An extension of Willard's Finite Basis Theorem: Congruence meet-semidistributive varieties of finite critical depth

KIRBY A. BAKER, GEORGE F. MCNULTY, AND JU WANG

In Celebration of the Sixtieth Birthday of Ralph N. McKenzie

ABSTRACT. Ross Willard proved that every congruence meet-semidistributive variety of algebras that has a finite residual bound and a finite signature can be axiomatized by some finite set of equations. We offer here a simplification of Willard's proof, avoiding its use of Ramsey's Theorem. This simplification also extends Willard's theorem by replacing the finite residual bound with a weaker condition.

1. Introduction

Over the last thirty years the following theorems have been established which draw the conclusion that a variety of finite signature is finitely based from the stipulation that the variety has a finite residual bound coupled with a condition on the congruence lattices.

Baker's Finite Basis Theorem(Baker, 1977). *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is congruence distributive and has a finite residual bound, then $\mathcal{V}$ is finitely based.*

McKenzie's Finite Basis Theorem(McKenzie, 1987). *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is congruence modular and has a finite residual bound, then $\mathcal{V}$ is finitely based.*

Willard's Finite Basis Theorem(Willard, 2000). *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is congruence meet-semidistributive and has a finite residual bound, then $\mathcal{V}$ is finitely based.*

For lattices, modularity and meet-semidistributivity are familiar generalizations of distributivity. The prevalence of congruence distributive varieties (especially for algebras arising from logic) and of congruence modular varieties (for algebras possessing group operations) has led to a vigorous investigation of these congruence properties. Semilattices have meet-semidistributive congruence lattices. So does any algebra which includes among its basic operations (or even among its term-operations) the basic operation of semilattices, as shown, for example, by J. B. Nation in 1971. Such algebras have recently played prominent roles in McKenzie's solution of Tarski's Finite Basis Problem, see (McKenzie, 1996a; 1996b; 1996c),

Date: 2 December 2003.

and Székely's example of a finite algebra that generates of a variety with an NP-complete finite algebra membership problem, see (Székely, 1998; 2002).

The proof in (Willard, 2000) of Willard's Finite Basis Theorem relies on a new characterization of congruence meet-semidistributivity. It also shares many points in common with the proof in (Baker, 1977). In particular, Willard's proof, like Baker's, includes an involved combinatorial argument invoking Ramsey's Theorem. A number of alternate proofs of Baker's Finite Basis Theorem have been published. The most direct is that given recently by Baker and Wang (2002). It avoids the involved combinatorial argument and even the use of the Jónsson terms characterizing congruence distributivity. One of the objectives of the present paper is to provide a proof of Willard's Finite Basis Theorem that avoids the involved combinatorial argument.

Independently of this paper, Maróti and McKenzie (preprint) have recently given a different proof of Willard's Finite Basis Theorem that avoids the use of Ramsey's Theorem. Their argument depends on a variant of the Principal Meet Theorem below. This work of Maroti and McKenzie, in fact, investigates finite basis questions for congruence meet-semidistributive quasivarieties and for relatively congruence distributive varieties, in the presence of a finite residual bound.

An algebra $\mathbf{A}$ is said to be **finitely subdirectly irreducible** if and only if for every finite set X of pairs of distinct elements of A there are two distinct elements $a, b \in A$ so that any homomorphism that separates a and b also separates all the pairs in X. For any class $\mathcal{K}$ of algebras we let $\mathcal{K}_{\text{fsi}}$ denote the class of all finitely subdirectly irreducible algebras belonging to $\mathcal{K}$. Every subdirectly irreducible algebra is finitely subdirectly irreducible and every finite algebra which is finitely subdirectly irreducible is subdirectly irreducible.

Bjarni Jónsson (1979) provided a generalization of Baker's Finite Theorem.

Jónsson's Finite Basis Theorem(Jónsson, 1979). *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is congruence distributive and $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable, then $\mathcal{V}$ is finitely based.*

Jónsson's proof also avoids the involved combinatorial argument. Loosely speaking, we follow in Jónsson's footsteps and give a corresponding generalization of Willard's Finite Basis Theorem. Just as Jónsson's approach relied on key elements of Baker's original proof, so our approach relies on two key elements of Willard's proof.

Willard (2001) presents a discussion of the extension of Jónsson's Finite Basis Theorem to the congruence meet-semidistributive case. He presents several possible approaches, listing problems that these approaches encounter. We provide progress on several of these problems. In particular, our Principal Meet Theorem solves Willard's Problem 4.8 and also Problem 4.7 (but the latter under the additional stipulation that $\mathcal{V}$ has bounded critical depth—a concept explained several paragraphs below). We also offer some progress on Problem 5.2 and Problem 5.3, but with the notion of bounded critical depth replacing term-finite principal congruences (alias bounded Mal′cev depth).

Let $\mathbf{A}$ be an algebra. A **basic translation** of $\mathbf{A}$ is a one-place function on A of the form $Q^{\mathbf{A}}(a_0, a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{r-1})$ where Q is an operation symbol of positive rank r, where $a_j \in A$ for all $j < r$, and where $i < r$. The elements $a_0, a_1, \ldots, a_{r-1}$ are called the **coefficients** of the translation. This basic translation is associated with the term $Qz_0z_1 \ldots z_{i-1}xz_{i+1} \ldots z_{r-1}$. A **translation** is the composition of some finite sequence of basic translations. For a natural number ℓ, we say that a translation **has complexity** ℓ if it is the composition of a sequence of length no more than ℓ of basic translations. The identity map is the only translation of complexity 0. Each translation is associated with a term that is built from the terms associated to basic translations by means of repeated substitution.

Let $\mathbf{A}$ be an algebra and $a, b, q, r \in A$. We use the notation

$$\{a, b\} \looparrowright_{\ell}^{1} \{q, r\}$$

to mean

$$\{u(a), u(b)\} = \{q, r\} \text{ or } q = r$$

for some translation $u(x)$ of complexity ℓ. Further, we use the notation

$$\{a, b\} \looparrowright_{\ell}^{n} \{q, r\}$$

to mean that there are $g_0, g_1, \ldots, g_n \in A$ such that

$$q = g_0$$
$$\{a, b\} \looparrowright_{\ell}^{1} \{g_i, g_{i+1}\} \text{ for all } i < n$$
$$g_n = r$$

We also use $\{a, b\} \looparrowright_{k}^{m} \circ \looparrowright_{\ell}^{n} \{q, r\}$ to mean

$$\{a, b\} \looparrowright_{k}^{m} \{c, d\} \text{ and } \{c, d\} \looparrowright_{\ell}^{n} \{q, r\}$$

for some $c, d \in A$, in which case we can compose and conclude that

$$\{a, b\} \looparrowright_{k+\ell}^{mn} \{q, r\}.$$

According to the description of principal congruences implicit in (Mal′cev, 1954; 1963), we have

$$\langle q, r\rangle \in \mathrm{Cg}^{\mathbf{A}}(a, b) \text{ if and only if } \{a, b\} \looparrowright_{\ell}^{n} \{q, r\} \text{ for some } n \text{ and } \ell.$$

For finite signatures there are only finitely many terms associated to basic translations. Therefore, we regard $\{x, y\} \looparrowright_{\ell}^{n} \{z, w\}$ as an elementary (first-order) formula. When the signature is not finite, we must in addition specify a finite set F of operation symbols and construe

$$\{x, y\} \looparrowright_{\ell,F}^{n} \{z, w\}$$

as referring to translations of complexity ℓ built with the help only of operation symbols belonging to F.

A class $\mathcal{K}$ of algebras of the same finite signature is said to have **bounded critical depth** provided there is a natural number ℓ so that for every $\mathbf{A} \in \mathcal{K}_{\mathrm{si}}$ and all $a, b, c, d \in A$ such that $c \neq d$ and $\langle a, b\rangle$ is a critical pair of $\mathbf{A}$ we have $\{c, d\} \looparrowright_{\ell}^{n} \{a, b\}$ for some natural number n. When the signature is not finite in addition to the existence of ℓ there must also exist a finite set F of operation symbols to which this notion is relativized.

Main Theorem. *If $\mathcal{V}$ is a locally finite, congruence meet-semidistributive variety of finite signature and of bounded critical depth, and if $\mathcal{V}_{\rm fsi}$ is finitely axiomatizable, then $\mathcal{V}$ is finitely based.*

Actually, we prove something slightly stronger, since we only need to bound critical depth in the finite algebras in $\mathcal{V}_{\rm si}$.

The following Principal Meet Theorem, which is of independent interest, encapsulates the key step in the proof of the Main Theorem.

Principal Meet Theorem. *If $\mathcal{V}$ is a locally finite, congruence meet-semidistributive variety of finite signature and of bounded critical depth, then there is an elementary formula $\Pi(x, y, z, w)$ such that for all algebras $\mathbf{B} \in \mathcal{V}$ and all $a, b, c, d \in B$*

$$\mathrm{Cg}^{\mathbf{B}}(a, b) \cap \mathrm{Cg}^{\mathbf{B}}(c, d) \text{ is nontrivial if and only if } \mathbf{B} \models \Pi(a, b, c, d).$$

A proof of the Main Theorem occupies the following section, which includes a proof of The Principal Meet Theorem in the guise of a more detailed Principal Meet Lemma. The remainder of this introduction describes how Willard's Finite Basis Theorem follows from the Main Theorem; it goes on to describe two corollaries of the Main Theorem and concludes with an example of a finitely generated variety to which the Main Theorem applies but to which Willard's Finite Basis Theorem does not.

Willard's Finite Basis Theorem follows from the Main Theorem easily. In fact, congruence meet-semidistributivity plays no role in deducing Willard's Theorem from the Main Theorem (other than as a hypothesis that is passed along). In view of the finite residual bound, up to isomorphism, $\mathcal{V}_{\rm si}$ is a finite set of finite algebras. Evidently, $\mathcal{V}$ has bounded critical depth. Also, $\mathcal{V}$ is locally finite since it is generated by this finite set of finite algebras. Furthermore, $\mathcal{V}_{\rm si}$ is finitely axiomatizable. It only remains to establish that $\mathcal{V}_{\rm fsi}$ is also finitely axiomatizable. It turns out that $\mathcal{V}_{\rm si} = \mathcal{V}_{\rm fsi}$. As Ross Willard remarked, this is probably part of the folklore. A short proof is sketched by Ross Willard (2000) in the proof of his Lemma 4.2. We give here a different proof (which might also be part of the folklore).

Folklore Lemma. *Let $\mathcal{V}$ be a variety such that $\mathcal{V}_{\rm si}$ is axiomatizable by a set of elementary sentences. Every finitely subdirectly irreducible algebra in $\mathcal{V}$ is embeddable into some subdirectly irreducible algebra in $\mathcal{V}$.*

Proof. Let $\mathbf{B}$ be a finitely subdirectly irreducible algebra belonging to $\mathcal{V}$ and let Φ be a set of elementary sentences which axiomatizes $\mathcal{V}_{\rm si}$. Expand the signature by adding a new constant to name each element of B. We use $\mathbf{B}^*$ to denote the corresponding expansion of $\mathbf{B}$. Let Δ be the atomic diagram of $\mathbf{B}$. To prove the Lemma we need to show that $\Delta \cup \Phi$ has a model. In view of the Compactness Theorem, we need only show that $\Gamma \cup \Phi$ has a model whenever Γ is a finite subset of Δ. So consider such a Γ. Without loss of generality, we assume that the only negated equations in Γ are of the form $c \neq d$ were c and d are constant symbols.

Let S be the set of all elements of B named by constants occurring in Γ. Since S is finite and $\mathbf{B}$ is finitely subdirectly irreducible, pick $p, q \in B$ with $p \neq q$ and so that $\langle p, q \rangle \in \mathrm{Cg}^{\mathbf{B}}(r, s)$ whenever r and s are distinct elements of S. Let θ be a maximal congruence of $\mathbf{B}$ which separates r and s. Then $\mathbf{B}/\theta$ is subdirectly irreducible. So $\mathbf{B}^*/\theta \models \Phi$. Now the equations in Γ hold in $\mathbf{B}^*/\theta$ since equations are preserved in the passage to quotient algebras. The negated equations in Γ also hold in $\mathbf{B}^*/\theta$ since θ separates all the elements of S. Therefore, $\mathbf{B}^*/\theta \models \Gamma \cup \Phi$, as desired. □

Thus in a variety $\mathcal{V}$ of finite signature with a finite residual bound b, every finitely subdirectly irreducible algebra must have cardinality bounded by b as well. It follows that every finitely subdirectly irreducible algebra is finite and so it is subdirectly irreducible. Thus, $\mathcal{V}_{\mathrm{fsi}} = \mathcal{V}_{\mathrm{si}}$ and therefore $\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable.

In this way, Willard's Finite Basis Theorem follows from the Main Theorem.

A class $\mathcal{K}$ of algebras sharing a finite signature is said to have **term-finite principal congruences** provided there is a natural number ℓ so that for every $\mathbf{A} \in \mathcal{K}$ and all $a, b, c, d \in A$ we have $\langle a, b \rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d)$ if and only if $\{c, d\} \looparrowright_{\ell}^{n} \{a, b\}$ for some natural number n. For infinite signatures, we must also demand the existence of a finite set F of operation symbols from which to build the required translations. For finite signatures, this notion was introduced by Baker (1983). Independently, the notion (for arbitrary signatures) was introduced recently by David Clark, Brian Davey, Ralph Freese, and Marcel Jackson (preprint). Ju Wang (1988), (1990)(see also (Baker and Wang, to appear)) has shown that finitely generated congruence distributive varieties of finite signature have term-finite principal congruences. Wang employed the name *finite principal length*. It is an enticing open problem whether the same holds in the case of congruence meet-semidistributive varieties with finite residual bounds (or which have bounded critical depth). Evidently, if $\mathcal{V}_{\mathrm{si}}$ has term-finite principal congruences, then $\mathcal{V}$ has bounded critical depth. So the following corollary is obtained.

Corollary 1. *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, $\mathcal{V}_{\mathrm{si}}$ has term-finite principal congruences, and $\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable, then $\mathcal{V}$ is finitely based.*

In view of the results in (Wang, 1990), (Baker and Wang, to appear) and in (Clark et al., preprint), Corollary 1 gives a generalization of Jónsson's Finite Basis Theorem, in the case of locally finite varieties of finite signature.

A class $\mathcal{K}$ of algebras of the same finite signature is said to have **bounded critical diameter** provided there is a natural number ℓ so that for every $\mathbf{A} \in \mathcal{K}_{\mathrm{si}}$ such that A is finite and all $a, b, c, d \in A$ such that $\langle a, b \rangle$ and $\langle c, d \rangle$ are critical pairs of $\mathbf{A}$ we have $\{c, d\} \looparrowright_{\ell}^{n} \{a, b\}$ for some natural number n. As above, this notion can be extended to arbitrary signatures by requiring the existence of some finite set F of operation symbols from which to devise the required translations.

Corollary 2. *Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is a locally finite, congruence meet-semidistributive variety with bounded critical diameter such that $\mathcal{V}_{\mathrm{si}}$ is elementary and $\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable, then $\mathcal{V}$ is finitely based.*

Corollary 2 is an immediate consequence of the Main Theorem and the Lemma below. For each natural number k let σ_k be the following elementary sentence:

$$\exists z, w[z \not\approx w \mathbin{⩓} \forall x, y(x \not\approx y \Rightarrow (\{x,y\} \hookrightarrow^k_k \{z,w\}))].$$

Thus, σ_k asserts, among other things, that there is a critical pair.

Lemma. *Let $\mathcal{V}$ be an elementary class of finite signature. The following conditions are equivalent:*

(i) *$\mathcal{V}_{\rm si}$ is an elementary class.*
(ii) *$\mathcal{V}_{\rm si}$ is closed with respect to ultraproducts.*
(iii) *There is a natural number ℓ such that $\mathbf{A} \in \mathcal{V}_{\rm si}$ if and only if $\mathbf{A} \in \mathcal{V}$ and $\mathbf{A} \models \sigma_\ell$.*
(iv) *$\mathcal{V}_{\rm si}$ is finitely axiomatizable relative to $\mathcal{V}$.*

Proof. Taking the implications cyclically, only (ii) $\Longrightarrow$ (iii) is not immediate.

Consider condition (iii). Evidently, if $\mathbf{A} \in \mathcal{V}$ and $\mathbf{A} \models \sigma_\ell$, then $\mathbf{A} \in \mathcal{V}_{\rm si}$, no matter what value ℓ has.

To prove the converse of (iii), suppose for the sake of contradiction, that $\mathcal{V}_{\rm si}$ is an closed with respect to ultraproducts but that $\mathcal{V}_{\rm si} \not\models \sigma_k$ for all k.

For each k pick $\mathbf{S}_k \in \mathcal{V}_{\rm si}$ so that $\mathbf{S}_k \models \neg\sigma_k$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on the set ω of natural numbers. Finally, let $\mathbf{S} = \prod_{k\in\omega} \mathbf{S}_k/\mathcal{U}$. Since $\mathcal{V}_{\rm si}$ is closed with respect to the formation of ultraproducts, we see $\mathbf{S} \in \mathcal{V}_{\rm si}$.

Let $\langle \overline{c}, \overline{d}\rangle$ be a critical pair for $\mathbf{S}$. For each $k \in \omega$ pick $c_k, d_k \in S_k$ so that

$$\overline{c} = \langle c_0, c_1, \dots\rangle/\mathcal{U} \text{ and}$$
$$\overline{d} = \langle d_0, d_1, \dots\rangle/\mathcal{U}$$

For each $k \in \omega$ we pick distinct a_k and b_k in S_k so that $\mathbf{S}_k \models \{a_k, b_k\} \not\hookrightarrow^k_k \{c_k, d_k\}$. Let

$$\overline{a} = \langle a_0, a_1, \dots\rangle/\mathcal{U} \text{ and}$$
$$\overline{b} = \langle b_0, b_1, \dots\rangle/\mathcal{U}$$

Since $\{k \mid a_k \neq b_k\} = \omega \in \mathcal{U}$, we have that $\overline{a}$ and $\overline{b}$ are distinct elements of S. Since $\langle \overline{c}, \overline{d}\rangle$ is a critical pair for $\mathbf{S}$, pick a natural number ℓ so that

$$\{\overline{a}, \overline{b}\} \hookrightarrow^\ell_\ell \{\overline{c}, \overline{d}\}.$$

By Łos' Theorem,

$$\{k \mid \{a_k, b_k\} \hookrightarrow^\ell_\ell \{c_k, d_k\}\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is nonprincipal, $\{k \mid k \geq \ell\} \in \mathcal{U}$. It follows that

$$\{k \mid \{a_k, b_k\} \hookrightarrow^\ell_\ell \{c_k, d_k\} \text{ and } k \geq \ell\} \in \mathcal{U}.$$

On the other hand,

$$\{k \mid \{a_k, b_k\} \hookrightarrow^\ell_\ell \{c_k, d_k\} \text{ and } k \geq \ell\} \subseteq \{k \mid \{a_k, b_k\} \hookrightarrow^k_k \{c_k, d_k\} \text{ and } k \geq \ell\}.$$

But this last set is empty, by the way in which we selected the a_k's and b_k's. Thus we find that the empty set belongs to the nonprincipal ultrafilter $\mathcal{U}$. This is a contradiction. □

A version of this Lemma holds with $\mathcal{V}_{\mathrm{fsi}}$ in place of $\mathcal{V}_{\mathrm{si}}$ and τ_ℓ in place of σ_ℓ, where τ_ℓ is the following sentence:

$$\forall x, y, z, w[(x \not\approx y \mathbin{⩕} z \not\approx w) \implies \exists u, v(u \not\approx v \mathbin{⩕} (\{x, y\} \looparrowright_\ell^\ell \{u, v\}) \mathbin{⩕} (\{z, w\} \looparrowright_\ell^\ell \{u, v\}))]$$

The proof of this variant requires only modest changes in the proof given above.

It may be that having a bounded critical diameter is a consequence of the other hypotheses of Corollary 2—or at least those hypotheses strengthened so that the $\mathcal{V}_{\mathrm{si}}$ is finitely axiomatizable. One attempt to prove this was to find a finite bound b so that the number of critical pairs in any algebra in $\mathcal{V}_{\mathrm{si}}$ never exceeded b. Ralph McKenzie shared with us a counterexample showing that such a b need not exist. We observe here that this same example provides a case to which Corollary 2 applies, even though Willard's Finite Basis Theorem does not.

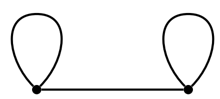
McKenzie's example is a flat graph algebra. Graph algebras originated in the Ph.D. dissertation of Caroline Shallon (1979), see also (McNulty and Shallon, 1983). Flat algebras gained prominence in (McKenzie, 1996a; 1996b; 1996c) and their theory was developed further in (Willard, 1996). Flat graph algebras are discussed in (Székely, 1998; 2002; Delić, 2001; Lampe et al., 2001).

Given a graph $\mathbf{G}$ with vertex set V the **flat graph algebra $\mathbf{A}_{\mathbf{G}}$** of $\mathbf{G}$ is the algebra with universe $V \cup \{0\}$, where $0 \notin V$, and two binary operations $\cdot$ and $\wedge$ defined via

$$x \cdot y = \begin{cases} x & \text{if there is an edge of } \mathbf{G} \text{ joining } x \text{ and } y \\ 0 & \text{otherwise} \end{cases}$$

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{A}$ be the flat graph algebra for the graph displayed below:



Let $\mathcal{V}$ be the variety generated by $\mathbf{A}$. McKenzie observed that

i. $\mathcal{V}_{\mathrm{si}}$ consists of those flat graph algebras associated with complete graphs with loops at each vertex.
ii. Thus, $\mathcal{V}_{\mathrm{si}}$ is finitely axiomatizable.
iii. Also, $\mathcal{V}_{\mathrm{si}}$ consists entirely of simple algebras.
iv. Hence, every pair of distinct elements of any member of $\mathcal{V}_{\mathrm{si}}$ is critical.

v. There is no finite bound on the number of critical pairs, since $\mathcal{V}$ has simple algebras of every cardinality larger than 1.

The later conclusions all follow from (i), which can be found, in essence, in (Delić, 2001). Similar arguments, although restricted to the finite members of $\mathcal{V}_{\mathrm{si}}$ can be found in (Székely, 1998). Dejan Delić (2001) actually characterizes the finitely based flat graph algebras ($\mathbf{A}$ is one such) and shows in fact that they generate varieties with definable principal congruences. As a consequence our particular variety $\mathcal{V}$ has a bounded critical diameter. It is also easy to verify this directly. Indeed, suppose that $\mathbf{S} \in \mathcal{V}_{\mathrm{si}}$. We regard the elements of S as vertices of a complete graph together with the default element 0. Let $\{a, b\}$ and $\{c, d\}$ be two-element subsets of S with $a \neq 0$. Then

$$\{a, b\} \looparrowright_1^1 \{a, 0\} \looparrowright_1^2 \{c, d\}$$

where the translation on the left is $a \wedge x$ and the translations on the right are $c \cdot x$ and $d \cdot x$. So 2 bounds the critical diameter.

Finally, observe that $\mathcal{V}_{\mathrm{si}}$ is closed under the formation of nontrivial subalgebras. It follows from the Folklore Lemma that $\mathcal{V}_{\mathrm{si}} = \mathcal{V}_{\mathrm{fsi}}$. So $\mathcal{V}_{\mathrm{fsi}}$ is finitely based. Hence, $\mathcal{V}$ satisfies all the hypotheses of Corollary 2 but it is residually large. This means, in particular, that the Main Theorem applies to finitely generated varieties that do not fall under Willard's Finite Basis Theorem.

2. Proof of the Main Theorem

The proof depends on a sequence of lemmas. Building on work of Czédli (1983), Kearnes and Szendrei (1998), Hobby and McKenzie (1988), and Lipparini (1998), Willard (2000) characterized congruence meet-semidistributive varieties by six equivalent conditions. The following theorem and lemma encapsulate a portion of that information, in modified language.

By a *bracket expression*, let us mean a string of bracket symbols, for example $\langle\,\langle\,\langle\,\rangle\,\langle\,\rangle\,\rangle\,\langle\,\rangle\,\rangle$, constructed recursively by the rules (a) $\langle\,\rangle$ is a bracket expression and (b) for $k \geq 1$ if $\beta_1, \ldots, \beta_k$ are bracket expressions then so is $\langle \beta_1 \ldots \beta_k \rangle$. If $\beta = b_0 ... b_n$ is a bracket expression, let us say that a left bracket $b_i = \langle$ and right bracket $b_j = \rangle$ are *matched* if they bound a subexpression involved in the construction; equivalently, we may say that their indices i, j are matched.

Theorem 1 (Willard). *A variety $\mathcal{V}$ is congruence meet-semidistributive if and only if there are a bracket expression β and ternary terms $t_0, \ldots, t_n$ in the language of $\mathcal{V}$ obeying the set Σ_β of laws*

$t_0(x, y, z) \approx x$	$t_n(x, y, z) \approx z$
$t_i(x, x, y) \approx t_{i+1}(x, x, y)$	*for each even* $i < n$
$t_i(x, y, y) \approx t_{i+1}(x, y, y)$	*for each odd* $i < n$
$t_i(x, y, x) \approx t_j(x, y, x)$	*for each matched pair of indices* i, j

Here the bracket expressions correspond to trees in Willard's presentation and parity of subscripts corresponds to colors of nodes.

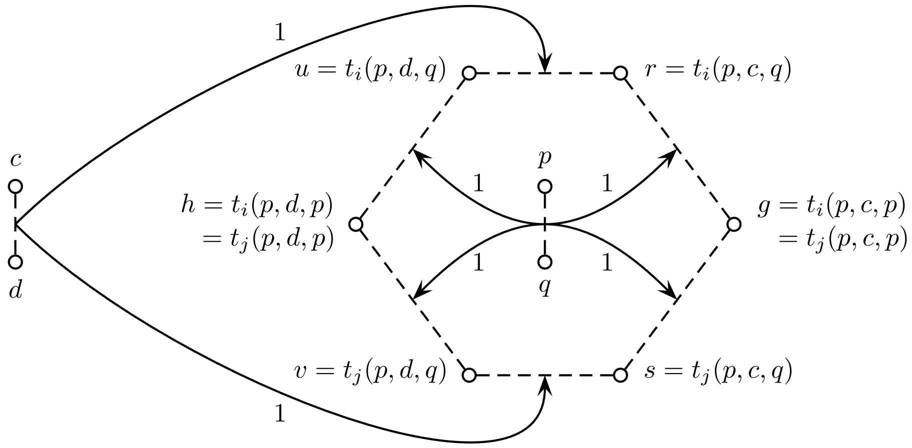


FIGURE 1. A hexagonal configuration

It simplifies matters to assume that the Willard terms t_i are basic operations of $\mathcal{V}$, and we do so freely in the following. For example, for $\mathbf{B} \in \mathcal{V}$ and matched bracket indices i, j, any $c, d, p, q \in B$ generate the hexagonal configuration of Figure 1, or a "homomorphic image" of it, where the arrows represent translations of length 1 and $\{p, q\} \looparrowright_1^2 \{r, s\}$, $\{p, q\} \looparrowright_1^2 \{u, v\}$. A possible image, for example, would be the diamond configuration of Figure 2.

Lemma 1 (Willard, 2000, Corollary 3.3 for $N = 1$). *Suppose that $\mathcal{V}$ is congruence meet-semidistributive, with Willard terms assumed to be basic operations, and let $\mathbf{A} \in \mathcal{V}$. If $\langle p, q \rangle \in \mathrm{Cg}^{\mathbf{A}}(a, b)$ with $p \neq q$, so that $\{a, b\} \looparrowright_m^n \{p, q\}$ for some m, n, then there exist matched bracket indices i, j, elements $c, d \in A$, and a translation $\{a, b\} \looparrowright_m^1 \{c, d\}$ giving the diamond configuration of Figure 2, with*

$$u = v, \quad r \neq s, \quad \{a, b\} \looparrowright_m^1 \{c, d\} \looparrowright_1^2 \{r, s\}, \quad \{p, q\} \looparrowright_1^2 \{r, s\}.$$

In this event, let us say that $\{a, b\} \looparrowright_m^1 \circ \looparrowright_1^2 \{r, s\}$ via the diamond configuration generated by c, d, p, q with indices i, j. (In adapting Willard's proof to this notation, notice that by interchanging the roles of c and d if necessary, one can always obtain $u = v$ and $r \neq s$ rather than the other way around.)

It is helpful to observe that such translations through a diamond configuration are preserved by inverse homomorphisms provided that $\mathrm{Cg}^{\mathbf{A}}(p, q)$ is atomic:

Lemma 2. *Let $\mathcal{V}$ be congruence meet-semidistributive, with Willard terms assumed to be basic operations, and let $\varphi : \mathbf{A} \to \mathbf{B}$ be a surjection in $\mathcal{V}$. Suppose that a, b, c, d, p, q, r, s in A map respectively to $\bar{a}, \bar{b}$, etc., in B, in such a way that $\{\bar{a}, \bar{b}\} \looparrowright_m^1 \{\bar{c}, \bar{d}\} \looparrowright_1^2 \{\bar{r}, \bar{s}\}$ via the diamond configuration with indices i, j generated by $\bar{c}, \bar{d}, \bar{p}, \bar{q}$. If $\mathrm{Cg}^{\mathbf{A}}(p, q)$ is atomic in $\mathrm{Con}(\mathbf{A})$, then in $\mathbf{A}$ we can replace c, d, r, s by new elements of the same name (without modifying their images) so*

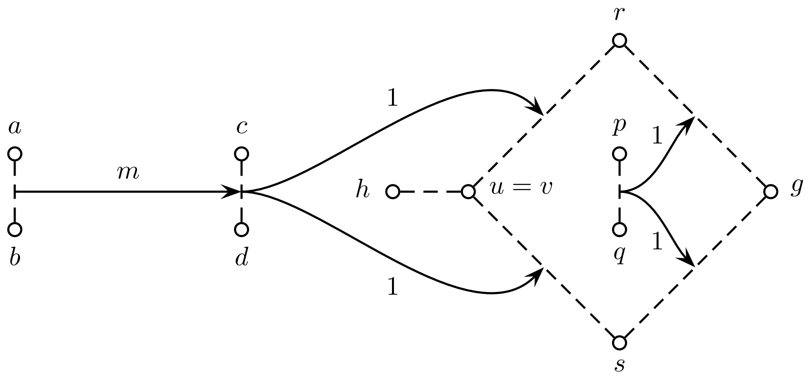


FIGURE 2. A diamond (collapsed-hexagon) configuration

that $\{a,b\} \looparrowright^1_m \{c,d\} \looparrowright^2_1 \{r,s\}$ *via the diamond configuration with indices* i,j *generated by* c,d,p,q.

Proof. By choosing any pre-images of the constants involved in the unary polynomial giving $\{a,b\} \looparrowright^1_m \{\bar{c},\bar{d}\}$, we obtain new $c,d \in A$ so that $\{a,b\} \looparrowright^1_m \{c,d\}$ in $\mathbf{A}$ and $\bar{c} = \bar{c}, \bar{d} = \bar{d}$. Then c,d,p,q at least generate a hexagonal configuration as in Figure 1, with possible equalities between vertices, in which r,s are recomputed as values of term functions. Since $\mathrm{Cg}^{\mathbf{A}}(p,q)$ is atomic and not under $\ker\varphi$, and since $\{p,q\} \looparrowright^2_1 \{u,v\}$ we have $\langle u,v\rangle \in \ker\varphi \cap \mathrm{Cg}^{\mathbf{A}}(p,q) = 0$, so that $u = v$. Also $\bar{r} \neq \bar{s}$ entails $r \neq s$. Therefore the hexagonal configuration must collapse to a diamond configuration, as required. □

We now turn to the Principal Meet Lemma (Lemma 3), which gives criteria for nontriviality of meets of principal congruence relations in terms of a translation condition with depth parameter m. The first assertion describes the general situation, in which m depends on particular pairs of elements. The second assertion applies for given m to those algebras in which m is a sufficient length uniformly over all pairs. The third assertion gives a condition enabling us to choose a single value $m = \ell$ applying uniformly throughout the whole variety.

Lemma 3 (Principal Meet Lemma). *Let* $\mathcal{V}$ *be a congruence meet-semidistributive variety, with Willard terms assumed to be basic operations. For algebras* $\mathbf{A} \in \mathcal{V}$ *and for elements* $a_1, b_1, a_2, b_2 \in A$, *consider the conditions*

$$\mathrm{Cg}^{\mathbf{A}}(a_1,b_1) \cap \mathrm{Cg}^{\mathbf{A}}(a_2,b_2) > 0, \tag{*}$$

$$\{a_i,b_i\} \looparrowright^1_m \circ \looparrowright^4_2 \{r,s\} \text{ for some } r \neq s \text{ in } A \text{ and for } i = 1,2, \tag{$\dagger_m$}$$

for each $m = 1,2,\ldots$ *These conditions are related as follows.*

(1) For all $\mathbf{A} \in \mathcal{V}$ *and for all* $a_1,b_1,a_2,b_2 \in A$, (*) *holds if and only if for some* m $(\dagger_m)$ *holds.*

(2) If $\mathcal{V}$ *has finite signature, then for each* $m = 1,2,\ldots,$

(a) the condition $(\dagger_m)$ *can be expressed as* $\Pi_m(a_1, b_1, a_2, b_2)$*, where* $\Pi_m(x_1, y_1, x_2, y_2)$ *is an elementary formula in the language of* $\mathcal{V}$*;*

(b) if we denote by $\mathcal{V}^{[m]}$ *the subclass of* $\mathcal{V}$ *consisting of all* $\mathbf{A} \in \mathcal{V}$ *in which* (*) *and* $(\dagger_m)$ *are equivalent for all choices of* a_1, b_1, a_2, b_2*, then* $\mathcal{V}^{[m]}$ *is finitely axiomatizable relative to* $\mathcal{V}$*.*

(3) If $\mathcal{V}$ *is locally finite with critical depth bounded by* ℓ*, then* $\mathcal{V}^{[\ell]} = V$*, i.e.,* (*) *is equivalent to* $(\dagger_\ell)$ *for all choices of* a_1, b_1, a_2, b_2 *in all algebras of* $\mathcal{V}$*.*

Proof. For (1): If $(\dagger_m)$ holds for some m, observe that $0 \neq \{r, s\} \in \mathrm{Cg}^{\mathbf{A}}(a_1, b_1) \cap \mathrm{Cg}^{\mathbf{A}}(a_2, b_2)$, so that (*) holds. Suppose conversely that (*) holds. We reason as in the case $N = 2$ of (Willard, 2000, Corollary 3.3): Choose $p, q \in A$ with $p \neq q$ and $\mathrm{Cg}^{\mathbf{A}}(p, q) \subseteq \mathrm{Cg}^{\mathbf{A}}(a_1, b_1) \cap \mathrm{Cg}^{\mathbf{A}}(a_2, b_2)$. Then by Lemma 1, $\{a_1, b_1\} \looparrowright^1_{m_1} \{c_1, d_1\} \looparrowright^2_1 \{r_1, s_1\}$ and $\{p, q\} \looparrowright^2_1 \{r_1, s_1\}$ for suitable $c, d, r_1, s_1 \in A$ via a diamond configuration generated by c_1, d_1, p, q. Again by Lemma 1, using r_1, s_1 in place of p, q and noting that $\langle r_1, s_1 \rangle \in \mathrm{Cg}^{\mathbf{A}}(a_2, b_2)$, we see that for some m_2 there exist $c_2, d_2, r_2, s_2 \in A$ with $\{a_2, b_2\} \looparrowright^1_{m_2} \{c_2, d_2\} \looparrowright^2_1 \{r_2, s_2\}$. Then composing arrows and setting $r = r_2, s = s_2$, we have $\{a_1, b_1\} \looparrowright^1_m \circ \looparrowright^2_1 \circ \looparrowright^2_1 \{r, s\}$ and also $\{a_2, b_2\} \looparrowright^1_m \circ \looparrowright^2_1 \{r, s\}$, which yield $(\dagger_m)$, with m being the larger of m_1, m_2.

For (2a): As mentioned earlier, in a variety $\mathcal{V}$ of finite signature an assertion $\{x, y\} \looparrowright^n_m \{z, w\}$ is expressible as an elementary formula. Then $(\dagger_m)$ is the assertion $\Pi_m(a_1, b_1, a_2, b_2)$, where

$$\Pi_m(x_1, y_1, x_2, y_2) := (\exists r)(\exists s)(\neg r \approx s \wedge (\wedge^2_{i=1}\{x_i, y_i\} \looparrowright^1_m \circ \looparrowright^4_2 \{r, s\})).$$

For (2b): By an induction, $\mathcal{V}^{[m]}$ consists of the models in $\mathcal{V}$ of the sentence

$$\psi_m := (\forall x_1)(\forall y_1)(\forall x_2)(\forall y_2)(\Pi_{m+1}(x_1, y_1, x_2, y_2) \rightarrow \Pi_m(x_1, y_1, x_2, y_2)).$$

For (3): By Mal'tsev's construction, the presence of a nontrivial pair in the intersection can be determined within a finitely generated, and hence finite, subalgebra of $\mathbf{A}$. Without loss of generality we may assume that $\mathbf{A}$ is finite to begin with. Choose an atom $\alpha = \mathrm{Cg}^{\mathbf{A}}(p, q)$. Let $\theta \in \mathrm{Con}(\mathbf{A})$ be maximal splitting p, q, so that $\mathbf{S} = \mathbf{A}/\theta$ is subdirectly irreducible with critical pair $\{\bar{p}, \bar{q}\}$. Since the critical depth of $\mathbf{S}$ is bounded by ℓ, by Lemmas 1 and 2 we have $\{\bar{a}_1, \bar{b}_1\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{\bar{r}_1, \bar{s}_1\}$ in $\mathbf{S}$ via a diamond configuration that is the image of arrows $\{a_1, b_1\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{r_1, s_2\}$ in $\mathbf{A}$. Observe that $\mathrm{Cg}^{\mathbf{A}}(r_1, s_1)$ is contained in the atom α and so equals α. Therefore as in the proof of (1) we may repeat the process for $\{a_2, b_2\}$, to obtain the result of (1) but with $m = \ell$.

□

We are now able to complete the proofs of theorems stated in the Introduction:

Proof of the Principal Meet Theorem. By (3) of Lemma 3, the conditions (*) and $(\dagger_m)$ are equivalent throughout $\mathcal{V}$ for $m = \ell$. By (2a), the condition $(\dagger_m)$ is expressible as $\Pi_\ell(a_1, b_1, a_2, b_2)$, where $\Pi_\ell(x_1, y_1, x_2, y_2)$ is an elementary formula. □

Proof of the Main Theorem. Let $\mathcal{V}$ be congruence meet-semidistributive, locally finite, of critical depth bounded by ℓ, and of finite signature, with $\mathcal{V}_{\mathrm{fsi}}$ being finitely axiomatizable. It is harmless to assume that the Willard terms are basic operations.

Let $\mathcal{W}$ be the class of the same signature as $\mathcal{V}$ determined by the set Σ_β of Willard laws of $\mathcal{V}$ and consider the class $\mathcal{K} = \mathcal{W}^{[\ell]}$, in the notation of Lemma 3 applied to $\mathcal{W}$ in place of $\mathcal{V}$. Then we observe the following.

(i) $\mathcal{V}$ is contained in $\mathcal{K}$, since by (3) of Lemma 3 we have $V = V^{[\ell]}$.
(ii) $\mathcal{K}$ is finitely axiomatizable, since $\mathcal{W}$ is finitely axiomatizable by definition and $\mathcal{K}$ is finitely axiomatizable relative to $\mathcal{W}$, by (2b) of Lemma 3 applied to $\mathcal{W}$ in place of $\mathcal{V}$,
(iii) $\mathcal{K}_{\text{fsi}}$ is finitely axiomatizable via the sentence

$$(\forall x_1)(\forall y_1)(\forall x_2)(\forall y_2)(\Pi_\ell(x_1, y_1, x_2, y_2) \leftrightarrow x_1 \approx y_1 \vee x_2 \approx y_2),$$

by (2a) of Lemma 3 for $\mathcal{W}$.

We now have the ingredients to apply the technique of Jónsson (Jónsson, 1979b): If a variety $\mathcal{V}$ is contained in a finitely axiomatizable class $\mathcal{K}$ such that $\mathcal{K}_{\text{SI}}$ is contained in an axiomatizable class $\mathcal{C}$ (here $\mathcal{K}_{\text{fsi}}$) whose intersection (here $\mathcal{V}_{\text{fsi}}$) with $\mathcal{V}$ is finitely axiomatizable (here by hypothesis), then $\mathcal{V}$ is finitely based. □

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Department of Mathematics, University of California, Los Angeles, CA, USA

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

College of Mathematics and Computer Science, GuangXi Normal University, Guilin, 541005, China