# PROBLEMS FROM LINEAR ALGEBRA

In the following $\mathbb{R}$ denotes the field of real numbers while $\mathbb{C}$ denotes the field of complex numbers. In general, $\mathbf{U}, \mathbf{V}$, and $\mathbf{W}$ denote vector spaces. The set of all linear transformations from $\mathbf{V}$ into $\mathbf{W}$ is denoted by $\mathcal{L}(\mathbf{V}, \mathbf{W})$, while $\mathcal{L}(\mathbf{V})$ denotes the set of linear operators on $\mathbf{V}$. For a linear transformation $T$, the null space of $T$ (also known as the kernel of $T$) is denoted by $\operatorname{null} T$, while the range space of $T$ (also known as the image of $T$), is denoted by $\operatorname{range} T$.

Problem 0.
Let $\mathbf{V}$ be a finite-dimensional vector space and let $T$ be a linear operator on $\mathbf{V}$. Suppose that $T$ commutes with every diagonalizable linear operator on $\mathbf{V}$. Prove that $T$ is a scalar multiple of the identity operator.

Problem 1.
Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and let $T$ be a linear transformation from $\mathbf{V}$ into $\mathbf{W}$. Suppose that $\mathbf{V}$ is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$.

Problem 2.
Let $A$ and $B$ be $n \times n$ matrices over a field $\mathbf{F}$

(1) Prove that if $A$ or $B$ is nonsingular, then $AB$ is similar to $BA$.
(2) Show that there exist matrices $A$ and $B$ so that $AB$ is *not* similar to $BA$.
(3) What can you deduce about the eigenvalues of $AB$ and $BA$. Prove your answer.

Problem 3.
Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$, where $D$ and $G$ are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$.

Problem 4.
Let $V$ be a finite dimensional vector space. Can $V$ have three distinct proper subspaces $W_0$, $W_1$ and $W_2$ such that $W_0 \subseteq W_1$, $W_0 + W_2 = V$, and $W_1 \cap W_2 = \{0\}$?

Problem 5.
Let $n$ be a positive integer. Define

$$G = \{A : \ A \text{ is an } n \times n \text{ matrix with only integer entries and } \det A \in \{-1, +1\}\},$$

$$H = \{A : \ A \text{ is an invertible } n \times n \text{ matrix and both } A \text{ and } A^{-1} \text{ have only integer entries}\}.$$

Prove $G = H$.

Problem 6.
Let $V$ be the vector space over $\mathbb{R}$ of all $n \times n$ matrices with entries from $\mathbb{R}$.

(1) Prove that $\{I, A, A^2, \ldots, A^n\}$ is linearly dependent for all $A \in V$.
(2) Let $A \in V$. Prove that $A$ is invertible if and only if $I$ belongs to the span of $\{A, A^2, \ldots, A^n\}$.

Problem 7.
Is every $n \times n$ matrix over the field of complex numbers similar to a matrix of the form $D + N$ where $D$ is a diagonal matrix, $N^{n-1} = 0$, and $DN = ND$. Why?

Problem 8.
Let $V$ and $W$ be vector spaces and let $T$ be a linear operator from $V$ into $W$. Suppose that $V$ is finite-dimensional. Prove $\operatorname{rank}(T)+\operatorname{nullity}(T)=\dim V$.

Problem 9.
Let $T\in L(V,V)$, where $V$ is a finite dimensional vector space. (For a linear operator $S$ denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of $S$.)

(1) Prove there is a least natural number $k$ such that $\mathcal{N}(T^k)=\mathcal{N}(T^{k+1})=\mathcal{N}(T^{k+2})\cdots$ Use this $k$ in the rest to this problem.
(2) Prove that $\mathcal{R}(T^k)=\mathcal{R}(T^{k+1})=\mathcal{R}(T^{k+2})\cdots$
(3) Prove that $\mathcal{N}(T^k)\cap\mathcal{R}(T^k)=\{0\}$.
(4) Prove that for each $\alpha\in V$ there is exactly one vector in $\alpha_1\in\mathcal{N}(T^k)$ and exactly one vector $\alpha_2\in\mathcal{R}(T^k)$ such that $\alpha=\alpha_1+\alpha_2$.

Problem 10.
Let $\mathbf{F}$ be a field of characteristic 0 and let

$$W=\left\{A=[a_{ij}]\in\mathbf{F}^{n\times n}:\operatorname{tr}(A)=\sum_{i=1}^{n}a_{ii}=0\right\}.$$

For $i,j=1,\ldots,n$ with $i\neq j$, let $E_{ij}$ be the $n\times n$ matrix with $(i,j)$-th entry 1 and all the remaining entries 0. For $i=2,\ldots,n$ let $E_i$ be the $n\times n$ mat-ix with $(1,1)$ entry $-1$, $(i,i)$-th entry $+1$, and all remaining entries 0. Let

$$S=\{E_{ij}:i,j=1,\ldots,n\text{ and }i\neq j\}\cup\{E_i:i=2,\ldots,n\}.$$

[Note: You can assume, without proof, that $S$ is a linearly independent subset of $\mathbf{F}^{n\times n}$.]

(1) Prove that $W$ is a subspace of $\mathbf{F}^{n\times n}$ and that $W=\operatorname{span}(S)$. What is the dimension of $W$?

(2) Suppose that $f$ is a linear functional on $\mathbf{F}^{n\times n}$ such that

(a) $f(AB) = f(BA)$, for all $A, B \in \mathbf{F}^{n\times n}$.

(b) $f(I) = n$, where $I$ is the identity matrix in $\mathbf{F}^{n\times n}$.

Prove that $f(A) = \mathrm{tr}(A)$ for all $A \in \mathbf{F}^{n\times n}$.

Problem 11.
Let $V$ be a vector space over $\mathbb{C}$. Suppose that $f$ and $g$ are linear functionals on $V$ such that the functional

$$h(\alpha) = f(\alpha)g(\alpha) \quad \text{for all} \quad \alpha \in V$$

is linear. Show that either $f = 0$ or $g = 0$.

Problem 12.
Let $C$ be a $2 \times 2$ matrix over a field $\mathbf{F}$. Prove: There exists matrices $C = AB - BA$ if and only if $\mathrm{tr}(C) = 0$.

Problem 13.
Prove that if $A$ and $B$ are $n \times n$ matrices from $\mathbb{C}$ and $AB = BA$, then $A$ and $B$ have a common eigenvector.

Problem 14.
Let $\mathbf{F}$ be a field and let $V$ be a finite dimensional vector space over $\mathbf{F}$. Let $T \in L(V, V)$. If $c$ is an eigenvalue of $T$, then prove there is a nonzero linear functional $f$ in $L(V, \mathbf{F})$ such that $T^* f = cf$. (Recall that $T^* f = fT$ by definition.)

Problem 15.
Let $\mathbf{F}$ be a field, $n \geq 2$ be an integer, and let $V$ be the vector space of $n \times n$ matrices over $\mathbf{F}$. Let $A$ be a fixed element of $V$ and let $T \in L(V, V)$ be defined by $T(B) = AB$.

(1) Prove that $T$ and $A$ have the same minimal polynomial.
(2) If $A$ is diagonalizable, prove, or disprove by counterexample, that $T$ is diagonalizable.
(3) Do $A$ and $T$ have the same characteristic polynomial? Why or why not?

Problem 16.
Let $M$ and $N$ be $6 \times 6$ matrices over $\mathbb{C}$, both having minimal polynomial $x^3$.

(1) Prove that $M$ and $N$ are similar if and only if they have the same rank.
(2) Give a counterexample to show that the statement is false if 6 is replaced by 7.

Problem 17.
Give an example of two $4 \times 4$ matrices that are not similar but that have the same minimal polynomial.

Problem 18.
Let $(a_1, a_2, \ldots, a_n)$ be a nonzero vector in the real $n$-dimensional space $\mathbb{R}^n$ and let $P$ be the hyperplane

$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = 0 \right\}.$$

Find the matrix that gives the reflection across $P$.

Problem 19.
Let $V$ and $W$ be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Prove that that exists a basis $\mathcal{A}$ of $V$ and a basis $\mathcal{B}$ of $W$ so that the matrix $[T]_{\mathcal{A},B}$ has the block from $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 20.
Let $V$ be a finite-dimensional vector space and let $T$ be a diagonalizable linear operator on $V$. Prove that if $W$ is a $T$ invariant subspace then the restriction of $T$ to $W$ is also diagonalizable.

Problem 21.
Let $T$ be a linear operator on a finite-dimensional vector. Show that if $T$ has no cyclic vector then, then there exists an operator $U$ on $V$ that commutes with $T$ but is *not* a polynomial in $T$.

Problem 22.
Exhibit two real matrices with no real eigenvalues which have the same characteristic polynomial and the same minimal polynomial but are not similar.

Problem 23.
Let $V$ be a vector space, not necessarily finite-dimensional. Can $V$ have three distinct proper subspaces $A$, $B$, and $C$, such that $A \subset B$, $A + C = V$, and $B \cap C = \{0\}$?

Problem 24.

Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Problem 25.

(1) Prove that if $A$ and $B$ are linear transformations on an $n$-dimensional vector space with $AB = 0$, then $r(A) + r(B) \leq n$ where $r(\cdot)$ denotes rank.

(2) For each linear transformation $A$ on an $n$-dimensional vector space, prove that there exists a linear transformation $B$ such that $AB = 0$ and $r(A) + r(B) = n$.

Problem 26.

(1) Prove that if $A$ is a linear transformation such that $A^2(I - A) = A(I - A)^2 = 0$, then $A$ is a projection.

(2) Find a non-zero linear transformation so that $A^2(I - A) = 0$ but $A$ is *not* a projection.

Problem 27.

If $S$ is an $m$-dimensional vector space of an $n$-dimensional vector space $V$, prove that $S^\circ$, the annilihilator of $S$, is an $(n - m)$-dimensional subspace of $V^*$.

Problem 28.
Let $A$ be the $4 \times 4$ real matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

(1) Determine the rational canonical form of $A$.
(2) Determine the Jordan canonical form of $A$.

Problem 29.
Let $T$ be the linear operator on $\mathbb{R}^3$ which is represented by

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

in the standard basis. Find matrices $B$ and $C$ which represent respectively, in the standard basis, a diagonalizable linear operator $D$ and a nilpotent linear operator $N$ such that $T = D + N$ and $DN = ND$.

Problem 30.
Suppose $T$ is a linear operator on $\mathbb{R}^5$ represented in some basis by a diagonal matrix with entries $-1$, $-1$, 5, 5, 5 on the main diagonal.

(1) Explain why $T$ can not have a cyclic vector.
(2) Find $k$ and the invariant factors $p_i = p_{\alpha_i}$ in the cyclic decomposition $\mathbb{R}^5 = \bigoplus_{i=1}^{k} Z(\alpha_i; T)$.
(3) Write the rational canonical form for $T$.

Problem 31.
Suppose that $V$ in an $n$-dimensional vector space and $T$ is a linear map on $V$ of rank 1. Prove that $T$ is nilpotent or diagonalizable.

Problem 32.
Let $M$ denote an $m \times n$ matrix with entries in a field. Prove that

$$\begin{aligned} &\text{the maximum number of linearly independent rows of } M \\ = &\text{ the maximum number of linearly independent columns of } M \end{aligned}$$

(Do not assume that $\operatorname{rank} M = \operatorname{rank} M^t$.)

Problem 33.
Prove the Cayley-Hamilton Theorem, using only basic properties of determinants.

Problem 34.
Let $V$ be a finite-dimensional vector space. Prove there a linear operator $T$ on $V$ is invertible if and only if the constant term in the minimal polynomial for $T$ is non-zero.

Problem 35.

(1) Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Find a matrix $T$ (with entries in $\mathbf{C}$) such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.
(2) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is $4$.

Problem 36.
Suppose $A$ and $B$ are linear operators on the same finite-dimensional vector space $V$. Prove that $AB$ and $BA$ have the same characteristic values.

Problem 37.
Let $M$ denote an $n \times n$ matrix with entries in a field $\mathbf{F}$. Prove that there is an $n \times n$ matrix $B$ with entries in $\mathbf{F}$ so that $\det(M + tB) \neq 0$ for every non-zero $t \in \mathbf{F}$.

Problem 38.
Let $W_1$ and $W_2$ be subspaces of the finite dimensional vector space $V$. Record and prove a formula which relates $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$, $\dim(W_1 \cap W_2)$.

Problem 39.
Let $M$ be a symmetric $n \times n$ matrix with real number entries. Prove that there is an $n \times n$ matrix $N$ with real entries such that $N^3 = M$.

Problem 40.
TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) If two nilpotent matrices have the same rank, the same minimal polynomial and the same characteristic polynomial, then they are similar.

Problem 41.
Suppose that $T : V \to W$ is a injective linear transformation over a field $\mathbf{F}$. Prove that $T^* : W^* \to V^*$ is surjective. (Recall that $V^* = L(V, \mathbf{F})$ is the vector space of linear transformations from $V$ to $\mathbf{F}$.)

Problem 42.
If $M$ is the $n \times n$ matrix

$$M = \begin{bmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{bmatrix}$$

then prove that $\det M = [x + (n-1)a](x-a)^{n-1}$.

Problem 43.
Suppose that $T$ is a linear operator on a finite dimensional vector space $V$ over a field $\mathbf{F}$. Prove that $T$ has a cyclic vector if and only if

$$\{U \in L(V,V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

Problem 44.
Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3)$$

(1) Compute the characteristic polynomial of $T$.
(2) Compute the minimal polynomial of $T$.
(3) The vector space $\mathbb{R}^4$ is the direct sum of two proper $T$-invariant subspaces. Exhibit a basis for one of these subspaces.

Problem 45.
Let $V$, $W$, and $Z$ be finite dimensional vector spaces over the field $\mathbf{F}$ and let $f : V \to W$ and $g : W \to Z$ be linear transformations. Prove that

$$\text{nullity}(g \circ f) \le \text{nullity}(f) + \text{nullity}(g)$$

Problem 46.
Prove that

$$\det \begin{bmatrix} A & 0 & 0 \\ B & C & D \\ 0 & 0 & E \end{bmatrix} = \det A \det C \det E$$

where $A$, $B$, $C$, $D$ and $E$ are all square matrices.

Problem 47.
Let $A$ and $B$ be $n \times n$ matrices with entries on the field $\mathbf{F}$ such that $A^{n-1} \neq 0$, $B^{n-1} \neq 0$, and $A^n = B^n = 0$. Prove that $A$ and $B$ are similar, or show, with a counterexample, that $A$ and $B$ are not necessarily similar.

Problem 48.
Let $A$ and $B$ be $n \times n$ matrices with entries from $\mathbb{R}$. Suppose that $A$ and $B$ are similar over $\mathbb{C}$. Prove that they are similar over $\mathbb{R}$.

Problem 49.
Let $A$ be an $n \times n$ with entries from the field $\mathbf{F}$. Suppose that $A^2 = A$. Prove that the rank of $A$ is equal to the trace of $A$.

Problem 50.
TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) Let $W$ be a vector space over a field $\mathbf{F}$ and let $\theta : V \to V'$ be a fixed surjective transformation. If $g : W \to V'$ is a linear transformation then there is linear transformation $h : W \to V$ such that $\theta \circ h = g$.

PROBLEM 51.
Let $V$ be a finite dimensional vector space and $A \in L(V,V)$.

(1) Prove that there exists and integer $k$ such that $\ker A^k = \ker A^{k+1} = \cdots$
(2) Prove that there exists an integer $k$ such that $V = \ker A^k \oplus \text{image } A^k$.

PROBLEM 52.
Let $V$ be the vector space of $n \times n$ matrices over a field $\mathbf{F}$, and let $T : V \to V^*$ be defined by $T(A)(B) = \operatorname{tr}(A^t B)$. Prove that $T$ is an isomorphism.

PROBLEM 53.
Let $A$ be an $n \times n$ matrix and $A^k = 0$ for some $k$. Show that $\det(A+I) = 1$.

PROBLEM 54.
Let $V$ be a finite dimensional vector sauce over a field $F$, and $T$ a linear operator on $V$. Suppose the minimal and characteristic polynomials of of $T$ are the same power of an irreducible polynomial $p(x)$. Show that no non-trivial $T$-invariant subspace of $V$ has a $T$-invariant complement.

PROBLEM 55.
Let $V$ be the vector space of all $n \times n$ matrices over a field $\mathbf{F}$, and let $B$ be a fixed $n \times n$ matrix that ti not of the form $cI$. Define a linear operator $T$ on $V$ by $T(A) = AB - BA$. Exhibit a not-zero element in the kernel of the transpose of $T$.

PROBLEM 56.
Let $V$ be a finite dimensional vector space over a field $\mathbf{F}$ and suppose that $S$ and $T$ are triangulable operators on $V$. If $ST = TS$ prove that $S$ and $T$ have an eigenvector in common.

Problem 57.
Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. If trace $A^i = 0$ for all $i > 0$, prove that $A$ is nilpotent.

Problem 58.
Let $V$ be a finite dimensional vector space over a field $\mathbf{F}$, and let $T$ be a linear operator on $V$ so that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$. Prove that $V$ is the direct sum of the range of $T$ and the null space of $T$.

Problem 59.
Let $V$ be the vector space of all $n \times n$ matrices over a field $\mathbf{F}$, and suppose that $A$ is in $V$. Define $T : V \to V$ by $T(AB) = AB$. Prove that $A$ and $B$ have the same characteristic values.

Problem 60.
Let $A$ and $B$ be $n \times n$ over the complex numbers.

(1) Show that $AB$ and $BA$ have the same characteristic values.
(2) Are $AB$ and $BA$ similar matrices?

Problem 61.
Let $V$ be a finite dimensional vector space over a field of characteristic 0, and $T$ be a linear operator on $V$ so that $\operatorname{tr}(T^k) = 0$ for all $k \geq 1$, where $\operatorname{tr}(\cdot)$ denotes the trace function. Prove that $T$ is a nilpotent linear map.

Problem 62.
Let $A = [a_{ij}]$ be an $n \times n$ matrix over the field of complex numbers such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad \text{for} \qquad i = 1, \ldots, n.$$

Then show that $\det A \neq 0$. ($\det$ denotes the determinant.)

PROBLEM 63.
Let $A$ be an $n \times n$ matrix, and let $\operatorname{adj}(A)$ denote the adjoint of $A$. Prove the rank of $\operatorname{adj}(A)$ is either $0$, $1$, or $n$.

PROBLEM 64.
Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$$

(1) Determine the rational canonical form of $A$.
(2) Determine the Jordan canonical form of $A$.

PROBLEM 65.
If

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then prove that there does not exist a matrix with $N^2 = A$.

PROBLEM 66.
Let $A$ be a real $n \times n$ matrix which is symmetric, i.e. $A^t = A$. Prove that $A$ is diagonalizable.

PROBLEM 67.
Give an example of two nilpotent matrices $A$ and $B$ such that

(1) $A$ is not similar to $B$,
(2) $A$ and $B$ have the same characteristic polynomial,

(3) $A$ and $B$ have the same minimal polynomial, and
(4) $A$ and $B$ have the same rank.

Problem 68.
Let $A$ be an $n \times n$ matrix over a field $\mathbf{F}$. Show that $\mathbf{F}^n$ is the direct sum of the null space and the range of $A$ if and only if $A$ and $A^2$ have the same rank.

Problem 69.
Let $A$ and $B$ be $n \times n$ matrices over a field $\mathbf{F}$.

(1) Show $AB$ and $BA$ have the same eigenvalues.
(2) Is $AB$ similar to $BA$? (Justify your answer).

Problem 70.
Given an exact sequence of finite-dimensional vector spaces

$$0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} 0$$

that is the range of $T_i$ is equal to the null space of $T_{i+1}$, for all $i$. What is the value of $\sum_{i+1}^{n} (-1)^i \dim(V_i)$? (Justify your answer).

Problem 71.
Let $\mathbf{F}$ be a field with $q$ elements and $V$ be a $n$-dimensional vector space over $\mathbf{F}$.

(1) Find the number of elements in $V$.
(2) Find the number of bases in of $V$.
(3) Find the number of invertible linear operators on $V$.

PROBLEM 72.
Let $A$ and $B$ be $n \times n$ matrices over a field $\mathbf{F}$. Suppose that $A$ and $B$ have the same trace and the same minimal polynomial of degree $n-1$. Is $A$ similar to $B$? (Justify your answer.)

PROBLEM 73.
Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = 1$ for all $i$ and $j$. Find its characteristic and minimal polynomial.

PROBLEM 74.
Give an example of a matrix with real entries whose characteristic polynomial is $x^5 - x^4 + x^2 - 3x + 1$.

PROBLEM 75.
TRUE or FALSE. (If true prove it. If false give a counterexample.) Let $A$ and $B$ be $n \times n$ matrices with minimal polynomial $x^4$. If $\operatorname{rank} A = \operatorname{rank} B$, and $\operatorname{rank} A^2 = \operatorname{rank} B^2$, then $A$ and $B$ are similar.

PROBLEM 76.
Suppose that $T$ is a linear operator on a finite-dimensional vector space $V$ over a field $\mathbf{F}$. Prove that the characteristic polynomial of $T$ is equal to the minimal polynomial of $T$ if and only if

$$\{U \in L(V,V) : TU = UT\} = \{f(T) : f \in \mathbf{F}[x]\}.$$

PROBLEM 77.

(1) Prove that if $A$ and $B$ are $3 \times 3$ matrices over a field $\mathbf{F}$, a necessary and sufficient condition that $A$ and $B$ be similar over $\mathbf{F}$ is that that have the same characteristic and the same minimal polynomial.
(2) Give an example to show this is not true for $4 \times 4$ matrices.

Problem 78.
Let $V$ be the vector space of $n \times n$ matrices over a field. Assume that $f$ is a linear functional on $V$ so that $f(AB) = f(BA)$ for all $A, B \in V$, and $f(I) = n$. Prove that $f$ is the trace functional.

Problem 79.
Suppose that $N$ is a $4 \times 4$ nilpotent matrix over $\mathbf{F}$ with minimal polynomial $x^2$. What are the possible rational canonical forms for $n$?

Problem 80.
Let $A$ and $B$ be $n \times n$ matrices over a field $\mathbf{F}$. Prove that $AB$ and $BA$ have the same characteristic polynomial.

Problem 81.
Suppose that $\mathbf{V}$ is an $n$-dimensional vector space over $\mathbf{F}$, and $T$ is a linear operator on $\mathbf{V}$ which has $n$ distinct characteristic values. Prove that if $S$ is a linear operator on $\mathbf{V}$ that commutes with $T$, then $S$ is a polynomial in $T$.

Problem 82.
Let $A$ and $B$ be $n \times n$ matrices over a field $\mathbf{F}$. Show that $AB$ and $BA$ have the same characteristic values in $\mathbf{F}$.

Problem 83.
Let $P$ and $Q$ be real $n \times n$ matrices so that $P + Q = I$ and $\operatorname{rank}(P) + \operatorname{rank}(Q) = n$. Prove that $P$ and $Q$ are projections. (Hint: Show that if $Px = Qy$ for some vectors $x$ and $y$, then $Px = Qy = 0$.)

Problem 84.
Suppose that $A$ is an $n \times n$ real, invertible matrix. Show that $A^{-1}$ can be expressed as a polynomial in $A$ with real coefficients and with degree at most $n-1$.

Problem 85.
Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Determine the rational canonical form and the Jordan canonical form of $A$.

Problem 86.

(1) Give an example of two $4 \times 4$ nilpotent matrices which have the same minimal polynomial but are not similar.
(2) Explain why $4$ is the smallest value that can be chosen for the example in part (a), i.e. if $n \leq 3$, any two nilpotent matrices with the same minimal polynomial are similar.

Problem 87.
**This is a very basic and important fact.** Let $V$ be a finite dimensional vector space and $f$ and $g$ two linear functionals on $V$. If $\ker f = \ker g$ show $g$ is a scalar multiple of $f$.

Problem 88.
This problem makes explicit some facts that are used several times in solving some of the problems above.

(1) Prove that if $V$ is a finite dimensional vector space over the field $\mathbf{F}$ and $T \in L(V,V)$ and $V$ is cyclic for $T$ that any $S \in L(V,V)$ that commutes with $T$ is a polynomial in $T$. That is $ST = TS$ implies that $S = p(T)$ for some $p(x) \in \mathbf{F}[x]$. Hint: Let $\dim V = n$. Then because $V$ is cyclic for $T$ there is a vector $v_0 \in V$ so that $v_0, Tv_0, \ldots, T^{n-1}v_0$ is a basis for $V$. Thus there are scalars $a_0, a_1, \ldots, a_{n-1}$ so that $Sv_0 = a_0v_0 + a_1Tv_0 + a_2T^2v_0 + \cdots + a_{n-1}T^{n-1}v_0$. Then letting $p(x) = a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1}$ we have $Sv_0 = p(T)v_0$. Now use that $S$ commutes with $T$ (and thus also $p(T)$) to show that $ST^iv_0 = p(T)T^iv_0$ for $i = 0, 1, \ldots, n-1$. Thus the two linear maps $S$ and $p(T)$ agree on a basis and hence are equal.

(2) If the minimal polynomial $f(x)$ of $T$ has $\deg f(x) = \dim V$ then $V$ is cyclic for $T$. Hint: I don't know any particularly easy way to do this. The basic idea is to factor $f(x) = p_1(x)^{k_1} \cdots p_l(x)^{k_l}$ into powers of primes and consider the corresponding primary decomposition $V = \ker(p_1(T)^{k_1}) \oplus \cdots \oplus \ker(p_l(T)^{k_l})$ and show that if $\deg f(x) = \dim V$ then each of the primary factors $\ker(p_i(T)^{k_i})$ is cyclic (this in turn uses that each of the $\ker(p_i(T)^{k_i})$ is a sum of cyclic subspaces). Now let $v_i$ be a cyclic for $T$ in $\ker(p_i(T)^{k_i})$ for $i = 1, \ldots, l$. Then show the vector $v_0 = v_1 + v_2 + \cdots + v_l$ is cyclic for $T$.

Problem 89.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct elements of the field $\mathbf{F}$. Then the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

is invertible. Hint: If $A$ is singular then it has rank less than $n$ and thus there is a nontrivial linear relation between the rows of $A$. This would in turn imply that there is a nonzero polynomial $p(x)$ of degree $\leq n-1$ than had the $n$ scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ as roots. But this is impossible.

Problem 90.
**This is another set of facts that anyone who has had a graduate linear algebra class should know.** Let $D$ be a diagonal matrix that has all its diagonal elements distinct. That is

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \text{where } \lambda_i \neq \lambda_j \text{ for } i \neq j.$$

Then show

(1) The only matrices that commute with $D$ are diagonal matrices.
(2) If $A$ is any other diagonal matrix then $A$ is a polynomial in $D$. That is there is a polynomial $p(x)$ so that $A = p(D)$.
(3) If $A$ is any matrix that commutes with $D$ then $A$ is a polynomial in $D$.

(4) There is a cyclic vector for $D$. HINT: Let $e_1, \dots, e_n$ be the standard coordinate vectors. Then as $D$ is diagonal $De_i = \lambda_i e_i$. Let $v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$. Then show that $v$ is a cyclic vector for $D$ if and only if $a_i \neq 0$ for all $i$ (One way to do this is use the last problem). In particular $v = e_1 + e_2 + \cdots + e_n$ is a cyclic vector for $T$.

PROBLEM 91.
This is another standard problem. Let $V$ be a finite dimensional vector space over a field $\mathbf{F}$ and let $T \in L(V, V)$. Let $\lambda$ be an eigenvalue of $T$ and let $V_\lambda := \{v \in V : Tv = \lambda v\}$ be the corresponding eigenspace.

(1) Let $S \in L(V, V)$ commute with $T$. Then show that $V_\lambda$ is invariant under $S$. (That is show $v \in V_\lambda$ implies $Sv \in V_\lambda$.)
(2) Show that if $A$ and $B$ are $n \times n$ matrices over the complex numbers that commute, then they have a common eigenvector. HINT: As $A$ is a complex matrix it has at least one eigenvalue $\lambda$. Let $V_\lambda$ be the corresponding eigenspace. Then by what we have just done $V_\lambda$ is invariant under $B$. But then the restriction of $B$ to $V_\lambda$ has an eigenvector in $V_\lambda$.
(3) This is a different way of looking at Problem 90(4) above. Assume $V$ has an basis of eigenvectors $e_1, e_2, \dots, e_n$ of eigenvectors of $T$, that is $Te_i = \lambda_i e_i$. Also assume the eigenvalues are distinct: $\lambda_i \neq \lambda_j$ for $i \neq j$. Then show if $S$ commutes with $T$ then for some scalars $c_i$ there holds $Se_i = c_i e_i$, and thus $S$ is also diagonal in the basis $e_1, \dots, e_n$. HINT: Let $V_{\lambda_i} := \{v : Te_i = \lambda_i v\}$. Then by the assumptions $V_{\lambda_i}$ is one dimensional with basis $e_i$. Part (1) of this problem then implies that $V_{\lambda_i}$ is invariant under $S$. As $V_{\lambda_i}$ is one dimensional this in turn implies $e_i$ is an eigenvector of $S$.