Solutions to Unique Readability Homework Set
30 August 2011

In the problems below L is a signature and X is a set of variables.

Problem 0.
Define a function λ from the set of finite nonempty sequences of elements of $X \cup L$ into the integers as follows:

$$\lambda(w) = \begin{cases} -1 & \text{if } w \in X, \\ r-1 & \text{if } w \text{ is an operation symbol of rank } r, \\ \sum_{i<n} \lambda(u_i) & \text{if } w = u_0 u_1 \dots u_{n-1} \text{ where } u_i \in X \cup L \text{ and } n > 1. \end{cases}$$

Prove that w is a term if and only if $\lambda(w) = -1$ and $\lambda(v) \geq 0$ for every nonempty proper initial segment v of w.

Solution
Let us first tackle the implication from left to right. We will prove, by induction on the complexity of terms w that

$$\lambda(w) = -1 \text{ and } \lambda(v) \geq 0 \text{ for every nonempty proper initial segment } v \text{ of } w$$

For the base step of the induction, w either belongs to X or it is an operation symbol of rank 0. In either case, $\lambda(w) = -1$. Also, w has no nonempty proper initial segments, so the second requirement holds vacuuously.

For the inductive step we have $w = Qt_0 \dots t_{r-1}$, where $r > 0$ is the rank of the operation symbol Q and $t_0, \dots, t_{r-1}$ are terms less complex than w. By the definition of λ and the induction hypothesis, we have

$$\lambda(w) = (r-1) + \sum_{k<r} \lambda(t_k) = (r-1) + \sum_{k<r} -1 = (r-1) + (-1)r = -1.$$

This gets us the first requirement. We prove the second requirement by induction on the length of the nonempty proper initial segment v. The base step of this induction is $v = Q$. In this case, $\lambda(v) = r - 1 \geq 0$. The induction step has two cases. In the first, $v = Qt_0 \dots t_{k-1}$, where $k < r$. In the case, $\lambda(v) = (r-1) + (-1)k = (r-k) - 1 \geq O$. The second case is that $v = Qt_0 \dots t_{k-2}u$ where u is a nonempty proper initial segment of t_{k-1}. So we know $k \leq r$. In this case,

$$\lambda(v) = (r-1) + (-1)(k-1) + \lambda(u) = (r-k) + \lambda(u).$$

Since u is a nonempty proper initial segment of t_{k-1} and t_{k-1} is less complex than w, our induction hypothesis yields that $\lambda(u) \geq 0$. So $\lambda(v) \geq 0$, as desired. This completes the second induction and secures the left-to-right implication in the problem.

For the right-to-left implication, let

$$\begin{aligned} W := \{w \mid & w \text{ is a finite nonempty sequence of elements of } X \cup L \text{ so that} \\ & \lambda(w) = -1 \text{ and } \lambda(v) \geq 0 \text{ for all nonempty proper initial segments } v \text{ of } w\} \end{aligned}$$

We want to show that everything in W is a term. Pick $w \in W$.

If w has length 1 then w is a term since $\lambda(w) = -1$ and the only symbols with this value are the variables and the operation symbols of rank 0. These are terms. If w is longer then the leftmost symbol in w must have a nonnegative λ, making it an operation symbol of positive rank.

Claim

If Q is an operation symbol of positive rank occurring in w, then the final segment of w starting at Q (and going rightward to the end) is a sequence of terms.

To see this, consider the leftmost occurrence of such a Q and let r be its rank. So $w = AQB$, where A and B are strings of symbols. Observe that B must be a string consisting of symbols that are either variables or operation symbols of rank 0. Then $-1 = \lambda(w) = \lambda(A)+(r-1)-|B|$. Hence, $|B| = r + \lambda(A) \geq r$. Let $B = CD$ where $|C| = r$. Then D is a sequence, possibly empty, of terms and QC is a term. This means QB is a sequence of terms. This was secretly the first step of a process. The idea is now to find the next operation symbol of positive rank to the left of Q and continue. The general situation is $w = AQBs_0 \ldots s_{m-1}$, where the s_j's are terms and Q is the first operation symbol of positive rank to the left of s_0. This means that B is either empty or it is a string of variables and/or operation symbols of rank 0. This gives

$$-1 = \lambda(w) = \lambda(A) + (r-1) - |B| - m.$$

So $|B| + m = r + \lambda(A) \geq r$. This means that $Bs_0 \ldots s_{m-1}$ is a sequence of at least r terms. So once again $QBs_0 \ldots s_{m-1}$ is a sequence of terms. So we can continue all the way to the leftmost symbol, which is an operation symbol of positive rank. This means w is a sequence of terms. So $w = s_0 \ldots s_{m-1}$ for some terms $s_0, \ldots, s_{m-1}$.

$$\begin{aligned} -1 &= \lambda(w) = \lambda(s_0) + \cdots + \lambda(s_{m-1}) \\ -1 &= -m \\ 1 &= m \end{aligned}$$

So we find that w is a sequence of one term, another way of saying that w is a term.

PROBLEM 1.

Let $w = u_0 u_1 \ldots u_{n-1}$, where $u_i \in X \cup L$ for all $i < n$. Prove that if $\lambda(w) = -1$, then there is a unique cyclic variant $\hat{w} = u_i u_{i+1} \ldots u_{n-1} u_0 \ldots u_{i-1}$ of w that is a term.

SOLUTION

Pick $k_0 < n$ as small as possible so that $\lambda(u_0 \ldots u_{k_0-1}) = -1$. This is possible since at each step the value of λ can decrease by at most 1 and we are assured that $\lambda(w) = -1$. So $u_0 \ldots u_{k_0-1}$ is a term according to Problem 0. Now begin at u_{k_0} and move rightward until the value of λ is -1. In this way we decompose w into a sequence of terms and a final segment:

$$w = s_0 s_1 \ldots s_{m-1} B$$

so that all the initial segments of B have nonnegative λ's. Because $-1 = \lambda(w) = -m + \lambda(B)$ we see that $\lambda(B) = m - 1$. But then $Bs_0 s_1 \ldots s_{m_1}$ is a cyclic variant of w and the λ values of all the proper initial segments of this variant are nonnegative. So it is a term according to Problem 0.

To see the uniqueness, suppose t is a term. Let $t = uv$ where neither of the strings u and v are empty. Then $\lambda(u) \geq 0$ and $\lambda(v) = -1 - \lambda(u)$. It follows that vu cannot be a term since it has a proper initial segment with a negative λ. So no proper cyclic variant of a term is a term.

PROBLEM 2.
Prove that if w is a term and w' is a proper initial segment of w, then w' is not a term.

SOLUTION
Proper initial segments of terms must have nonnegative λ values but terms must have λ value -1.

PROBLEM 3.
Let $\mathbf{T}$ be the term algebra of L over X. Prove

> If Q and P are operation symbols, and $P^{\mathbf{T}}(p_0, p_1, \ldots, p_{n-1}) = Q^{\mathbf{T}}(q_0, q_1, \ldots, q_{m-1})$, then $P = Q$, $n = m$, and $p_i = q_i$ for all $i < n$.

SOLUTION
According to the hypotheses of the implication, $Pp_0 \ldots p_{n-1} = Qq_0 \ldots q_{m-1}$. So $P = Q$ since these are the leftmost symbols in the strings. We also see that $n = m$ since these are the ranks of the operation symbols. Next, we see that either the term p_0 is an initial segment of the term q_0 or that q_0 is an initial segment of p_0. Since proper initial segments of terms are not terms, this mean $p_0 = q_0$. But now the same reasoning applies to p_1 and q_1. Moving rightward a step at a time, we see that $p_i = q_i$ for each $i < n$.

SOLUTIONS FOR THE FIRST COMPACTNESS PROBLEM SET
22 SEPTEMBER 2011

PROBLEM 4.
Let L be the signature for group theory with operation symbols $\cdot, ^{-1}$, and 1. Let T be a set of L-sentences which includes all the group axioms (so every model of T will be a group). Suppose that for each n, there is a model of T which has no elements, other than 1, of order smaller than n. Prove that there is a model of T such that 1 is the only element of finite order.

SOLUTION
For each natural number $n > 1$ let φ_n be an L-sentence expressing the notion that every element other than 1 has order larger than n. Here is one sentence that will serve for φ_4:

$$\forall x[\neg x \approx 1 \rightarrow (\neg x^2 \approx 1 \wedge \neg x^3 \approx 1 \wedge \neg x^4 \approx 1)]$$

Now let $\Sigma = \{\varphi_n \mid n > 1\}$. Now $T \cup \Gamma$ has a model whenever Γ is a finite subset of Σ. (This is just what we have assumed about T.) By the Compactness Theorem $T \cup \Sigma$ has a model. But all the sentences in Σ, taken together, assert that each element other than 1 cannot have a finite order. So any model of $T \cup \Sigma$ will serve our need.

PROBLEM 5.
Suppose that G is a group which has elements of arbitrarily large finite order. Prove that G is elementarily equivalent to a group with an element of infinite order.

SOLUTION
Let T be the elementary theory of G. So any model of T will be elementarily equivalent to G. Expand the signature by adjoining one new constant symbol c. For each natural number $n > 1$, let $\varphi_n(c)$ express the notion that the element named by c has order larger than n. Here is a sentence which will serve as $\varphi_4(c)$:

$$\neg c^2 \approx 1 \wedge \neg c^3 \approx 1 \wedge \neg c^4 \approx 1$$

Let $\Sigma = \{\varphi_n(c) \mid n > 1\}$. Then $T \cup \Gamma$ has a model (indeed, we can just expand G by naming an element of large enough finite order) whenever Γ is a finite subset of Σ. By the Compactness Theorem we know that $T \cup \Sigma$ has a model. Now the sentences in Σ, taken together, assert that the element named by c is not of finite order. So the reduct to the signature of group theory of any model of $T \cup \Sigma$ will be a group elementarily equivalent to G and which has an element (the one named by c in the expanded signature) which is of infinite order.

PROBLEM 6.
Let $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$ be the familiar structure consisting of the natural numbers equipped with addition, multiplication, the two distinguished elements 0 and 1, and the usual order relation. Let T consist of all the sentences true in $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$. Prove T has a model $\mathbf{M}$ with an element ω so that all the following are true in $\mathbf{M}$:

$$0 \leq \omega, 1 \leq \omega, 2 \leq \omega, \ldots.$$

SOLUTION
Such a nonstandard model of arithmetic can be obtained in a number of ways. Perhaps the quickest is to invoke the Upward Löwenhein-Skolem-Tarski Theorem. This gives an uncountable elementary extension M of the structure $N = \langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$. Evidently, $M \models T$. Among the sentences in T one finds those expressing that 0 names the least element, that $\leq$ is a linear ordering, and that for each x there are no elements properly between x and $x + 1$. Now the elements of M named by $0, 1, 1+1, 1+1+1, \ldots$ will constitute an initial segment of the ordering of M (as we shall see shortly). This list is countable. Since M is uncountable there must be an element ω of M which is not listed. It cannot come before the element named by 0, nor can it fall between the element named by n and that named by $n + 1$. So the only alternative is that it comes after all the listed elements.

We can also obtain this result by following the ideas used to solve Problem 1. Namely, expand the signature with a new constant symbol c (for naming our pathological element).

Let $\theta_n(c)$ be the sentence $n \leq c$. Here we understand n to be the correct term of the form $(\dots(((1+1)+1)\dots)+1)+1$. Let $\Sigma = \{\theta_n(c) \mid n \geq 0\}$. Then once more argue that $T \cup \Sigma$ has a model. In that model take ω to be the element named by c.

Problem 7.

Let L be the signature of rings. Find a set Σ of L-sentences such that $\operatorname{Mod}\Sigma$ is the class of algebraically closed fields. Then prove that there is no finite set of L-sentences which will serve the same purpose.

Solution

A field K is algebraically closed provided every polynomial in one variable of positive degree with coefficients from K—that is the every member of $K[x]$ of postive degree—has a root in K. In essence, there is a finite set of sentences which capture the notion of a field in the obvious way. This finite set is then supplemented by an infinite list of sentences of the form φ_n:

$$\forall y_0 y_1 \dots y_{n-1} \exists x [x^n + y_{n-1}x^{n-1} + \dots + y_1 x + y_0 = 0]$$

The sentence displayed above asserts that every (monic) polynomial of degree n has a root. The set Σ consists of the field axioms and the sentences φ_n for each $n > 0$. So Σ is a countably infinite set of sentences.

Suppose there were a finite set Γ of sentences serving the same purpose. So Σ and Γ are logically equivalent—they have the same models. Then $\Sigma \models \Gamma$ and $\Gamma \models \Sigma$. By a corollary of the Compactness Theorem, there is a finite set $\Delta \subset \Sigma$ so that $\Delta \models \Gamma$. This entails that $\Delta \models \Sigma$, since $\Gamma \models \Sigma$. But plainly, $\Sigma \models \Delta$ since Δ is a subset of Σ. This means that if some finite set Γ serves to axiomatize the class of algebraically closed fields, then some finite subset Δ of Σ itself would work as well. To show that this is impossible what we need is the following claim.

Claim

For arbitrarily large natural numbers n there is a field K which is not algebraically closed but such that every polynomial of positive degree no more than n and with coefficients in K has a root in K.

This claim is an exercise in the theory of fields. I will sketch a construction, leaving aside the demonstration of those points which can be regarded as well-known (to those who know something of fields...). First, here is some notation and the Key Facts from the theory of fields that I will use. Suppose F is a field and L and M are subfields of F. The field F can be construed as a vector space over the field M. The dimension of F as a vector space over M is denoted by $[F : M]$. If $a \in F$ then $L(a)$ denotes the smallest subfield of F containing $L \cup \{a\}$. We use $L \vee M$ to denote the smallest subfield of F that contains $L \cup M$. We use $L \leq M$ to denote that L is a subfield of M and e use $L[x]$ to denote the ring of polynomials with coeffecients from L.

Here are the Key Facts:

Key Fact

If $L \leq M \leq F$, then $[F : L] = [F : M][M : L]$.

Key Fact

If S, T and M are subfields of F with $S \leq T$, then $[T \vee M : S \vee M] \leq [T : S]$.

Key Fact

Suppose L is a subfield of the field F and $a \in F$ is a root of an irreducible polynomial in $L[x]$ with degree r. Then $[L(a) : L] = r$. Consequently, if a is a root of $q(x) \in L[x]$ (irreducible or not), then $[L(a) : L]$ is less than or equal to the degree of $q(x)$.

Key Fact

Suppose L is a subfield of the field F and $a \in F$. If $[F : L]$ is finite, then a is the root of an irreducible polynomial in $L[x]$.

The construction I have in mind happens within the field of complex numbers. Let n be a postive integer. Call a subfield L of $\mathbb{C}$ n-**algebraically closed** provided every polynomial in $L[x]$ with positive degree no greater than n has a root in L. So what we are looking for is a subfield K which is n-algebraically closed but is not algebraically closed. Let $\mathcal{K}$ be the set of all n-algebraically closed subfields of $\mathbb{C}$. Observe that $\mathbb{C} \in \mathcal{K}$, so we know that $\mathcal{K}$ is not empty. Let $K = \bigcap \mathcal{K}$. It is straightforward to check that K is the smallest n-algebraically closed subfield of $\mathbb{C}$. This is the field we want.

It remains only to show that K is not algebraically closed. For this purpose, rather than the "shrinkwrapped" description of K given above, we use one that builds K up from the inside.

We start with the field $\mathbb{Q}$ of rational numbers and add roots of polynomials of degree no larger than n. We do this as much as possible to get K. Here are the details. We will say that a subfield L of the complex numbers is **reachable** provided there is a finite sequence of subfields $L_0 \leq \cdots \leq L_m$ so that

- $L_0 = \mathbb{Q}$ and $L_m = L$, and
- $[L_{i+1} : L_i] \leq n$ for all $i < m$.

Let $\mathcal{R}$ be the set of reachable subfields of $\mathbb{C}$.

Suppose that L and M are reachable subfields of $\mathbb{C}$ with the sequences $L_0 \leq \cdots \leq L_m$ and $M_0 \leq \cdots \leq M_r$ witnessing the reachability. Then our Key Facts tell us that the sequence

$$M_0 \leq \cdots \leq M_r = M \leq M \vee L_0 \leq M \vee L_1 \leq \cdots \leq M \vee L_m = M \vee L$$

witnesses that $M \vee L$ is also reachable. This means that $\mathcal{R}$ is upward directed by the subfield relation. So $\bigcup \mathcal{R}$ is also a subfield of $\mathbb{C}$. Call it K'. An easy application of one of the Key Facts reveals that K' is n-algebraically closed. So $K \leq K'$. On the other hand, our Key Facts also reveal that each reachable subfield must be a subfield of K. So $K = K' = \bigcup \mathcal{R}$.

To show that K is not algebraically closed, we must find a polynomial over K which has no root in K. We give this polynomial explicitly. Let p be a prime number larger than n. By Eisenstein's Criterion, the polynomial $x^p - 2$ is irreducible over $\mathbb{Q}$. Since the rationals are contained in K, this is also a polynomial over K. We will show that K contains no root of this polynomial.

Why did I pick this polynomial? One of the Key Facts asserts that if r is a complex number which is a root of this polynomial, then $[\mathbb{Q}(r) : \mathbb{Q}] = p$. On the other hand, p cannot divide any positive integer less than or equal to n.

Let $a \in K$. So we can find subfields $\mathbb{Q} = L_0 \leq L_1 \leq \cdots \leq L_m$ so that $a \in L_m$ and $[L_{i+1} : L_1] \leq n$ for all $i < m$.

So now we observe

$$[L_m : \mathbb{Q}] = [L_m : L_{m-1}][L_{m-1} : L_{m-2}] \cdots [L_2 : L_1][L_1 : L_0]$$

The factors occurring on the right are each no greater than n. This means the prime factors of the dimension on the left all have to be less than or equal to n. But now, since $\mathbb{Q} \leq \mathbb{Q}(a) \leq L_m$, we see that $[\mathbb{Q}(a) : \mathbb{Q}]$ is a divisor of $[L_m : \mathbb{Q}]$. So the prime factors of $[\mathbb{Q}(a) : \mathbb{Q}]$ must all be less than or equal to n. Since p is a prime larger than n, we see that a cannot be a root of $x^p - 2$. This means that K is not algebraically closed, as desired.

There are two remarks I would like to add. First, this argument is very similar to the standard argument that there is no straightedge-and-compass method for trisecting angles. Second, I worked inside of the field of complex numbers to make the exposition a bit simpler. Actually, this argument can be carried out without reference to such an ambient algebraically closed field. One point needs some adjustment if the smallest field L_0 that came up is not chosen to be the rationals. Any field will serve as L_0 provided $L_0[x]$ has irreducible polynomials for arbitrarily large prime degree. I used Eisenstein's Criterion above to establish this for the field of rational numbers. This property does not always hold. For example $\mathbb{C}[x]$ only has irreducible polynomials of degree 1 and $\mathbb{R}[x]$ has no irreducible polynomials of degree greater than 2. On the other hand, it is a standard exercise in field theory to show that if F is any finite field, then in $F[x]$ there are irreducible polynomials of every positive degree. So any finite field would work for L_0 in the construction above.

Problem 8.
Let L be the signature of ordered sets. Prove that there is no set Σ of L-sentences such that $\operatorname{Mod}\Sigma$ is the class of all well-ordered sets.

Solution
We take $<$ to be the relation symbol in the signature L. We intend to show that there is no set Σ of L-sentences so that the models of Σ are exactly the sets equipped with a strict well-ordering. So suppose that Σ is a set of L-sentences such that every well-ordered set is a model of Σ. We aim to produce a model of Σ which is not a well ordered set. So we make a model with a nonempty set that has no least element. Our paradigm is the set of nonpositive integers. So we expand the language by adding countably many new constants $c_0, c_{-1}, c_{-2}, \ldots$. Let ψ_n be the sentence $c_{-n-1} < c_{-n}$. Let $\Psi = \{\psi_n \mid n \geq 0\}$. Notice that $\Sigma \cup \Phi$ has a model whenever Φ is a finite subset of Ψ. For example, the natural numbers with their usual ordering is a model of Σ which has arbitrarily long finite chains, so it is easy to name elements so that everything is Φ will be true. By the Compactness Theorem, $\Sigma \cup \Psi$ has a model. In this model, the set of elements named by the new constants is nonempty but has no least element. This means Σ has a model which is not a well ordered set. Thus Σ does not constitute a set of elementary acioms for the notion of well ordering. In short, there is no set of elementary axioms for the notion of well ordering.

Solutions: Second Problem Set About the Compactness Theorem
18 October 2011

Problem 9.
Let L be a signature and $\mathcal{K}$ be a class of L-structures. We say that $\mathcal{K}$ is *axiomatizable* provided $\mathcal{K} = \operatorname{Mod}\Sigma$ for some set Σ and L-sentences. $\mathcal{K}$ is *finitely axiomatizable* provided there is a finite such Σ. Prove that $\mathcal{K}$ is finitely axiomatizable if and only if both $\mathcal{K}$ and $\{A \mid A \text{ is an } L\text{-structure and } A \notin \mathcal{K}\}$ are axiomatizable.

Solution
First suppose that $\mathcal{K}$ is finitely axiomatizable. Then it is axiomatizable by some single sentence σ. (This σ can be just the conjunction of some finite set of sentences axiomatizing $\mathcal{K}$.) Then, as easily checked, the sentence $\neg\sigma$ axiomatizes $\{A \mid A \text{ is an } L\text{-structure and } A \notin \mathcal{K}\}$, the complement of $\mathcal{K}$. Thus both $\mathcal{K}$ and its complement are axiomatizable (and even finitely axiomatizable).

For the converse, suppose Σ axiomatizes $\mathcal{K}$ and Δ axiomatizes its complement. Then $\Sigma \cup \Delta$ can have no models, since no structure can belong to both $\mathcal{K}$ and to its complement. So by the Compactness Theorem, there is a finite $\Sigma' \subseteq \Sigma$ and a finite $\Delta' \subseteq \Delta$ so that $\Sigma' \cup \Delta'$ has no model. This means that $\operatorname{Mod}\Sigma'$ and $\operatorname{Mod}\Delta'$ are disjoint classes of L-structures. But $\operatorname{Mod}\Sigma \subseteq \operatorname{Mod}\Sigma'$ and $\operatorname{Mod}\Delta \subseteq \operatorname{Mod}\Delta'$ and $\operatorname{Mod}\Sigma \cup \operatorname{Mod}\Delta$ is the class of all L-structures. This forces $\operatorname{Mod}\Sigma = \operatorname{Mod}\Sigma'$ (and $\operatorname{Mod}\Delta = \operatorname{Mod}\Delta'$). Thus $\mathcal{K}$ is finitely axiomatizable (as is its complement).

Problem 10.

Show that the class of fields of finite characteristic is not axiomatizable.

Solution

Let χ_p be the sentence

$$\neg(\underbrace{1+\cdots+1}_{p-\text{times}} \approx 0).$$

Let Γ be the set of all sentences which are true in every field of finite characteristic. If any set axiomatizes the class of field of finite characteristic, then the set Γ must also axiomatize this class.

We contend that $\Gamma \cup \{\chi_p \mid p \text{ is and prime number}\}$ has a model. To see this, by the Compactness Theorem it is enough to see that $\Gamma \cup \{\chi_p \mid p < n \text{ and } p \text{ is prime}\}$ has a model for each natural number n. But this is easy. Let $r > n$ be prime. Then the field $\mathbb{Z}_r$ is a model of $\Gamma \cup \{\chi_p \mid p < n \text{ and } p \text{ is prime}\}$.

Now any field which is a model of χ_p for all primes p is a field of characteristic 0. So Γ must have a model which is of characteristic 0. Hence, Γ does not axiomatize the class of fields of finite characteristic.

Problem 11.

Show that the class of fields of characteristic 0 is not finitely axiomatizable.

Solution

Let ϕ be a sentence which axiomatizes the theory of fields and let χ_p be the sentences defined in the solution to Problem 6. Then $\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number }\}$ axiomatizes the class of fields of characteristic 0. (This is just a formalization of the standard definition.). Let Δ be any finite set of sentences true in every field of characteristic O. Then

$$\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number}\} \models \Delta$$

and, since Δ is finite, the Compactness Theorem entails that

$$\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number and } p < n\} \models \Delta$$

for some natural number n. Let r be any prime larger than n. Then $\mathbb{Z}_r \models \{\phi\} \cup \{\chi_p \mid p \text{ is a prime number and } p < n\}$. So also $\mathbb{Z}_r \models \Delta$. Hence Δ has a model which is not of characteristic 0. This means Δ cannot axiomatize the class of fields of characteristic 0. So this class has no finite axiom set.

Note: It is possible to use Problem 5 to help with the solutions to Problem 6 and Problem 7. Here is one way. With Problem 7 in hand we prove Problem 6 with the help of Problem 5. Suppose to the contrary that Δ axiomatizes the class of fields of finite characteristic. Then $\{\phi \to \delta \mid \delta \in \Delta\}$ axiomatizes the complement of the class of fields of characteristic 0 (notice that there are structures in this complement which are not fields. . .). In Problem 7, we saw that the class of fields of characteristic 0 is axiomatizable as well. By Problem 5 we would get that the class of fields of characteristic 0 is finitely axiomatizable. But in Problem 7 we proved it was not. So there can be no such Δ.

Problem 12.

Let θ be any sentence in the signature of fields. Prove that if θ is true in every field of characteristic 0, then there is a natural number n so that θ is true in every field of characteristic p for all primes $p > n$.

Solution

Let the sentence ϕ axiomatize the theory of fields and let χ_p be the sentence described in the solution to Problem 6. So we know $\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number }\}$ axiomatizes the class of fields of characteristic 0. Hence $\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number }\} \models \theta$. By the Compactness Theorem there is some natural number n so that $\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number and } p < n\} \models \theta$. Now any field of characteristic r where $r > n$ is a model of $\{\phi\} \cup \{\chi_p \mid p \text{ is a prime number and } p < n\}$ and so it is also a model of θ, as desired.

Problem 13.
Let L be a signature and for each natural number n suppose that T_n is a set of L-sentences closed with respect to logical consequence. Further, suppose that $T_0 \subset T_1 \subset T_2 \subset \dots$ is strictly increasing. Let $T = \bigcup_{n\in\omega} T_n$. Prove that

(1) T has a model.
(2) T is closed under logical consequence.
(3) T is not finitely axiomatizable.

Solution
First we contend that T_n has a model for each n. Since T_n is a proper subset of T_{n+1} pick $\sigma_n \in T_{n+1}$ with $\sigma_n \notin T_n$. Since T_n is closed under logical consequence we see that $T_n \not\models \sigma_n$. This means that T_n has a model in which σ_n is false. But in particular, T_n has amodel.

To see that T has a model we show that each finite subset Δ of T has a model and then appeal to the Compactness Theorem. Since $\Delta \subseteq T = \bigcup_{n\in\omega} T_n$ we see that $\Delta \subseteq T_n$ for some natural number n, because Δ is finite and the T_k's form an ascending chain. Since T_n has a model we see that Δ has a model (the same one).

Now suppose that $T \models \theta$. By the Compactness Theorem, there is a finite $\Delta \subseteq T$ so that $\Delta \models \theta$. But then, as above, there is a natural number n so that $\Delta \subseteq T_n$. Since T_n is closed under logical consequence, we see that $\theta \in T_n \subseteq T$. So T is closed under logical conquence.

To see that T is not finitely axiomatizable, let $\Delta \subseteq T$ with Δ finite. As before, pick a natural number n so that $\Delta \subseteq T_n$ and pick $\sigma_n \in T_{n+1}$ with $\sigma_n \notin T_n$. Then $\Delta \not\models \sigma_n$ since T_n is closed under logical consequence, but $\sigma_n \in T_{n+1} \subseteq T$. So Δ cannot axiomatize T.

Solutions to Problem Set About Infinite Models of Complete Theories
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Suppose $\mathbf{A}$ is a structure. The group $\operatorname{Aut}\mathbf{A}$ of all automorphisms of $\mathbf{A}$ partitions A into orbits. [Elements $a, b \in A$ belong to the same orbit iff there is an automorphism f such that $f(a) = b$.] Notice that the same applies the n-tuples from A: the group $\operatorname{Aut}\mathbf{A}$ partitions A^n into orbits.

Problem 14.
Let L be a countable signature and let T be a complete set of L-sentences with infinite models. Prove that T is ω-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits for every natural number n, for every countable $\mathbf{A} \models T$.

Problem 15.
Let L be a countable signature and let T be a complete set of L-sentences with infinite models. Prove that T is ω-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits for every natural number n, for some countable $\mathbf{A} \models T$.

Solution
We will handle Problem 10 and Problem 11 at once. Recall, as part of that multipart characterization theorem we did in class, we know that the following are equivalent:

(0) T is ω-categorical.
(1) T has a countably infinite model which is both saturated and atomic.
(2) Every complete n-type of T is supported, for each natural number n.
(3) The number of complete n-types of T is finite, for each natural number n.

The two additional conditions given in Problem 10 and Problem 11 are

(a) $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits, for each natural number n and for each countable model $\mathbf{A}$ of T.
(b) $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits, for each natural number n and for some countable model $\mathbf{A}$ of T.

Our aim is to include these last two conditions on the list of equivalent statements. We will do this by showing that $(0) \Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow (2)$.

$(0) \Rightarrow$ (a).
Because (0) implies (1) and (3) we can use all three of these to get (a). So let $\mathbf{A}$ be a countably infinite model of T and let n be a natual number. Because of (1) we know that T has a countably infinite model which is saturated. Because of (0) we know that any two countably infinite models of T are isomorphic. So we find that $\mathbf{A}$ is saturated. Now let $\Phi(\bar{x})$ be a complete n-type of T. Because T is complete, we know $T = \operatorname{Th}\mathbf{A}$. So $\Phi(\bar{x})$ is realized in $\mathbf{A}$ since $\mathbf{A}$ is saturated.

Suppose that two n-tuple $\bar{a}$ and $\bar{b}$ of $\mathbf{A}$ both realize $\Phi(\bar{x})$. Then $(\mathbf{A}, \bar{a}) \equiv (\mathbf{A}, \bar{b})$. But since $\mathbf{A}$ is saturated, it follows from the definition of saturation, that both $(\mathbf{A}, \bar{a})$ and $(\mathbf{A}, \bar{b})$ are saturated. But then $(\mathbf{A}, \bar{a}) \cong (\mathbf{A}, \bar{b})$. Let f be such an isomorphism. Then $f \in \operatorname{Aut}\mathbf{A}$ and f takes the tuple $\bar{a}$ coordinatewise to the tupe $\bar{b}$.

What we have just established is that if $\bar{a}$ and $\bar{b}$ are two n-tuples of $\mathbf{A}$ which realize the same complete n-type, then they lie in the same orbit.

Now we define a function G which assigns to each complete n-type of T an orbit of A^n. This function will even be onto the set of orbits. Once we have this, since (3) tells us that the number of complete n-types in finite, we will know that the number of orbits is finite.

Here is how to evaluate G at the complete n-type $\Phi(\bar{x})$ of T: Since $\mathbf{A}$ is a countable saturated model of T and T is complete, we know that $\Phi(\bar{x})$ is realized in $\mathbf{A}$. Suppose $\bar{a}$ in an n-tuple of $\mathbf{A}$ which realizes $\Phi(\bar{a})$. As the value of G upon input of $\Phi(\bar{x})$ output the orbit to which $\bar{a}$ belongs. This definition of G is sound since if $\bar{b}$ also realizes $\Phi(\bar{x})$ then we know that $\bar{a}$ and $\bar{b}$ belong to the same orbit. Of course, G is onto the set of orbits since for each orbit $\mathcal{O}$ we can select $\bar{a} \in \mathcal{O}$ and let $\Phi(\bar{x})$ be the type realized by $\bar{a}$. The G will send $\Phi(\bar{x})$ to $\mathcal{O}$.

This establishes $(0) \Rightarrow ((a))$.■

(a)⇒(b)
This is just logically true.■

(b)⇒(2)
Let $\mathbf{A}$ be a countably infinite model of T so that $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits, for each natural number n.

Fix a natural number n. Suppose $\bar{a}$ and $\bar{b}$ are two n-tuples which lie in the same orbit. Then there is $f \in \operatorname{Aut}\mathbf{A}$ so that f carries $\bar{a}$ coordinatewise to $\bar{b}$. Because isomorphisms preserve the satisfaction of all formulas (this can be proved by a routine induction on the complexity of formulas) we see that the complete n-type realized by $\bar{a}$ is the same as the complete n-type realized by $\bar{b}$. Thus, we can define a function F which assigns to each orbit $\mathcal{O}$ of A^n the complete n-type $\Phi(\bar{x})$ as follows: Pick $\bar{a} \in \mathcal{O}$ and let $\Phi(\bar{x})$ be the complete n-type realized by $\bar{a}$. Since any other $\bar{b} \in \mathcal{O}$ gives the same $\Phi(\bar{x})$, this definition is sound. The function F is onto the set of n-types realized in $\mathbf{A}$ since F sends the orbit to which a n-tuple belongs to the type realized by the n-tuple. This means that the number of complete n-types realized in $\mathbf{A}$ is bounded above by the number of orbits. Since the number of orbits is finite, then only finitely many complete n-types are realized in $\mathbf{A}$.

Let $\Phi_0(\bar{x}), \Phi_1(\bar{x}), \ldots, \Phi_{m-1}(\bar{x})$ be a list of the distinct complete n-types realized in $\mathbf{A}$. Now for each $i, j < m$ with $i \neq j$, pick $\varphi_{ij}(\bar{x}) \in \Phi_i(\bar{x})$ so that $\neg\varphi_{ij}(\bar{x}) \in \Phi_j(\bar{x})$. We can do this since our n-types are complete and distinct. Now put

$$\theta_i(\bar{x}) \text{ equal to } \bigwedge_{j \neq i} \varphi_{ij}(\bar{x}).$$

Evidently, $\mathbf{A} \models \theta_i(\bar{a})$ whenever $\bar{a}$ realizes $\Phi_i(\bar{x})$. This means $\theta_i(\bar{x}) \in \Phi_i(\bar{x})$. But we also see that $\theta_i(\bar{x}) \notin \Phi_j(\bar{x})$ for any $j \neq i$ with $j < m$. Let $\bar{b}$ be any n-tuple such that $\mathbf{A} \models \theta_i(\bar{b})$. Then $\bar{b}$ cannot realize any $\Phi_j(\bar{x})$ with $j \neq i$ and $j < m$. But $\bar{b}$ must realize some complete n-type. The only remaining one is $\Phi_i(\bar{x})$. Another way to say all this is

(i) $\mathbf{A} \models \exists \bar{x} \theta_i(\bar{x})$ for each $i < m$.
(ii) $\mathbf{A} \models \forall \bar{x}[\theta_0(\bar{x}) \vee \cdots \vee \theta_{m-1}(\bar{x})]$.
(iii) $\mathbf{A} \models \forall \bar{x}[\theta_i(\bar{x}) \to \psi(\bar{x})]$ for all $\psi(\bar{x}) \in \Phi_i(\bar{x})$ for all $i < m$.

Now (i) and (iii) yield that each $\theta_i(\bar{x})$ is a complete formula for $\operatorname{Th}\mathbf{A}$. (This means that $\mathbf{A}$ is atomic, since every n-type realized in $\mathbf{A}$ is supported.) Since T is complete and $\mathbf{A} \models T$ we have that $T = \operatorname{Th}\mathbf{A}$. So each $\theta_i(\bar{x})$ is a complete formula for T. We also have $T \models \forall \bar{x}[\theta_0(\bar{x}) \vee \cdots \vee \theta_{m-1}(\bar{x})]$. Now let $\mathbf{B} \models T$ and let $\bar{b}$ be an n-tuple from $\mathbf{B}$. Then $\bar{b}$ must satisfy one of the complete formulas $\theta_i(\bar{x})$ where $i < m$. Thus the complete n-type realized by $\bar{b}$ contains a complete formula. This means that every complete n-type of T is supported and we have established (2), as desired. We have also established that the only complete n-types of T are the ones realized in $\mathbf{A}$—so they are finite in number giving us (3). This also shows that $\mathbf{A}$ is saturated and we already noted that $\mathbf{A}$ is atomic—hence we also have (1).

The two conditions from Problem 10 and Problem 11 were the contribution of Svenonius to the characterization of ω-categorical theories. Actually, in the course of this proof we have come across yet another equivalent condition:

> There is a countably infinite model of T which, for each natural number n, realizes only finitely many complete n-types.

Problem 16.
Let T be an elementary theory in a countable signature and suppose that T is κ-categorical for some infinite cardinal κ. Let $\mathcal{K} = \{\mathbf{A} \mid \mathbf{A} \models T \text{ and } A \text{ is infinite}\}$. Prove that $\mathcal{K}$ is axiomatizable and that $\operatorname{Th}\mathcal{K}$ is complete.

Solution
This statement is called Vaught's Test for Completeness.

For each natural number $n > 1$, let γ_n be the sentence which asserts that there are at least n elements. These sentences can be written without the help of relation symbols and function symbols, so they are sentences of every signature. For example

$$\gamma_3 \text{ can be taken to be } \exists x_0, x_1, x_2[\neg(x_0 \approx x_1) \wedge \neg(x_0 \approx x_2) \wedge \neg(x_1 \approx x_2)].$$

Let $\Gamma = \{\gamma_n \mid n > 1\}$. Evidently, $\mathcal{K}$ is axiomatized by $T \cup \Gamma$.

Now since T is κ-categorical it must have a model $\mathbf{M}$ of cardinality κ. The theory of any structure is complete. We will prove that $\operatorname{Th}\mathcal{K}$ is complete by showing that $\operatorname{Th}\mathcal{K} = \operatorname{Th}\mathbf{M}$. Since κ is infinite, we see that $\mathbf{M} \in \mathcal{K}$. This means that $\operatorname{Th}\mathcal{K} \subseteq \operatorname{Th}\mathbf{M}$.

To see the reverse inclusion, let σ be a sentence with $\sigma \notin \operatorname{Th}\mathcal{K}$. So we can pick $\mathbf{A} \in \mathcal{K}$ so that $\mathbf{A} \not\models \sigma$. Now $\mathbf{A} \models T$ and $\mathbf{A}$ is infinite. Since the signature is countable, the Löwenheim-Skolem-Tarski theorems give us $\mathbf{B}$ with $\mathbf{A} \equiv \mathbf{B}$ and $\mathbf{B}$ of cardinality κ. (We use the upward theorem if the cardinality of $\mathbf{A}$ is smaller than κ and the downward theorem otherwise.) Because T is κ-categorical we have that $\mathbf{B} \cong \mathbf{M}$. This gives $\mathbf{A} \equiv \mathbf{M}$. Thus $\mathbf{M} \not\models \sigma$. So $\sigma \notin \operatorname{Th}\mathbf{M}$, as desired.

Problem 17.

Construct an example of a complete theory in a countable signature which has, up to isomorphism, exactly 3 models. (Well, I posed the problem so as to give a hint...)

Solution

Our signature will provide one 2-place relation symbol $\leq$ and a countably infinite list $c_0, c_1, \dots$ of operation symbols of rank 0. Let Δ be the theory of dense linear orders without endpoints, and T be axiomatized by $\Delta \cup \{c_k < c_{k+1} \mid k \in \omega\}$. I claim that T is a complete theory and that, up to isomorphism it has exactly 3 countable models.

For completeness, let $\theta(\bar{c})$ be a sentence so that $T \nvDash \theta(\bar{c})$. So there must be a countable model of $T \cup \{\neg\theta(\bar{c})\}$. It is harmless to suppose this countable model is just the ordered set of rational in which infinitely many elements have been designated. Let $\bar{y}$ be a tuple of new variables to match the tuple $\bar{c}$ of constant symbols. Let $\phi(\bar{y})$ be a formula expressing that the variables are ordered the same way that the elements named by the constant symbols are ordered. Because the ordered set of rationals is homogenous (any two tuples of elements ordered the same way can be mapped to each other by automorphisms), we see that $\forall \bar{y}[\phi(\bar{y}) \rightarrow \neg\theta(\bar{y})]$ is true in the ordered set of rationals. But the theory of dense linear orders without endpoints is complete, so $\Delta \models \forall \bar{y} \rightarrow \neg\theta(\bar{y})$. But then $T \models \neg\theta(\bar{c})$ since $T \models \phi(\bar{c})$. This establishes that T is a complete theory.

What are the three countable models? Well, any countable model amounts to the ordered set of rationals with a designated increasing infinite sequence of elements named by the constant symbols. Because every dominated increasing sequence of real numbers converges, there are three possibilities:

(a) The designated sequence increases without bound.
(b) The designated sequence converges to a rational.
(c) The designated sequence converges to an irrational.

So to complete the proof we need to argue two things. First that any two countable models of T that are any one of these three types are isomorphic. Second, that models of different types are not isomorphic.

For the first task, let $\mathbf{A}$ and $\mathbf{B}$ be countable models of one of the three kinds. We suppose $A = B = \mathbb{Q}$ and the $a_0 < a_1 < \dots$ is the sequence of designated elements of $\mathbf{A}$ and that $b_0 < b_1 < \dots$ are the designated elements of $\mathbf{B}$. (And in handling alternative (b) that d and e are the respective rational limit points.) The isomorphism F we want must give us $F(a_i) = b_i$

for all i (and $F(d) = e$ in the middle alternative). Observe that the elements strictly below the element named by c_0 in each model is a dense linear order without endpoints. So that part of $\mathbf{A}$ is isomorphic to the corresponding part of $\mathbf{B}$. Likewise, the parts of the models that lie between the elements named by c_i and c_{i+1} are dense linear orders without endpoints, so they are isomorphic. What is left over? Nothing in alternative (a). In the other alternatives, we have the parts above the limit points: again isomorphic dense linear orders without endpoints. So just put together these isomorphisms with what we already said about F to show that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

For the second task, suppose that $\mathbf{A}$ falls under alternative (a) and that $\mathbf{B}$ falls under one of the other alternatives. Let $F : B \to A$ be one-to-one and let $b \in B$ so that b is an upper bound on all the named elements. Now $F(b) \in A$ and there must be a named element strictly above $F(b)$. Say it is named by the constant symbol c. Then we have in $\mathbf{B}$ that $c^{\mathbf{B}} < b$ and in $\mathbf{A}$ that $F(b) < c^{\mathbf{A}}$. But $F(c^{\mathbf{B}}) = c^{\mathbf{A}}$ in the event that F is an isomorphism—this would lead to $c^{\mathbf{A}} = F(c^{\mathbf{B}}) < F(b) < c^{\mathbf{A}}$, a contradiction. So F cannot be an isomorphism.

It remains to show that if $\mathbf{A}$ falls under alternative (b) and $\mathbf{B}$ fall under alternative (c), then $\mathbf{A}$ and $\mathbf{B}$ are not isomorphic. Let us suppose not. Let $F : \mathbf{A} \to \mathbf{B}$ is an isomorphism and that d is the limit point for the designated sequence in $\mathbf{A}$. The $F(d)$ must lie strictly above all the named elements of $\mathbf{B}$. In $\mathbf{B}$ these named elements do not have a rational least upper bound. So there is $r \in B$ with $r < F(d)$ and r strictly above each named element of $\mathbf{B}$. Let $t \in A$ be the element so that $F(t) = r$. Because F is an isomorphism we must have in $\mathbf{A}$ that $t < d$ and that t is strictly larger than all the named elements of $\mathbf{A}$. This is impossible, since d is the least upper bound of all the named elements in $\mathbf{A}$.