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A modular inherently nonfinitely based lattice

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This paper is dedicated to Walter Taylor.

In [5] we gave a construction of inherently nonfinitely based lattices which produced a wide variety of examples. But none of these examples was modular and we asked in Problem 1 for a modular example. Here we shall show that $\mathbf{L}_\infty$ of Figure 1 is such an example.

Theorem 1. $\mathbf{L}_\infty$ *is an inherently nonfinitely based modular lattice.*

Proof. As observed in McNulty [7], a locally finite variety $\mathcal{V}$ of finite type is inherently nonfinitely based if and only if for infinitely many natural numbers N, there is a non-locally-finite algebra each of whose N-generated subalgebras belongs to $\mathcal{V}$. We prove the theorem by establishing these facts. We assume the reader is familiar with the basic facts of modular lattices; see [1], [2], [6].

Let $\mathbf{B}$ (for bottom) be the sublattice of $\mathbf{L}_\infty$ consisting of all elements of finite height and let $\mathbf{T}$ consist of all elements of finite depth. Of course $\mathbf{L}_\infty$ is the ordinal (or linear) sum $\mathbf{B} + \mathbf{T}$ of these sublattices.

Lemma 2. *The variety* $\mathbf{V}(\mathbf{L}_\infty)$ *generated by* $\mathbf{L}_\infty$ *is locally finite.*

Proof. To prove this we need to show that for every finite n there is a bound on the size of the n-generated subalgebras of $\mathbf{L}_\infty$. We do this by induction on n. Suppose that $x_1, \dots, x_n$ are elements of $\mathbf{L}_\infty$ and let $\mathbf{S}$ be the sublattice they generate. We may assume that all of these elements either lie in $\mathbf{B}$ or they all lie in $\mathbf{T}$ since otherwise $\mathbf{S}$ is the ordinal sum of two sublattices with fewer generators. By duality we may assume they all lie in $\mathbf{B}$. Thus each x_k has a rank (or height) r_k.

Observe that if a and b are elements of $\mathbf{B}$ with ranks r_a and r_b and $r_b - r_a \geq 4$ then the meet of all elements with rank at least r_b is greater than or equal to the join of all elements with rank at most r_a. So if we let r_k be the rank of x_k and (re)order the x_k's so that $r_1 \leq r_2 \leq \cdots \leq r_n$ then we may assume $r_{k+1} - r_k \leq 3$ since otherwise $\mathbf{S}$ is a ordinal sum of sublattices with fewer generators. Thus $\mathbf{S}$ lies

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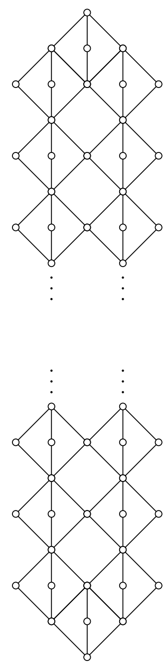

FIGURE 1. $\mathbf{L}_\infty$, an inherently nonfinitely based modular lattice

in an interval of $\mathbf{L}_\infty$ which has length at most $3(n+2)$. All intervals of $\mathbf{L}_\infty$ of fixed length have a bound on their size. Thus $\mathbf{S}$ is of bounded size. □

Let $\mathbf{M}_3$ be the five element modular, nondistributive lattice and let $\mathbf{Z}$ be the integers as a chain. Let $\mathbf{M}_3[\mathbf{Z}]$ be the lattice of all order-preserving functions from $\mathbf{Z}$ to $\mathbf{M}_3$. If $x \in \mathbf{M}_3$ let $\overline{x}$ denote the constant map. The following facts are easily established:

(1) $\mathbf{M}_3[\mathbf{Z}]$ is a subdirect power of $\mathbf{M}_3$ and so is in the variety generated by $\mathbf{M}_3$.
(2) $x \mapsto \overline{x}$ embeds $\mathbf{M}_3$ in $\mathbf{M}_3[\mathbf{Z}]$.
(3) If $v \prec u$ in $\mathbf{M}_3$, then the interval $[\overline{v}, \overline{u}]$ is $\mathbf{1} + \mathbf{Z} + \mathbf{1}$ (the ordinal sum).
(4) If a is an atom of $\mathbf{M}_3$ and 0 is its least element, then $\mathbf{M}_3[\mathbf{Z}]$ is generated by the constant maps and the interval $[\overline{0}, \overline{a}]$.
(5) The (left) shift operator σ $[\,(\sigma f)(i) = f(i+1)\,]$ is an automorphism of $\mathbf{M}_3[\mathbf{Z}]$.

We wish do something similar with $\mathbf{L}_\infty$ and other modular lattices. So let $\mathbf{L}$ be a modular lattice. We start by forming $\mathbf{M}$, the lattice of all order-preserving maps from $\mathbf{Z}$ into $\mathbf{L}$. This lattice is bigger than we want; we would like that intervals of $\mathbf{L}$ which are chains remain chains in $\mathbf{M}$ under the natural (diagonal) embedding (denoted $x \mapsto \overline{x}$) of **L** into **M**. To do this we take the sublattice of **M** whose universe is

$$\{x \in \mathbf{M} : \overline{v} \leq x \leq \overline{u}, \text{ for some } u,\ v \in \mathbf{L} \text{ with } [v,u] \text{ complemented}\}$$

The following lemma proves that this is (the universe of) a sublattice of **M**. We denote this lattice by $\mathbf{L}[\mathbf{Z}]$.

Lemma 3. *If* $[a_0, a_1]$ *and* $[b_0, b_1]$ *are complemented intervals in a modular lattice* **L**, *then* $[a_0 \vee b_0, a_1 \vee b_1]$ *is also complemented.*

Proof. Let $c_0 = a_0 \vee b_0$ and $c_1 = a_1 \vee b_1$. Let $x \in [c_0, c_1]$. Let $d = a_1 \vee b_0$ and $e = a_0 \vee b_1$. Since $[c_0, d]$ is a transpose of a subinterval of $[a_0, a_1]$, it is complemented, and similarly $[c_0, e]$ is complemented. Let y be a complement of $x \wedge d$ in $[c_0, d]$ and let z be a complement of $(x \vee d) \wedge e$ in $[c_0, e]$. Then $y \vee z$ is a complement of x in $[c_0, c_1]$. Indeed

$$\begin{aligned}
x \wedge (y \vee z) &= x \wedge (x \vee d) \wedge (y \vee z) = x \wedge [y \vee ((x \vee d) \wedge z)] \\
&= x \wedge [y \vee ((x \vee d) \wedge e \wedge z)] = x \wedge [y \vee c_0] \\
&= x \wedge y = x \wedge d \wedge y = c_0 \\
x \vee y \vee z &= x \vee (x \wedge d) \vee y \vee z \\
&= x \vee d \vee z = x \vee d \vee e = c_1
\end{aligned}$$

□

Now we turn to constructing non-locally-finite lattices whose N-generated sublattices lie in $\boldsymbol{V}(\mathbf{L}_\infty)$. For n an even integer at least 4, **B** has 5 elements of height n, only one of which is join reducible (the middle one). We let $\mathbf{K}_n$ denote the principal ideal generated by this join reducible element. $\mathbf{K}_n$ has two coatoms, and we let $\mathbf{L}_n$ be the lattice obtained from $\mathbf{K}_n$ by adding another coatom which is above the meet of these two coatoms thus forming an $\mathbf{M}_3$. $\mathbf{L}_{10}$ is diagrammed in Figure 2.

We shall modify $\mathbf{L}_n[\mathbf{Z}]$ into a lattice $\mathbf{L}_n[\mathbf{Z}]^*$ using the Hall–Dilworth gluing construction, which we now review.

If a lattice **L** has a filter which is isomorphic to the ideal of another lattice **K**, then we can identify each element of the filter with corresponding element of the ideal and order the elements of $L \cup K$ (with these identifications) by the transitive closure of the orders on **L** and **K**. The result is a lattice **M**, and if both **L** and **K** are modular, then **M** is also. This is the famous Hall–Dilworth gluing construction. Now, **L** is an ideal of **M**, **K** is a filter, and $L \cup K = M$. Conversely, if a lattice **M** is the set union of an ideal **L** and a filter **K** and $L \cap K \neq \emptyset$, then **M** is the Hall–Dilworth gluing of **L** and **K** over their intersection using the identity map.

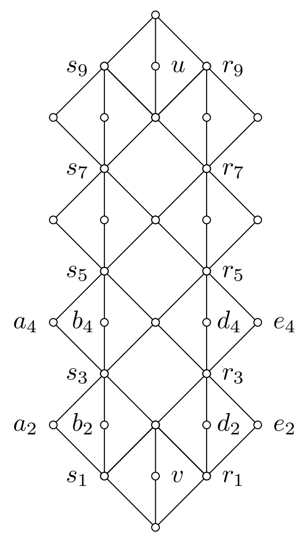

FIGURE 2. $\mathbf{L}_{10}$

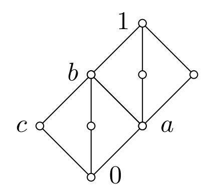

FIGURE 3. $\mathbf{M}_{33}$

If both $\mathbf{L}$ and $\mathbf{K}$ are copies of $\mathbf{M}_3$ and a is an atom of $\mathbf{L}$ and b is an atom of $\mathbf{K}$, then we can apply this construction using the filter of $\mathbf{L}$ generated by a and the ideal of $\mathbf{K}$ generated by b to obtain the lattice $\mathbf{M}_{33}$ of Figure 3.

Similarly $\mathbf{M}_{33}[\mathbf{Z}]$ is the Hall–Dilworth gluing of two copies of $\mathbf{M}_3[\mathbf{Z}]$ over the interval from $\overline{a}$ to $\overline{b}$. To prove this using the remarks above one only needs to verify that every element of $\mathbf{M}_{33}[\mathbf{Z}]$ is either below $\overline{b}$ or above $\overline{a}$, because the complemented intervals of $\mathbf{M}_{33}$ are either below b or above a.

A typical element of $[\overline{a}, \overline{b}]$ is c_i, where c_i is the function $c_i(k) = a$ if $k \leq i$ and b otherwise. Instead of using the identity map when gluing the two $\mathbf{M}_3[\mathbf{Z}]$'s, we could use the shift map: $c_i \mapsto c_{i-1}$, obtaining a lattice $\mathbf{M}_{33}[\mathbf{Z}]^*$. It is easy to check that $\mathbf{M}_{33}[\mathbf{Z}]^* \cong \mathbf{M}_{33}[\mathbf{Z}]$.

In $\mathbf{L}_n - \{u, v\}$ there are two elements of each odd dimension. We can arrange these elements into two chains $s_1 < s_3 < \cdots < s_{n-1}$ and $r_1 < r_3 < \cdots < r_{n-1}$; see Figure 2.

We have $s_1 < r_3$, and one can verify that every element in $\mathbf{L}_n[\mathbf{Z}]$ is either in the principal ideal $\mathbf{I}$ generated by $\overline{r}_3$ or in the principal filter $\mathbf{F}$ generated by $\overline{s}_1$. Thus $\mathbf{L}_n[\mathbf{Z}]$ is the Hall–Dilworth gluing of $\mathbf{I}$ and $\mathbf{F}$. $\mathbf{I}$ is isomorphic to $\mathbf{M}_{33}[\mathbf{Z}]$. Let $\mathbf{L}_n[\mathbf{Z}]^*$ be the result of gluing $\mathbf{M}_{33}[\mathbf{Z}]^*$ to $\mathbf{F}$ using the identity map on the interval $[\overline{s}_1, \overline{r}_3]$.

Unlike the situation with $\mathbf{M}_{33}[\mathbf{Z}]^*$, $\mathbf{L}_n[\mathbf{Z}]^*$ is not isomorphic to $\mathbf{L}_n[\mathbf{Z}]$. In fact $\mathbf{L}_n$ has the sequence of transpositions

$$\begin{aligned}[0, v] \nearrow [a_2, s_3] \searrow [s_1, b_2] \nearrow [e_4, r_5] \searrow [r_3, d_4] \\ \cdots \nearrow [u, 1] \searrow \cdots \searrow [r_1, d_2] \nearrow [e_2, r_3] \searrow [0, v]\end{aligned}$$

where a_i, $b_i \in [s_{i-1}, s_{i+1}]$ and d_i, $e_i \in [r_{i-1}, r_{i+1}]$ are the four irreducible elements of rank i with i even.

Of course this defines an automorphism on $[\overline{0}, \overline{v}]$ in $\mathbf{L}_n[\mathbf{Z}]^*$ but since the sequence of transpositions goes through the shifted interval, this automorphism is the shift, sending each element x with $\overline{0} < x < \overline{v}$ to its lower cover. Thus the sublattice of $\mathbf{L}_n[\mathbf{Z}]^*$ generated by $\mathbf{L}_n$ and any element $\overline{0} < x < \overline{v}$ is infinite (in fact it is all of $\mathbf{L}_n[\mathbf{Z}]^*$, but we do not use this fact).

In order to complete the proof of Theorem 1 it suffices to show that for each N we can choose n large enough so that the N-generated sublattices of $\mathbf{L}_n[\mathbf{Z}]^*$ lie in $\boldsymbol{V}(\mathbf{L}_\infty) = \boldsymbol{V}(\mathbf{L}_\infty[\mathbf{Z}])$. As usual, we view $\mathbf{L}_n$ as embedded in $\mathbf{L}_n[\mathbf{Z}]^*$ by the diagonal map. Every element of $x \in \mathbf{L}_n[\mathbf{Z}]^*$ lies in a uniquely determined complemented interval $[z_x, t_x]$ of $\mathbf{L}_n$ of minimal dimension, where z_x is just the join of all elements of $\mathbf{L}_n$ below x, and t_x is defined dually. Of course, the dimension (in $\mathbf{L}_n$) of $[z_x, t_x]$ is at most 2. Thus we may assign to each element x of $\mathbf{L}_n[\mathbf{Z}]^*$ a *lower* and *upper rank* (the ranks in $\mathbf{L}_n$ of z_x and t_x), and these differ by at most 2. So if we let $\mathbf{S}$ be an N-generated sublattice of $\mathbf{L}_n[\mathbf{Z}]^*$, then an argument similar to the proof of Lemma 2 shows that, if $\mathbf{S}$ is linearly indecomposable, it lies in an interval $[\overline{a}, \overline{b}]$ of $\mathbf{L}_n[\mathbf{Z}]^*$ where the dimension of $[a, b]$ in $\mathbf{L}_n$ is at most $5(N + 2)$. It follows that, for any N-generated sublattice $\mathbf{S}$, if $n > 5(N + 2) + 4$ then either $\mathbf{S}$ lies in the filter $[\overline{r}_1 \vee \overline{s}_1, \overline{1}]$ or the ideal $[\overline{0}, \overline{r}_{n-1} \wedge \overline{s}_{n-1}]$ or is the linear sum of two such lattices. So it suffices to show that this filter and ideal are isomorphic to sublattices of $\mathbf{L}_\infty[\mathbf{Z}]$. For the filter this is straightforward, since the filter does not contain the shifted interval.

Let $\mathbf{I}$ denote the ideal. To see that it is also embeddable into $\mathbf{L}_\infty[\mathbf{Z}]$ let $\mathbf{P}_n$ be the filter $[s_1, 1]$ of $\mathbf{K}_n$. (Recall $\mathbf{K}_n$ is $\mathbf{L}_n$ with u removed.)

Now $\mathbf{I}$ is isomorphic to $\mathbf{K}_{n-2}[\mathbf{Z}]^*$ and $\mathbf{K}_{n-2}[\mathbf{Z}]$ is naturally embeddable into $\mathbf{B}[\mathbf{Z}]$ (recall $\mathbf{B}$ is bottom half of $\mathbf{L}_\infty$) and so into $\mathbf{L}_\infty[\mathbf{Z}]$. Thus it suffices to show that

$\mathbf{K}_n[\mathbf{Z}] \cong \mathbf{K}_n[\mathbf{Z}]^*$. In $\mathbf{P}_n$ every prime interval is projective with either $[r_1 \vee s_1, s_3]$ or $[r_1 \vee s_1, r_3]$, but not both. So $\mathbf{P}_n$ is a subdirect product of two lattices, say $\mathbf{Q}$ and $\mathbf{R}$, and $\mathbf{P}_n[\mathbf{Z}]$ is a subdirect product of $\mathbf{Q}[\mathbf{Z}]$ and $\mathbf{R}[\mathbf{Z}]$. So $\mathbf{P}_n[\mathbf{Z}]$ has an automorphism τ which is the shift operator on one of these factors and the identity on the other. To make τ explicit, let $x \in \mathbf{P}_n[\mathbf{Z}]$ and let z_x and t_x be the elements of $\mathbf{P}_n$ defined above. If $z_x = t_x$ then $x \in \mathbf{P}_n$ and $\tau(x) = x$. Suppose the dimension of $[z_x, t_x]$ is 1. If $[z_x, t_x]$ projects to $[r_1 \vee s_1, r_3]$ then we apply the shift operator on $[\overline{z}_x, \overline{t}_x]$ to x; that is, $\tau(x)$ is the unique upper cover of x in $[\overline{z}_x, \overline{t}_x]$. If $[z_x, t_x]$ projects to $[r_1 \vee s_1, s_3]$ then $\tau(x) = x$. If the dimension of $[z_x, t_x]$ is 2, then $[z_x, t_x]$ is isomorphic to either $\mathbf{M}_3$ or $\mathbf{2} \times \mathbf{2}$. In the former case, if the prime quotients of $[z_x, t_x] \cong \mathbf{M}_3$ project to $[r_1 \vee s_1, r_3]$, then we apply the shift operator to x, otherwise x is fixed. Finally, if $[z_x, t_x] \cong \mathbf{2} \times \mathbf{2}$ and r and s are the atoms of $[z_x, t_x]$, then $x = (x \wedge \overline{r}) \vee (x \wedge \overline{s})$ and $\tau(x) = \tau(x \wedge \overline{r}) \vee \tau(x \wedge \overline{s})$.

Now we define $\rho \colon \mathbf{K}_n[\mathbf{Z}] \to \mathbf{K}_n[\mathbf{Z}]^*$. If $x \geq \overline{s}_1$ we let $\rho(x) = \tau(x)$. If $x \ngeq \overline{s}_1$ then $x \leq \overline{r}_3$. If such an x is above $\overline{r}_1$, then ρ applies the shift operator to x; otherwise $\rho(x) = x$. Since $[\overline{0}, \overline{r}_3]$ in $\mathbf{K}_n[\mathbf{Z}]^*$ is obtained from gluing two copies of $\mathbf{M}_3$ by applying the shift operator to the top copy, ρ restricted to $[\overline{0}, \overline{r}_3]$ is an isomorphism. Finally, one checks that ρ is one-to-one and onto and that it preserves order and so is an isomorphism.

This completes the proof of Theorem 1. □

In [5] we gave an example of a locally finite variety which is not finitely based but fails to be inherently nonfinitely based. In order to produce modular variety with these properties we use some results of [4]. There it is shown that the variety $\mathcal{M}_4^\infty$ generated by all modular lattices of width four is finitely based and locally finite. It is also shown that $\mathcal{M}_4^\infty$ has uncountably many subvarieties. Of course not all of these are finitely based, but being subvarieties of $\mathcal{M}_4^\infty$, none of them can be inherently nonfinitely based.

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