 2)$.
b. $(x + 1, y - 2)$.
c. $(y^2 - x^3, x^2 + 3)$.

Problem 47.
Let D be a commutative ring. Show that if $D[x]$ is a principal ideal domain, then D must be a field.

Problem 48.
Let D be a unique factorization domain and let I be a nonzero prime ideal of $D[x]$ which is minimal among all the nonzero prime ideals of $D[x]$. Prove that I is a principal ideal.

Problem 49.
Suppose that R is a commutative ring with 1, and that I is an ideal of R. Show that $(R/I)[x] \cong R[x]/I[x]$.

Problem 50.
If F is a field, prove that $F[x]$ is a principal ideal domain.

Problem 51.
a. Prove that the ideal $(2, x)$ in $\mathbb{Z}[x]$ is not a principal ideal.
b. Prove that the ideal (3) in $\mathbb{Z}[x]$ is not a maximal ideal.

Problem 52.
Let I be the kernel of the ring homomorphism $\mathbb{Z}[x] \to \mathbb{R}$ induced by the substitution $x \mapsto 1 + \sqrt{2}$. Show that I is a principal ideal and find a generator for it.

Problem 53.
Let F be an infinite field and let $f(x, y) \in F[x, y]$. Prove that if $f(\alpha, \beta) = 0$ for all $\alpha, \beta \in F$, then $f(x, y) = 0$.

Problem 54.
Let F be a field. Prove that the rings $F[x, y]$ and $F[x]$ are not isomorphic.

Problem 55.
Let R be a commutative ring and let $f(x) \in R[x]$. Prove that $f(x)$ is nilpotent in $R[x]$ if and only if each coefficient of $f(x)$ is a nilpotent element of R.

Problem 56.
Let R be a commutative ring. The nil radical of R is defined to be $N(R) = \{x \in R : x^n = 0 \text{ for some natural number } n\}$.

a. Show that $N(R)$ is an ideal of R.
b. Show that $N(R)$ is the intersection of all the prime ideals of R.

Problem 57.
Let F be a field, $p(x) \in F[x]$, and $R = \dfrac{F[x]}{(p(x))}$. The *nil radical* of R is equal to

$$\{r \in R : r^n = 0 \text{ for some positive integer } n\}$$

Fill in the blank with some property of the polynomial $p(x)$ and then prove the resulting statement: The nil radical of R is $\{0\}$ if and only if ____________.

Problem 58.
Let R be a commutative ring with 1, and let J denote an ideal of R. The set $\{a \in R : a^n \in J \text{ for some } n\}$ is denoted by $\operatorname{rad}(J)$.

a. Prove that $\operatorname{rad}(J)$ is an ideal of R.
b. Prove that if I is a finitely generated ideal included in $\operatorname{rad}(J)$, then $I^m \subseteq J$ for some positive integer m.

Problem 59.
Let R be a commutative ring with 1. The *nil radical* of R is the set $N = \{r \in R : r^k = 0 \text{ for some positive integer } k\}$.

a. Prove that N is an ideal of R.

b. Let a be an element of R which is not an element of N, let $S = \{1, a, a^2, a^3, \ldots\}$, and let I be an ideal which is maximal among all ideals disjoint from S. Prove that I is a prime ideal of R.

PROBLEM 60.
Let $\mathbb{F}$ be a finite field and let $\mathbb{F}^* = \mathbb{F} - \{0\}$. Show that $\prod_{a\in\mathbb{F}^*} a = -1$.

PROBLEM 61.
Construct a field with 8 elements.

PROBLEM 62.
Let $F = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z}_3\}$.

a. Prove that F is a ring that contains $\mathbb{Z}_3$.
b. Give a basis for F as a vector space over Z_3.
c. Show that the equation $x^2 + 1 = 0$ has a solution in F, and prove that F and $\mathbb{Z}_3[x]/(x^2 + 1)$ are isomorphic rings.
d. Prove that F is a field.

PROBLEM 63.
Let F be a finite field, and let $g : F \to F$. Prove that there are infinitely many polynomials $f(x) \in F[x]$ such that $f(a) = g(a)$ for all $a \in F$.

PROBLEM 64.

a. If R is a finite ring with exactly a prime number of elements, prove that R is commutative.
b. Give an example of a finite noncommutative ring.

Problem 65.
For any ring R, let $\mathrm{Aut}(R)$ denote the group of ring automorphisms of R.

a. Show that $\mathrm{Aut}(\mathbb{R}) = \{1\}$.
b. Find $\mathrm{Aut}(\mathbb{R}[x])$.

Problem 66.
The D be a commutative domain and F be its field of fractions. the domain D is said to be *neat* provided both $f(x)$ and $g(x)$ are in $D[x]$ whenever $f(x)$ and $g(x)$ are monic polynomials in $F[x]$ such that $f(x)g(x) \in D[x]$.

a. Prove that if D is a unique factorization domain, then D is neat.
b. Give an example of a domain that is not neat.

Problem 67.
Let F be a field, $R = F[x]$, and M be the ideal (x). If $I = (x^2)$ and $J = (x^2 - x^3)$, prove that $J \subseteq I$ in R, but $J_M = I_M$ in R_M. (As usual, R_M denotes the localization $S^{-1}R$, where $S = R - M$.)

Problem 68.
Let R be the ring of 2×2 matrices over the field of complex numbers. Find two left ideals I and J of R such that I and J are isomorphic as left R-modules, but $I \neq J$.

Problem 69.
Let

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms, with exact rows. Show that if α_1 is surjective, and α_2 and α_3 are injective, the α_3 is injective.

Problem 70.
Let R be a unique factorization domain and let K be the quotient field of R. An element $z \in K$ is said to be it integral over R if there exists a monic polynomial $F \in R[x]$ such that $f(z) = 0$. Prove that if z is integral over R, then $z \in R$.

Problem 71.
Let F be a field, let $n \geq 2$ be an integer, and let $R = M_n(F)$ be the ring of $n \times n$ matrices with entries from F.

a. Give an example of a left ideal I in R with $I \neq \{0\}$ and $I \neq R$.
b. Give an example of a simple left ideal I in R (i.e. a nontrivial ideal I such that $\{0\}$ is the only left ideal properly contained in I.)

Problem 72.
Let R be the ring of formal power series $F[[x]]$, where F is any field. T typical element of R looks like $\sum_{i=0}^{\infty} \alpha_i x^i$, where $\alpha_i \in F$ for all i. The elements of R and added and multiplied in the obvious manner.

a. Find all the units of R.
b. Find all the ideals of R.
c. Find all the maximal ideals of R.

Problem 73.
Let R be an integral domain of prime characteristic p, and define $\phi : R \to R$ by $\phi(x) = x^p$ for all $x \in R$.

a. Show that ϕ is a homomorphism.
b. Show by example that ϕ can be an isomorphism.
c. Show by example that ϕ can fail to be an isomorphism.

Problem 74.

Let R be a commutative ring with identity, let n be a positive integer, and let S be the ring of $n \times n$ matrices with entries from R. Prove that the center of S is the set of scalar matrices, namely $\{aI : a \in R\}$ where I denotes the identity matrix.