George F. McNulty

Elementary Model Theory

Notes for MATH 762

drawings by the author

University of South Carolina

Fall 2011

Model Theory
MATH 762

Fall 2011
TTh 12:30p.m.–1:45 p.m. in LeConte 303B
Office Hours: 2:00pm to 3:30 pm Monday through Thursday
Instructor: George F. McNulty

Recommended Text: *Introduction to Model Theory*
By Philipp Rothmaler

What We Will Cover

After a couple of weeks to introduce the fundamental concepts and set the context (material chosen from the first three chapters of the text), the course will proceed with the development of first-order model theory. In the text this is the material covered beginning in Chapter 4. Our aim is cover most of the material in the text (although not all the examples) as well as some material that extends beyond the topics covered in the text (notably a proof of Morley's Categoricity Theorem).

The Work

Once the introductory phase of the course is completed, there will be a series of problem sets to entertain and challenge each student. Mastering the problem sets should give each student a detailed familiarity with the main concepts and theorems of model theory and how these concepts and theorems might be applied. So working through the problems sets is really the heart of the course. Most of the problems require some reflection and can usually not be resolved in just one sitting.

Grades

The grades in this course will be based on each student's work on the problem sets. Roughly speaking, an A will be assigned to students whose problems sets eventually reveal a mastery of the central concepts and theorems of model theory; a B will be assigned to students whose work reveals a grasp of the basic concepts and a reasonable competence, short of mastery, in putting this grasp into play to solve problems. Students are invited to collaborate on the problem sets. Some of the problems will be designated as individual efforts. Students intending to use this course as part of the Comprehensive Exams should make particularly diligent efforts on the problems sets.

The Final

As the material is ill-suited to a sit down three hour writing effort, in place of a final examination we will instead have a party at my house. Everyone (and their partners) is invited. The party does have a little exit exam. . . .

I plan to offer a MATH 748V in the spring 2012 semester. The topic of that course will be *Varieties of Aglebras*, which is an algebraic counterpart to a portion of MATH 762.

MATH 762 and MATH 748V would form a course sequence upon which a Ph.D. Comprehensive Exam could be based.

Please feel free to drop by my office at any time. My office is 302 LeConte.

Contents

LECTURE 0

Mathematical Structures and Mathematical Formulas

A host of systems have attracted the attention of mathematicians. There are groups, graphs, and ordered sets. There are geometries, ordered fields, and topological spaces. There are categories, affine varieties, and metric spaces. The list is seemingly enormous, as is the list of concepts and theorems emerging from the efforts mathematicians have spent in their investigation.

It is the ambition of model theory to attach to such mathematical systems some formalized linguistic apparatus appropriate for framing concepts and stating theorems. This linguistic or syntactical apparatus, once it is given an unambiguous definition, can be subjected itself to a mathematical analysis. Roughly speaking, model theory is that branch of mathematics that exploits the connections between such a syntax and the appropriate mathematical systems. This means that model theory is a kind of mathematical semantics: it is centered on the *meanings* that syntactical expressions achieve in particular mathematical systems.

The ambition of model theory to make and to exploit such connections stands in a common mathematical tradition: by adjoining extra structure and developing its mathematics one hopes to be able to throw light on old problems and also to open up new parts of mathematics to further development. It is surely evident, for example, that the geometric, analytic, and topological aspects of the complex plane have greatly informed our understanding of ordinary addition and multiplication of whole numbers—an example of associating extra structure.

First observe that a linguistic apparatus suitable for the theory of groups need not be suitable for the theory of rings and may be quite unsuitable for the theory of ordered sets. So we envision the need for a wide assortment formal syntactical systems.

Next observe that groups, graphs, ordered sets, ordered fields, and certain geometries have the form of nonempty sets equipped with some system of operations and relations, each having some fixed finite number of places. For example a graph $\mathbf{G} = \langle G, E\rangle$ can be construed as a set G of vertices and a set E of edges—so that E is a symmetric irreflexive two-place relation on the set G of vertices. On the other hand, a topological space is a system $\langle X, \mathcal{T}\rangle$, where X is a set and $\mathcal{T}$ is a collection of subsets of X that satisfy certain properties making $\mathcal{T}$ into a topology of X.

In the part of model theory we will develop, we will allow mathematical systems like groups, graphs, ordered fields, and many others, but we will exclude systems like topological spaces, as well as many other very interesting mathematical systems.

0.1 Symbols for Operations, Symbols for Relations, and Signatures

Let us consider an example: the set of real numbers endowed with addition, multiplication, negation, zero, one, and the usual ordering. We could denote this structure by

$$\langle \mathbb{R}, +, \cdot . -, 0, 1, \leq \rangle.$$

We mean here, in part, that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function from the set of pairs of real numbers into the set of real numbers and that $\leq \subseteq \mathbb{R} \times \mathbb{R}$ is a two-place relation on the set of real numbers.

Consider another example: the set of real numbers that are algebraic over the rationals endowed with addition, multiplication, negation, zero, one, and the usual ordering. We could denote this structure by

$$\langle \mathbb{A}, +, \cdot, -, 0, 1, \leq \rangle.$$

In this example $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is again a two-place function but it differs from the operation $+$ of the first example—the domains are not the same. A sharper illustration of this point might use the set of all continuous functions from the unit interval into the set of real numbers, endowed with addition, multiplication, negation, the constantly zero function, the constantly one function, and the usual pointwise ordering.

In mathematical practice, we trust to context to resolve any ambiguities that arise in this way. At least tacitly, what happens is that we treat $+$ as a symbol that can be subjected to many interpretations requiring only that in every such interpretation that $+$ denote a two-place operation. In the same way, we treat $\leq$ as a symbol open to many interpretations, requiring only the in every such interpretation that $\leq$ denote a two-place relation.

We start our project by formalizing this common piece of mathematical practice. In the examples above, we used two 2-place operation symbols, one 1-place operation symbol (it was $-$), two 0-place operation symbols (these were 0 and 1) and one 2-place relation symbol. It is important for the development of the theory of the ordererd field of real number not to confuse addition and multiplication. Our arrangements must take that into account. Here is how we do it.

A **signature** is a pair $\langle \sigma, \rho \rangle$ of functions so that the domain of σ and the domain of ρ are disjoint, that σ assigns values that are natural numbers and the ρ assigns values that are positive natural numbers. The elements of the domain of σ are referred to as **operation symbols** and the elements of the domain of ρ are referred to as **relation symbols**. The functions σ and ρ assign **ranks** to the symbols. So we say, for example, that $+$ has rank 2. Operation symbols of rank 0 are also called **constant symbols**.

The examples above all had the same signature. In these examples, $\operatorname{dom} \sigma = \{+, \cdot, -, 0, 1\}$ and $\operatorname{dom} \rho = \{\leq\}$. Moreover,

$$\begin{aligned} \sigma(+) &= 2 \\ \sigma(\cdot) &= 2 \\ \sigma(-) &= 1 \\ \sigma(0) &= 0 \\ \sigma(1) &= 0 \\ \rho(\leq) &= 2 \end{aligned}$$

In a signature $\langle \sigma, \rho \rangle$ we allow either or both of σ and ρ to be the empty function—that is functions with empty domain. A signature appropriate for the theory of groups may have no relation symbols, while a signature appropriate to the theory of ordered sets may have no operation symbols.

0.2 Mathematical Structures of a Given Signature

Fix a signature $\langle \sigma, \rho \rangle$.

A **structure** for the signature is a system consisting of a nonempty set, called the **universe** of the structure, endowed with

- a system of **basic operations**, one for each operation symbol of the signature, so that the rank of each basic operation agrees with the rank of the correlated operation symbol, and
- a system of **basic relations**, one for each relation symbol of the signature, so that the rank of each basic relation agree with the rank of the correlated relation symbol.

Generally, we use the notation

$$\mathbf{A} = \langle A, Q^{\mathbf{A}}, R^{\mathbf{A}} \rangle_{Q \in \operatorname{dom} \sigma, R \in \operatorname{dom} \rho}$$

for structures. Here, $Q^{\mathbf{A}}$ is the basic operation of the structure $\mathbf{A}$ denoted by the operation symbol Q and $R^{\mathbf{A}}$ is the basic relation of $\mathbf{A}$ denoted by the relation symbol R. In case the signature is finite, we suppress the indexing and write things like

$$\mathbf{R} = \langle \mathbb{R}, +^{\mathbf{R}}, \cdot^{\mathbf{R}}, -^{\mathbf{R}}, 0^{\mathbf{R}}, 1^{\mathbf{R}}, \leq^{\mathbf{R}} \rangle.$$

Of course, all those superscripts are cumbersome, so where the ambiguities can be resolved by context, we write

$$\mathbf{R} = \langle \mathbb{R}, +, \cdot, -, 0, 1, \leq \rangle.$$

When the signature provides no relation symbols, we also refer to structures as **algebras**. By the misfortunes of mathematical history, the word *algebra* has a couple of other meanings. We meet it in school as a branch of mathematics, only to meet it again in university as the name of a broader branch of mathematics (which might be construed as the investigation of what we have here called algebras). More inconvenient for us, is that within algebra itself those structures that can be made from vector spaces by imposing an additional basic operation of rank 2, a product, perhaps satisfying some additional properties are also called *algebras*.

When the signature provides no operation symbols, we also refer to structures as **relational structures**.

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature. We say that $\mathbf{B}$ is a **substructure** of $\mathbf{A}$ and write $\mathbf{B} \leq \mathbf{A}$, provided

- $B \subseteq A$,
- $Q^{\mathbf{B}}$ is the restriction of $Q^{\mathbf{A}}$ to B^r, where r is the rank of Q, for each operation symbol Q, and
- $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^r$, where r is the rank of R, for each relation symbol R.

So the substructures of the structure $\mathbf{A}$ inherit their basic operations and basic relations from $\mathbf{A}$. Observe that the universe B of the substructure $\mathbf{B}$ of $\mathbf{A}$ is closed under the basic operations of $\mathbf{A}$. Indeed, any nonempty subset of A that is closed with respect to all the basic operations of $\mathbf{A}$ will be the universe of a uniquely determined substructure. We call the subsets of A that are closed with respect to all the basic operations of $\mathbf{A}$ **subuniverses** of $\mathbf{A}$. There is one thing to take note of here. If the signature provides no operation symbols of rank 0, that is no constant symbols, then the empty set will be a subuniverse of $\mathbf{A}$ but it will not be the universe of any substructure of $\mathbf{A}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be structures of the same signature. We say that a function $h : A \to B$ is a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, and we write $h : \mathbf{A} \to \mathbf{B}$, provided

- $h(Q^{\mathbf{A}}(a_0, \ldots, a_{r-1})) = Q^{\mathbf{B}}(h(a_0), \ldots, h(a_{r-1}))$ where r is the rank of Q and $a_0, \ldots, a_{r-1}$ are arbitrary elements of A, for all operation symbols Q, and
- if $(a_0, \ldots, a_{r-1}) \in R^{\mathbf{A}}$, then $(h(a_0), \ldots, h(a_{r-1})) \in R^{\mathbf{B}}$, where r is the rank of R and $a_0, \ldots, a_{r-1}$ are arbitrary elements of A, for all relation symbols R.

The homomorphism h is said to be a **strong homomorphism** when the last item listed above is replaced by

- $(a_0, \ldots, a_{r-1}) \in R^{\mathbf{A}}$ if and only if $(h(a_0), \ldots, h(a_{r-1})) \in R^{\mathbf{B}}$, where r is the rank of R and $a_0, \ldots, a_{r-1}$ are arbitrary elements of A, for all relation symbols R.

One-to-one strong homomorphisms are called **embeddings** of $\mathbf{A}$ into $\mathbf{B}$. If, in addition, they are onto B then they are called **isomorphisms**. When h is an embedding of $\mathbf{A}$ into $\mathbf{B}$ we write $h : \mathbf{A} \hookrightarrow \mathbf{B}$. When h is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$, we write $\mathbf{A} \overset{h}{\cong} \mathbf{B}$. When we write $\mathbf{A} \cong \mathbf{B}$, we mean there is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$.

Finally, let $\mathbf{A}_i$ be a structure for each $i \in I$. We require that these structures all have the same signature. By the direct product, denoted $\prod_I A_i$ of the system $< A_i \mid i \in I\rangle$ of sets, we mean the set

$$\{\bar{a} \mid \bar{a} : I \to \bigcup_I A_i \text{ such that } \bar{a}(i) \in A_i \text{ for all } i \in I\}.$$

We could render $\bar{a} = \langle \bar{a}(i) \mid i \in I\rangle$. It is convenient to think of $\bar{a}$ as an I-tuple such that the i^{th} coordinate of $\bar{a}$ (which is, in fact, $\bar{a}(i)$) comes from the i^{th} factor set A_i. So if $I = \{0, 1\}$, then $\bar{a}$ would, to all intents and purposes, be an "ordered pair". But actually, it is only really necessary to know that $0 \neq 1$ and not that $0 < 1$, so that the word "ordered" here is a little misleading. In general, we impose no ordering of any kind on the set I.

Let us take $A = \prod_I A_i$. We make a structure $\mathbf{A}$, which we call the **direct product** of the system of structures $\langle \mathbf{A}_i \mid i \in I\rangle$, and denote it as

$$\mathbf{A} = \prod_I \mathbf{A}_i$$

by imposing on the set A the basic operations and basic relations coordinatewise. That is for each operation symbol Q of the signature, let r be the rank of Q and let $\bar{a}_0, \ldots, \bar{a}_{r-1}$ be arbitrary elements of A. Define

$$Q^{\mathbf{A}}(\bar{a}_0, \ldots, \bar{a}_{r-1}) = \langle Q^{\mathbf{A}_i}(\bar{a}_0(i), \ldots, \bar{a}_{r-1}(i)) \mid i \in I\rangle.$$

And when R is a relation symbol of the signature, let r be the rank of R and let $\bar{a}_0, \dots, \bar{a}_{r-1}$ be arbitrary elements of A. Define $R^{\mathbf{A}}$ by

$$(\bar{a}_0, \dots, \bar{a}_{r-1}) \in R^{\mathbf{A}} \text{ if and only if } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i} \text{ for all } i \in I.$$

The notions of substructure, homomorphism, and direct product described here agree, for the most part, with mathematical practice. Perhaps the most significant departure comes in graph theory. The notion of substructure given above is commonly referred to among graph theorists as that of *induced* subgraph. This because it is convenient in graph theory to also omit edges to obtain a smaller graph. Of course one might like to omit both some edges and some vertices. So the graph $\mathbf{H} = \langle H, E\rangle$ is called a *subgraph* of the graph $\mathbf{G} = \langle G, F\rangle$ provided $H \subseteq G$ and $E \subseteq F$. Similar considerations apply to any relational structure (for example, to any ordered set). This means that some care must be taken is these cases.

0.3 Terms, Formulas, and Sentences

We turn now to describing the linguistic or syntactical apparatus that we will associate with each signature.

Fix a signature.

The things belonging to our syntax will generally be finite strings (i.e. sequences) of symbols. In addition to our operation symbols and our relation symbols we require a countably infinite list of **variable**, which are also symbols. Here is the official list of variables: $x_0, x_1, x_2, \dots$. We insist that these variables are always distinct from the operation and relation symbols of any signature. The variables will play the role of pronouns in our syntactical apparatus. For any structure $\mathbf{A}$ of our signature, these pronouns are intended to range over the *elements* of the universe A of $\mathbf{A}$. In particular, the variables are not intended to range over subsets of A nor over sequences of elements of A or over, say the natural numbers (unless the natural numbers happen to belong to A). It is this limitation that the variables are intended to range over elements of the universe that explains the prevalence of the word "elementary" in our subject. One might, of course, provide a richer syntax by including variables to range of subsets of the universe, or binary relations or functions on the universe, ... This would result in a nonelementary model theory and it lies outside to scope of our course.

We will say the **elementary language** of our signature is just the set consisting of all operation and relation symbols. Of course, taking this perspective carries the hidden assumption that the rank of any of these symbols is a trait of the symbol itself. So, while it is more pedantic it is also more proper to to identify a language L with its signature $\langle \sigma, \rho\rangle$.

Terms

We can combine variables and operation symbols to obtain terms.

The **set of terms** is the smallest set T of finite strings of symbols that

- contains all the variables, and
- for each operation symbol Q and any $t_0, \dots, t_{r-1} \in T$, where r is the rank of Q, the string $\langle Q, t_0, \dots, t_{r-1}\rangle$ also belongs to T.

As a matter of practice, we write $Qt_0 \dots t_{r-1}$ in place of $\langle Q, t_0, \dots, t_{r-1}\rangle$. Notice that if Q has rank 0, that is Q is a constant symbol, then the associated sequence is just $\langle Q\rangle$ which we typically write as Q. If the signature provides constant symbols, then they are, like the variables, terms of the simplest kind.

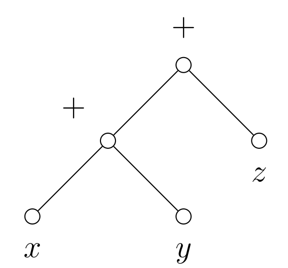
This style of definition is known as *definition by recursion* and its hallmark is to declare the simplest of the things being defined and them tell how to obtain the more complicated things from those that are simpler. As we shall see, definitions by recursion invite proofs by induction.

Our definition of terms invokes the *prefix* notation: the operation symbols occur to the left of their "arguments". This notation is sometimes called Polish notation, in honor of the Polish logician Jan Łukasiewicz who promoted its use in the 1920's. There is also postfix notation, which places the operation symbols to the right. The standard notation we are all so use to for dealing with $+$ and $\cdot$ is called *infix* notation. It has two disadvantages over the other two systems: it only works for binary operation symbols and it requires the use of parentheses, which would have to be added to our list of symbols and our rules of the correct formation of terms would have to take them into account. What we do in these notes is to adopt the prefix notation officially, but when dealing with binary operation symbols we will informally follow the traditional notation and rely of the diligent graduate students to put it all into prefix form at their pleasure.

Here is what one side of the associative law for $+$ would look like

$++xyz$	in our official prefix notation.
$((x+y)+z)$	in the unofficial infix or traditional notation.

There is another, highly informative, way to view terms. We can render them as rooted trees whose nodes have a left-to-right ordering. The internal nodes of such a tree are labelled with operation symbols (and a node labelled by an operation symbol of rank r must have r children). The leaves have to be labelled either with variables or with constant symbols. Here is the tree depiction of the term we used above:



The tree depicting the term $++xyz$.

Let T be the set of terms of our signature. There is a natural way to impose the structure of an algebra on T. We call it the **term algebra** and denote by $\mathbf{T}$. Of course, we are obliged to impose operations to obtain this algebra. To this end, let Q be an operation symbol and let r be the rank of Q. Let $t_0, \dots, t_{r-1}$ be any terms of our signature. We define $Q^{\mathbf{T}}$ by putting

$$Q^{\mathbf{T}}(t_0, \dots, t_r) = Qt_0 \dots t_{r-1}.$$

The set of terms has a key parsing property that is expressed in the following theorem.

The Unique Readability Theorem for Terms. *Let* $\mathbf{T}$ *be the term algebra for some fixed signature. For any operation symbols* P *and* Q *and any terms* $p_0, \dots, p_{n-1}$ *and* $q_0, \dots, q_{m-1}$*, where* n *is the rank of* P *and* m *is the rank of* Q*. if* $P^{\mathbf{T}}(p_0, \dots, p_{n-1}) = Q^{\mathbf{T}}(q_0, \dots, q_{m-1})$*, then* $P = Q, n = m$*, and* $p_i = q_i$ *for all* $i < n$*.*

We offer no proof here, but rather make this the object of the first set of exercises.

Of course, our intention is that just as the operation symbols are to name the basic operations in a structure, so the terms are intended to name certain derived operations in the structure. But even after we make this explicit, our syntax will be lacking everything but certain, perhaps very involved, names. We need more.

Formulas and Sentences

The operation and relation symbols have meanings that differ from structure to structure. The meanings of the logical symbols of our syntactical apparatus, on the other hand, will not vary from structure to structure—apart from, perhaps, a restriction to the universe of each structure. The logical symbols have essentially grammatical roles intended to be the same from structure to structure. We have already seen one sort of logical symbols, namely the variables $x_0, x_1, x_2, \dots$. There are four remaining logical symbols: $\approx$, which is referred to as the **equality symbol** $\approx$, $\neg$ and $\vee$, which are referred to as the **negation symbol** $\neg$ and the **disjunction symbol** $\vee$, and $\exists$, which is referred to as the **existential quantifier** $\exists$.

Just as operation symbols and variables can be put together to form terms, with the help of terms, relation symbols and the logical symbols, we can build more complicated strings of symbols called formulas.

Atomic formulas are those strings of symbols of the form $\approx st$ where s and t are terms and those of the form $Rt_0, t_1 \dots t_{r-1}$ where R is a relation symbol and $t_0, \dots, t_{r-1}$ are terms, where r is the rank of R. The **set of formulas** is the smallest set F of finite strings of symbols such that

- F contains all the atomic formulas,
- if $\varphi, \psi \in F$, then $\neg\varphi \in F$ and $\vee\varphi\psi \in F$, and
- if $\psi \in F$, then $\exists x_i \psi \in F$ for all natural numbers i.

Like the definition of the set of terms, this definition is recursive and it invites proofs by induction. You will observe that we have continued to use prefix notation here. This is our official definition. In practice, we never write things like $\approx st$ or $\vee\varphi\psi$, writing instead $s \approx t$ and $\varphi \vee \psi$.

Just as for terms, there is a Unique Readability Theorem for formulas. We leave not only the proof but even the formulation of this theorem to the eager graduate students.

We need one more notion, that of a variable occurring free in a formula. Let us define a function, FreeVar on the set of formulas, by the following recursion:

- If φ is an atomic formula, then $\operatorname{FreeVar}\varphi$ is the set of variables occurring in φ.
- If φ is $\neg\psi$, then $\operatorname{FreeVar}\varphi = \operatorname{FreeVar}\psi$.
- If φ is $\psi \vee \theta$, then $\operatorname{FreeVar}\varphi = \operatorname{FreeVar}\psi \cup \operatorname{FreeVar}\theta$.

- If φ is $\exists x_i \psi$, then $\operatorname{FreeVar}\varphi = \operatorname{FreeVar}\psi \setminus \{x_i\}$.

We call $\operatorname{FreeVar}\varphi$ the **set of variables occurring freely** in φ and we refer to the elements of $\operatorname{FreeVar}\varphi$ has the **free variables** of φ.

A formula with no free variables is called a **sentence**.

0.4 Satisfaction and Truth

Assigning an exact meaning to an English sentence like "It is snowing." is difficult largely because it is so indefinite. One must know something about the time and place to which the pronoun "it" refers. So it is also with the formulas and sentences of our syntactical apparatus. A structure of the signature gives meanings to the operation and relation symbols and the universe of the structure at least tells us the set of elements to which the variables, the pronouns of of syntax, refer. Still, even with, say the ring of integers designated as the structure of interest, a formula like $x \cdot y \approx 2 \cdot x + 2$ cannot be given a definite value as true or false—it depends on which integers are assigned to x and y.

Given a structure $\mathbf{A}$ for each operation symbol Q we have associated an operation $Q^{\mathbf{A}}$ on the set A. Likewise, for each relation symbol R we have associated a relation $R^{\mathbf{A}}$ on A. In essence, our plan is to associate with each term t an operation $t^{\mathbf{A}}$ on A and with each formula φ a relation $\varphi^{\mathbf{A}}$. We might want to assign to $t^{\mathbf{A}}$ a finite rank and we might want to do the same for $\varphi^{\mathbf{A}}$. Such a procedure has a natural appeal, and it can be carried out. However, there are some troubles with it as well. Consider the two terms $t = 2 \cdot x + 2$ and $s = x \cdot y$. We might decide that $t^{\mathbf{A}}$ should be an operation of rank 1, while $s^{\mathbf{A}}$ should be an operation of rank 2. On the other hand, a sentence like $\forall x \forall y \big(t \approx s\big)$ would seem to express that s and t denote the same function in the structure $\mathbf{A}$, provided of course that this sentence in true in $\mathbf{A}$. But no function of rank 1 can be equal to any function of rank 2, as long as A is not empty. A similar trouble arises with formulas and the meaning of disjunction. For example, the consider the formulas $\varphi = x \leq y$ and $\psi = z \approx z \cdot z$. In the structure $\mathbf{A}$ we might want φ to name a subset of $A \times A$ and ψ to name a subset of A. The same impulse would lead us to see that the formula $\varphi \vee \psi$ should name a subset of $A \times A \times A$, disrupting the idea that $\vee$ should correspond to the union of the two earlier relations. So we will drop the insistence on finite rank.

Let $\mathbf{A}$ be a structure. By an **assignment** for $\mathbf{A}$ we mean a function $\bar{a} : A^\omega \to A$. That is

$$\bar{a} = \langle a_0, a_1, a_2, \dots \rangle = \langle a_i \mid i \in \omega \rangle$$

for some $a_0, a_1, a_2, \dots \in A$. We define $t^{\mathbf{A}}$ for each term t by the following recursion.

- $x_i^{\mathbf{A}}(\bar{a}) = a_i$ for each natural number i and each assignment $\bar{a}$.
- $(Qt_0 \dots t_{r-1})^{\mathbf{A}}(\bar{a}) = Q^{\mathbf{A}}(t_0^{\mathbf{A}}(\bar{a}), \dots, t_{r-1}^{\mathbf{A}}(\bar{a}))$, for each operation symbol Q, for all terms $t_0, \dots, t_{r-1}$, where r is the rank of Q, and for all assignments $\bar{a}$.

According to the first condition above, we see that $\bar{a}$ actually ends up assigning an element of A to each variable. We also see that $t^{\mathbf{A}}$ is an operation on A, but of rank ω:

$$t^{\mathbf{A}} : A^\omega \to A.$$

Despite its infinite rank, $t^{\mathbf{A}}$ only depends on finitely many on its inputs—a fact the eager graduate students are invited to prove by induction on the complexity of the term t.

We define what it means to say that $\bar{a}$ **satisfies** φ **in A**, which we display as $\mathbf{A} \models \varphi[\bar{a}]$, by the following recursion on the complexity of the formula φ.

In case φ is $s \approx t$,

$$\mathbf{A} \models (s \approx t)[\bar{a}] \text{ if and only if } s^{\mathbf{A}}(\bar{a}) = t^{\mathbf{A}}(\bar{a}).$$

In case φ is $Rt_0 \dots t_{r-1}$,

$$\mathbf{A} \models (Rt_0 \dots t_{r-1})[\bar{a}] \text{ if and only if } (t_0^{\mathbf{A}}(\bar{a}), \dots, t_{r-1}^{\mathbf{A}}(\bar{a})) \in R^{\mathbf{A}}.$$

In case φ is $\neg\psi$,

$$\mathbf{A} \models (\neg\psi)[\bar{a}] \text{ if and only if } \mathbf{A} \models \psi[\bar{a}] \text{ fails.}$$

In case φ is $\psi \vee \theta$,

$$\mathbf{A} \models (\psi \vee \theta)[\bar{a}] \text{ if and only if } \mathbf{A} \models \psi[\bar{a}] \text{ or } \mathbf{A} \models \theta[\bar{a}].$$

In case φ is $\exists x_i \psi$,

$$\begin{aligned} \mathbf{A} \models (\exists x_i \psi)[\bar{a}] \text{ if and only if } \mathbf{A} \models \psi[\bar{b}] & \text{ for some assignment } \bar{b} \\ & \text{such that } \bar{a}(j) = \bar{b}(j) \text{ for all } j \neq i. \end{aligned}$$

This definition attaches to the logical symbols $\approx, \neg, \vee$, and $\exists$ the meanings of equality, negation, or (in the inclusive sense), and existence as we ordinarily use them in mathematical discourse.

At this point a word is in order about why we made this particular choice of logical symbols and not some other. In particular, we don't have symbols like $\wedge, \rightarrow, \leftrightarrow, \forall \dots$ to symbolize "and, implies, if and only if, for every..." and other forms of usage that arise so often in mathematics. The short answer is that were we to undertake the formalization of mathematics having a richer vocabulary would pay off, but our object here is different. Roughly speaking, we intend, rather, to prove things about what is expressible. These proofs are less involved—in short, more convenient—when the syntactical setup is less involved.

We attempt to have our cake and eat it too by adopting the following system of abbreviations:

$$\begin{aligned} \varphi \wedge \psi &\text{ abbreviates } \neg(\neg\varphi \vee \neg\psi) \\ \varphi \rightarrow \psi &\text{ abbreviates } \neg\varphi \vee \psi \\ \varphi \leftrightarrow \psi &\text{ abbreviates } (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\ \forall x_i \varphi &\text{ abbreviates } \neg\exists x_i \neg\varphi \end{aligned}$$

where φ and ψ are any formulas and i is a natural number.

In our syntax, our official symbols $\neg$ and $\vee$ and the unofficial $\wedge, \rightarrow$, and $\leftrightarrow$ are referred to as connectives. Grammatically, they are used to connect clauses. In school, my languages teachers called them *conjunctions* (this was a "part of speech" other parts being noun, verb, adjective,etc.). Here, however, we will say that $\wedge$ is the conjunction symbol (even though it is an abbreviation), that $\vee$ is the disjunction symbols, and so on.

As with terms, whether an assignment $\bar{a}$ for the structure $\mathbf{A}$ satisfies the formula φ can depend only on the values in A that $\bar{a}$ assigns to the free variables of φ. Graduate students should find delight in carrying out the appropriate induction on the complexity of φ to establish this.

We say a sentence φ is **true** in the structure $\mathbf{A}$ provided that any assignment $\bar{a}$ for $\mathbf{A}$ satisfies φ (or, what is the same, that *some* assignment satisfies φ). We also say that φ **holds** in $\mathbf{A}$ and that $\mathbf{A}$ is a **model** of φ. We write this as

$$\mathbf{A} \models \varphi.$$

More generally, we write $\mathbf{A} \models \Gamma$ and say that $\mathbf{A}$ is a model of Γ, in case Γ is a set of sentences and every sentence belonging to Γ is true in $\mathbf{A}$.

The notions of structure, signature, formula, sentence, satisfaction, and truth that are central in this lecture have a surprising intricate history. It wasn't until the early 1930's that Alfred Tarski was able to give an entirely explicit account of satisfaction and truth of elementary formulas in structures. Perhaps intuitively known for thousands of years and at times a clear goal—for example of Leibniz—these ideas emerged slowly. Structures, perhaps under a different guise, certainly arose in the 19$^{\text{th}}$ century—this notion is plainly visible in the work of Richard Dedekind, Wilhelm Weber, David Hilbert, and even Hermann Grassmann. The notion of satisfaction of a formula was implicit in the work of George Peacock as early as 1834 and was explicit in the work of 1847 of George Boole. However, no unambiguous notion of formula was available at the time. That notion arose out of the work of Ernst Schröder, Charles Saunders Pierce, and Gottlob Frege in the late 19$^{\text{th}}$ century.The manner in which these notions are laid out above would have been familiar in the 1950's.

0.5 Problem Set 0

Problem Set About Unique Readability
Due 30 August 2011

In the problems below L is the set of operation and relation symbols of same signature and X is a set of variables.

Problem 0.
Define a function λ from the set of finite nonempty sequences of elements of $X \cup L$ into the integers as follows:

$$\lambda(w) = \begin{cases} -1 & \text{if } w \in X, \\ r-1 & \text{if } w \text{ is an operation symbol of rank } r, \\ \sum_{i<n} \lambda(u_i) & \text{if } w = u_0 u_1 \ldots u_{n-1} \text{ where } u_i \in X \cup L \text{ and } n > 1. \end{cases}$$

Prove that w is a term if and only if $\lambda(w) = -1$ and $\lambda(v) \geq 0$ for every nonempty proper initial segment v of w.

Problem 1.
Let $w = u_0 u_1 \ldots u_{n-1}$, where $u_i \in X \cup L$ for all $i < n$. Prove that if $\lambda(w) = -1$, then there is a unique cyclic variant $\hat{w} = u_i u_{i+1} \ldots u_{n-1} u_0 \ldots u_{i-1}$ of w that is a term.

Problem 2.
Prove that if w is a term and w' is a proper initial segment of w, then w' is not a term.

Problem 3.
Let $\mathbf{T}$ be the term algebra of L over X. Prove

> If Q and P are operation symbols, and $P^{\mathbf{T}}(p_0, p_1, \ldots, p_{n-1}) = Q_1^{\mathbf{T}}(q_0, q_1, \ldots, q_{m-1})$, then $P = Q$, $n = m$, and $p_i = q_i$ for all $i < n$.

LECTURE 1

Models, Theories, and Logical Consequence

1.1 The Galois Connection Established by Truth

Fix a signature. Let Σ be the set of all sentences of the signature and let $\mathcal{S}$ be the class of all structures of the signature. The concept of truth establishes a binary relation between $\mathcal{S}$ and Σ:

$$\mathbf{A} \models \varphi$$

where φ is a sentence, that is $\varphi \in \Sigma$, and $\mathbf{A}$ is a structure, that is $\mathbf{A} \in \mathcal{S}$. As with any binary relation, there is an underlying Galois connection. The Problem Set attached to this lecture develops the properties of general Galois connections for those who meet this notion for the first time here.

The polarities of this particular Galois connection are the following

$$\begin{aligned}
\operatorname{Th} \mathcal{K} &= \{\varphi \mid \mathbf{A} \models \varphi \text{ for all } \mathbf{A} \in \mathcal{K}\} \\
\operatorname{Mod} \Gamma &= \{\mathbf{A} \mid \mathbf{A} \models \varphi \text{ for all } \varphi \in \Gamma\}
\end{aligned}$$

for any class $\mathcal{K} \subseteq \mathcal{S}$ and any set $\Gamma \subseteq \Sigma$.

For any set Γ of sentences we refer to $\operatorname{Mod} \Gamma$ as the **class of all models of** Γ or as the **elementary class axiomatized by** Γ. We refer to $\operatorname{Th} \mathcal{K}$ as the **elementary theory** of $\mathcal{K}$. In case $\mathcal{K} = \{\mathbf{A}\}$, we write $\operatorname{Th} \mathbf{A}$ and refer to it as the **elementary theory** of $\mathbf{A}$. In general, an **elementary theory** is a set of elementary sentences true in some class $\mathcal{K}$ of structures. Likewise, an **elementary class** is the class of all models of some set Γ of sentences.

As with every Galois connection, the polarities give rise to closure operations:

$$\begin{aligned}
&\operatorname{Mod} \operatorname{Th} \mathcal{K} \text{ which is the smallest elementary class including } \mathcal{K} \\
&\operatorname{Th} \operatorname{Mod} \Gamma \text{ which is the smallest elementary theory including } \Gamma
\end{aligned}$$

These closure operations provoke the question of finding convenient descriptions. For the first, one would like some means of combining the structures in $\mathcal{K}$ to obtain all the structures in the (usually) richer class $\operatorname{Mod} \operatorname{Th} \mathcal{K}$. Ideally, one could aspire to means of combination of some loosely algebraic kind that avoids any reference to the syntax. For the second, one would like some means of manipulating the sentences in Γ to obtain all the sentences in the (usually) richer set $\operatorname{Th} \operatorname{Mod} \Gamma$. Ideally, one would aspire to means of manipulation that take place entirely at the syntactic level and avoid any reference to structures.

The project of finding a reasonable and convenient description of $\operatorname{Mod} \operatorname{Th} \mathcal{K}$ quite clearly belongs to model theory, whereas a description of $\operatorname{Th} \operatorname{Mod} \Gamma$ turns out to belong to proof theory, another branch of mathematical logic. Nevertheless, let us take a look at $\operatorname{Th} \operatorname{Mod} \Gamma$.

We have

$$\varphi \in \operatorname{Th} \operatorname{Mod} \Gamma \text{ if and only if } \varphi \text{ is true in every model of } \Gamma.$$

Put only a slightly different way, any structure that makes all the sentences of Γ true necessarily makes φ true as well. For this reason, say that φ is a **logical consequence** of Γ provided $\varphi \in \operatorname{Th}\operatorname{Mod}\Gamma$. To denote this, widening the use of $\models$, we write $\Gamma \models \varphi$.

The notion of logical consequence just introduced is a semantical notion because it relies on the notion of a sentences (syntactical objects) being true in structures (objects of ordinary mathematical practice). A starting point for proof theory is the invention of a purely syntactical notion of **provable** so that "provable from" is exactly the same as "is a logical consequence of". Roughly speaking, proofs given in ordinary mathematical practice are certain strings of sentences (in English or some other natural language), each sentence in the string either belonging to our assumptions (that is belonging to Γ) or which follows from ealier sentences in the string by some rule of logical inference. So one of the tasks of proof theory is to cast this in our formal language, instead of the natural language, and to devise such rules of inference that will be sound and adequate for this task. But there is a lot more to proof theory than that. Kurt Gödel, in his 1929 Ph.D. thesis, succeeded in carrying out this task for countable elementary languages and the great bulk of proof theory has been developed since then, much in the last thirty years. Anatolĭi Mal'cev, working on his own Ph.D. in 1936, was able to remove the countability restriction. Without giving the details, let us write $\Gamma \vdash \varphi$ to mean that φ is provable from Γ. What Gödel proved in for countable languages and Mal'cev proved (based on Gödel's work) in general was

The Completeness Theorem. *Fix a signature. Let Γ be any set of sentences of the signature and let φ be any sentence of the signature. Then*

$$\Gamma \models \varphi \text{ if and only if } \Gamma \vdash \varphi.$$

In our development of elementary model theory, we will have little occasion to use the apparatus of formal proofs, so we will not give a precise definition of $\vdash$ nor a proof of the Completeness Theorem. Rather, we will rely on the semantical notion of logical consequence. But the Completeness Theorem has corollary that is crucial to the development of model theory.

Corollary. *Fix a signature. Let Γ by any set of sentences of the signature and let φ be any sentence of the signature. If $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

This corollary should at least be plausible. It turns out that any formal proof of φ from Γ will be a certain finite string of sentences. Such a finite string can only invoke finitely many of the assumptions, that is the sentences of Γ. So φ must be provable from the finite many assumptions that are used. The corollary above is essentially a reformulation of the Compactness Theorem, a crucial result in elementary model theory that is the focus of the next lecture. We will give a proof of the Compactness Theorem that does not depend on the Completeness Theorem.

Now let us look at the other closure operator arising from the Galois connection, $\operatorname{Mod}\operatorname{Th}\mathcal{K}$. Here $\mathcal{K}$ is a class of structures, all of the same signature, and $\operatorname{Mod}\operatorname{Th}\mathcal{K}$ is the smallest elementary class that includes $\mathcal{K}$. Suppose, for example, that $\mathcal{K}$ has just one structure: $\langle \mathbb{Q}, \leq \rangle$, the ordered set of rational numbers. What other structures $\mathbf{A}$ must belong to the larger class $\operatorname{Mod}\operatorname{Th}\mathcal{K}$? Certainly, all the properties of $\langle \mathbb{Q}, \leq \rangle$ that can expressed as elementary sentences, must be true in $\mathbf{A}$ as well. And, since the failure of a sentence in $\langle \mathbb{Q}, \leq \rangle$ entails that the negation of the sentence is true, the converse must also hold. That is $\mathbf{A} \in \operatorname{Mod}\operatorname{Th}\langle \mathbb{Q}, \leq \rangle$

if and only if the same sentences are true in $\mathbf{A}$ as in $\langle \mathbb{Q}, \leq \rangle$. After only a little reflection, the diligent graduate students should see that $\mathbf{A}$ must be a partially ordered, even a linearly ordered set. Moreover, the ordering has to be dense—between any two distinct elements there must be a third element. Finally, the ordering are no greatest element and no least element. At this point, it is difficult to cook up an elementary property of $\langle \mathbb{Q}, \leq \rangle$ that does not follow from these. Could this be the whole story? We will be able to find out the answer later.

In the reasoning just above, we found that if $\mathbf{A} \in \operatorname{Mod}\operatorname{Th}\langle \mathbb{Q}, \leq \rangle$ then $\operatorname{Th}\mathbf{A} = \operatorname{Th}\langle \mathbb{Q}, \leq \rangle$, that is the same sentences were true in $\mathbf{A}$ and in $\langle \mathbb{Q}, \leq \rangle$. This notion is important more generally. We say that two structures $\mathbf{A}$ and $\mathbf{B}$ of the same signature are **elementarily equivalent** provided $\operatorname{Th}\mathbf{A} = \operatorname{Th}\mathbf{B}$. We denote this by $\mathbf{A} \equiv \mathbf{B}$. The reasoning above gives us at least

Fact. Let $\mathbf{A}$ be a structure. Then $\operatorname{Mod}\operatorname{Th}\mathbf{A} = \{\mathbf{B} \mid \mathbf{A} \equiv \mathbf{B}\}$.

More generally, we see

Fact. Let $\mathcal{K}$ be a class of structures of the same signature. If $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A} \equiv \mathbf{B}$, then $\mathbf{B} \in \operatorname{Mod}\operatorname{Th}\mathcal{K}$.

So any attempt to understand the closure operator Mod Th would seem to involve a closer understanding of $\equiv$; that is, of elementary equivalence. This is actually a challenging undertaking. In the work of Fraïssé, Ehrenfeucht, Kochen, Keisler, and Shelah one can find good characrterizations of elementary equivalence (and consequently, descriptions of the closure operator Mod Th. This is one of the many topics that couldn't be accommodated in one semester.

Here is a theorem that is sometimes useful.

Theorem on Finite Structures. *Let $\mathbf{A}$ and $\mathbf{B}$ be structures of the same signature. If $\mathbf{A}$ is finite and $\mathbf{A} \equiv \mathbf{B}$, then $\mathbf{A} \cong \mathbf{B}$.*

Proof. Let us say that $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and that A has n distinct elements. Further we suppose, for the sake of contradiction, that $\mathbf{A} \equiv \mathbf{B}$ but that $\mathbf{A}$ and $\mathbf{B}$ are not isomorphic.

There is an elementary sentence, call it λ_k, that asserts that there are at least k distinct elements. Here is λ_3.

$$\exists x_0 \exists x_1 \exists x_2 \Big[\neg x_0 \approx x_1 \wedge \neg x_0 \approx x_2 \wedge \neg x_1 \approx x_2\Big]$$

In general, λ_k has the same form but there will be $\binom{k}{2}$ conjuncts. Likewise, there is an elementary sentence, call it β_k, that asserts that there are at most k elements. Here is β_3.

$$\forall x_0 \forall x_1 \forall x_2 \forall x_3 \Big[x_0 \approx x_1 \vee x_0 \approx x_2 \vee x_0 \approx x_3 \vee x_1 \approx x_2 \vee x_1 \approx x_3 \vee x_2 \approx x_3\Big]$$

In general, β_k is like β_3 but with $\binom{k+1}{2}$ disjuncts.

Since $\mathbf{A} \equiv \mathbf{B}$, we find that B must have exactly n elements, just like A. Now there are $n!$ one-to-one maps from A onto B and none of them is an isomorphism. So for each of these finitely many maps there must be either an operation symbol or a relation symbols so that the map in question does not preserve the basic operation or relation it denotes in $\mathbf{A}$ upon passage to $\mathbf{B}$. So for each of our $n!$ maps pick such a disruptive operation or relation symbol.

Now the idea is to write out a very long sentence true in $\mathbf{A}$ that describes exactly how these finitely many disruptive operation and relation symbols work out. We will arrange matters so that the element a_i will be associated with the variable x_i for each $i < n$.

Let $\theta(x_0, x_1, \ldots, x_{n-1})$ be a formula asserting that the n variables are all distinct.

Suppose Q is a disruptive operation symbol and, for simplicity, let its rank be 3. If

$$Q^{\mathbf{A}}(a_i, a_i, a_k) = a_\ell,$$

we will want the formula $\varphi_Q(x_i, x_j, x_k, x_\ell)$, which is

$$Qx_ix_kx_k \approx x_\ell.$$

Suppose R is a disruptive relation symbol and, for simplicity, let its rank be 3. If $(a_i, a_j, a_k) \in R^{\mathbf{A}}$, we will want the formula $\psi(x_i, x_j, x_k)$, which is

$$Rx_ix_jx_k.$$

On the other hand, if $(a_i, a_j, a_k) \notin R^{\mathbf{A}}$, we will want the formula $\mu(x_i, x_j, x_k)$, which is

$$\neg Rx_ix_jx_k.$$

Now let $\gamma(x_0, x_1, \ldots, x_{n-1})$ be the conjunction of $\theta(x_0, \ldots, x_{n-1})$ with all the formulas acquired as described above from the disruptive operation and relation symbols. Finally, let σ be the sentence $\exists x_0 \exists x_0 \ldots \exists x_{n-1} \gamma(x_0, \ldots, x_{n-1})$.

By the construction of σ, we see that σ is true in $\mathbf{A}$. So it must be true in $\mathbf{B}$. Let $b_0, b_1, \ldots, b_{n-1}$ be the elements that the variables $x_0, x_1, \ldots, x_{n-1}$ stand for in $\mathbf{B}$. Let $f : A \to B$ be defined via $f(a_i) = b_i$ for all $i < n$. Then f is a one-to-one map from A onto B and the truth of σ in $\mathbf{B}$ asserts that none of the disruptive operation and relation symbols actually disrupt f. This is the contradiction we were seeking. $\square$

Elementary theories of the form $\mathrm{Th}\,\mathbf{A}$ has an interesting property noted above. Namely, given an elementary sentence φ exactly one of φ and $\neg\varphi$ belonged to $\mathrm{Th}\,\mathbf{A}$. An elementary theory T with this property is said to be a **complete** theory. Most elementary theories are not complete. For instance, the elementary theory of rings fails to be complete—for example, the commutative law for multiplication does not belong to the theory nor does its negation. This is another way to say that some rings are commutative and some rings are not. The question left open a little above was whether the elementary theory of dense linear ordering without end points is a complete theory.

1.2 A Short Sampler of Elementary Classes

We conclude this lecture with a collection of examples of elementary classes. In each case we present a set of elementary sentences to axiomatize the class. The corresponding elementary theory is the set of all logical consequences of the set of sentences.

The Elementary Theory of Infinite Sets

In this example the signature is empty: in provides no relation or operation symbols. What we want is, for each postive natural number n, a sentence σ_n that asserts that there are at least n elements. Here is σ_4:

$$\exists x_0, x_1, x_2, x_3 \left[\neg x_0 \approx x_1 \wedge \neg x_0 \approx x_2 \wedge \neg x_0 \approx x_3 \wedge \neg x_1 \approx x_2 \wedge \neg x_1 \approx x_3 \wedge \neg x_2 \approx x_3\right].$$

In general, σ_n asserts the existence of n elements no two of which are equal. The number of disjuncts in σ_n will be $\binom{n}{2}$.

Let $\Sigma = \{\sigma_n \mid\ n \text{ is a positive natural number}\}$. Then $\operatorname{Mod}\Sigma$ is the class of infinite sets.

It is also easy to write down a sentence that asserts that there are no more than n elements. We leave this task to the eager graduate students. On the other hand, there is no set Γ of sentences so that $\operatorname{Mod}\Gamma$ is the class of all finite sets. This is a conseqence of the Compactness Theorem, which we take up in the next lecture.

The Elementary Theory of Graphs

We understand a graph to be a set of vertices endowed with an adjacency relation. So our signature this time with supple one binary relation symbol E. Here i s the list of sentences that captures the class of graphs:

(a) $\forall x\, [\neg Exx]$

(b) $\forall x, y\, [Exy \to Eyx]$

The sentence (a) asserts that no vertex is adjacent to itself—that is there are no loops. The sentence listed in (a) is also said to assert that E is irreflexive.

The sentence (b) asserts that the relation E is symmetric.

The sentences (a) and (b) together allow us to identify adjacencies are two-element sets of vertices.

In a like manner, the class of all ordered sets (sometimes called partially ordered sets), the class of sets endowed this an equivalence relation, and other similar classes can be described by sets of sentences.

The Elementary Theory of Dense Linear Orders

Again the signature supplies us this a single nonlogical symbol, a two-place relation symbol $\leq$, which by tradition we use in the infix notation.

The class of **dense linear orders** is axiomatized by the following list of sentences:

(a) $\forall x\, [x \leq x]$

(b) $\forall x, y, x\, [x \leq y \wedge y \leq z \to x \leq z]$

(c) $\forall x, y\, [x \leq y \wedge y \leq x \to x = y]$

(d) $\forall x, y\, [x \leq y \vee y \leq x]$

(e) $\forall x, y \exists z\, [(x \leq y \wedge \neg x \approx y) \to (x \leq z \wedge \neg x \approx z \wedge z \approx y \wedge z \leq y)]$

The models of the sentences (a),(b), and (c) are just the ordered sets and the models of (a), (b), (c), and (d) are the linearly ordered sets. There are lots of different dense linear orders. As an example consider the real number in $[0, 3]$. Through out all the irrational number between 1 and 2. The result is a dense linear order, using the usual order on the real numbers.

The Elementary Theory of Real Closed Fields

The paradigmatic example of a real closed field is $\langle \mathbb{R}, +, \cdot, -, 0, 1, \leq \rangle$, namely the ordered field of real numbers. The signature this time provides two 2-place operation symbols, one 1-place operation symbol, two constant symbols, and one two place relation symbols. A **real closed field** is an ordered field in which every positive element has a square root and every polynomial of odd degree has a root. As we have all seen the elementary sentences that axiomatize the class of fields, I won't write them out here again. For the ordering we need the sentences from the example above that assert that the ordering is linear. In addition we need the following two sentences:

$$\forall x, y, z\,[x \leq y \to x + z \leq y + z]$$
$$\forall x, y, z\,[x \leq y \wedge 0 \leq z \to x \cdot z \leq y \cdot z]$$

That every positive element has a square root is expressed by

$$\forall x \exists y\,[0 \leq x \to x \approx y \cdot y]\,.$$

The sentence π_n below expresses that every polynomial of degree n has a root:

$$\forall z_0, \ldots, z_{n-1} \exists x \left[z_0 + z_1 x + z_2 x^2 + \cdots + z_{n-1} x^{n-1} + x^n \approx 0\right]$$

So gather into the set Σ all the field axioms, the axioms of linear orders, the two sentences that say that the ordering is compatible in the expected way with $+$ and $\cdot$, and all the sentences π_n when n is odd. Then $\operatorname{Mod}\Sigma$ is the class of all real closed fields. While it is by no means easy, it turns out that the elementary theory of real closed fields is a complete theory—that is that any real closed field in elementarily equivalent to the ordered field of real numbers. Alfred Tarski had this conclusion in hand as early as 1930, although it did not appear in print until 1948.

The Elementary Theory of Sets

The theory of sets might more appropriately be called the theory of membership. For this theory we take a signature that provides only one nonlogical symbol, a two-place relation symbol $\in$. Zermelo-Fraenkel set theory, with choice (denoted by ZFC for short) has the following elementary sentences as axioms:

(a) $\forall x, y\,[x \approx y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)]$

(b) $\exists x \forall y(\neg y \in x)$

(c) $\forall x, y \exists z \forall u\,[u \in z \leftrightarrow (u \approx x \vee u \approx y)]$

(d) $\forall x \exists y \forall z\,[z \in y \leftrightarrow \exists u(u \in x \wedge z \in u)]$

(e) $\forall x \exists y \forall z \, [z \in y \leftrightarrow \forall u(u \in z \to u \in x)]$

(f) $\exists x \, [\exists y(y \in x) \wedge \forall y \, (y \in x \to \exists z(y \in z \wedge z \in x))]$

(g) $\forall x \, [\exists y(y \in x) \to \exists y \, (y \in x \wedge \neg \exists z(z \in y \wedge z \in x))]$

(h) $\forall u \forall \bar{v} \, [\forall x \exists ! y(\varphi(x, y, u, \bar{v})) \to \exists y \forall z \, [z \in y \leftrightarrow \exists x \, (x \in u \wedge \varphi(x, y, u, \bar{v}))]]$
Here $\exists!$ expresses "there is exactly one". We leave it to the delight of the graduate students invent a proper expansion of the this abbreviation.

(i) $\forall x \exists y \, [y$ "is a function with domain x" $\wedge \forall z((z \in x \wedge \exists u(u \in z)) \to y(z) \in z)]$

Each of these sentences assert some attribute of membership. For example, (a) asserts that two sets are the same if and only if they have the same members, while (e) asserts that the collection of all subsets of a x is also a set (we call it the power set). Notice that (h) is not actually one sentence but a whole infinite schema of sentences, each depending on the choice of the formula φ. This axiom asserts, intuitively, that if the formula φ defines a function, then the image of a set (here associated with u) with respect to this definable function is itself a set. The statement (h) is called the Axiom of Replacement. It is well worth the effort to decode into ordinary mathematical English what each of these sentences asserts. Try, for example, to see why (f) should be called the Axiom of Infinity. The statement (i) is the famous Axiom of Choice. I have left it is the capable hands of the graduate students to replace the part in quotations with an actual formula of a our signature.

This system of axioms arose in stages. In the last decades of the nineteenth century the effort to put analysis on a firm foundation led to the development of an informal theory of sets (or membership), principally at the hands of Georg Cantor. In 1899, Ernst Zermelo and Bertrand Russell independently noted that this informal theory led to a contradiction. There ensued decades of effort to secure the foundations of mathematics. The axiom system given above is one of the consequences of this effort. The largest part of the axioms for ZFC were worked out by Zermelo by 1908, although notably missing was the axiom (g), called the Axiom of Foundation or the Axiom of Regularity, and Zermelo used a weaker version of the Replacement Axiom that is known as the Axiom of Comprehension. The Axiom of Replacement was advanced by Abraham Fraenkel and independently by Thoralf Skolem in 1922. John von Neumann introduced the Axiom of Foundation (alias Axiom of Regularity) in 1925.

One might ask how successful ZFC is in capturing our intuitions about membership. The report, almost a century later is mixed but very positive. The most notable shortcoming of ZFC is that it provides not direct way to handle objects like the class of all groups. So ZFC has some cousins like the Gödel-Bernays-von Neumann theory of classes and the related Morse-Kelley theory of classes. Nevertheless, ZFC has proven adequate for the development of almost any kind of mathematics. But is it free of contradiction? After all, the construction of a systematic foundation for mathematics free of contradiction was the motivation for this elementary theory. Here the report is mixed, and necessarily so. We have no proof that it is free of contradiction. Instead, we have the extraordinary theorem of Kurt Gödel, that asserts, loosely speaking, that any comprehensive formal system in which proofs can be tested mechanically cannot prove its own consistency, unless it is inconsistent. We know, at least, the very nonobvious facts that, what some regard as the most questionable of the axioms, namely the Axiom of Choice and the Axiom of Foundation, cannot be a source of an inconsistency: If ZFC is inconsistent, then so is ZFC $-\{(h),(i)\}$. You might find it reassuring that the enormous range and depth of mathematical work over the past century has uncovered no inconsistencies. Of course, reassurance is not proof.

The Elementary Theory of Peano Arithmetic

We would like to write down a list of axioms for the elementary theory of $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$, the structure consisting of the natural numbers endowed with ordinary addition and multiplication, and with the numbers 0 and 1 given the status of named elements. In the last decades of the nineteenth century this task was taken on by Charles Saunders Peirce and Richard Dedekind and brought to completion by Giuseppe Peano in 1889. What Peano did was provide an axiomatization for the structure $\langle \mathbb{N}, S, 0\rangle$ where S denotes the successor operation. Here is a version of Peano's axioms:

$$\forall y[\neg Sy \approx 0]$$
$$\forall x, y[Sx \approx Sy \to x \approx y]$$
$$\text{For all sets } K \ \ [0 \in K \wedge \forall x(x \in K \to Sx \in K) \to \forall x(x \in K)]$$

One can prove that any two models of these axioms must be isomorphic. One can also give recursive definitions of $+$ and $\cdot$ and prove, on the basis of these axioms that they have all the expected properties. The trouble is, as you see, the third axiom has a variable K that ranges over subsets rather than just elements. So this axiom system is not elementary. Also, the definitions of $+$ and $\cdot$ are not given by elementary formulas.

One could simple take $\mathrm{Th}\langle \mathbb{N}, +, \cdot, 0, 1\rangle$ as a set of axioms, but without at least a way to list the elements of this set such a solution is not very helpful. It turns out, as shown that Alonzo Church in 1936, that there is no way the mechanically list the sentences belonging to this theory. This result lies outside the scope of our course. As a consequence, the best we can hope for is listable set of sentences that axiomatizes a substantial and interesting part of $\mathrm{Th}\langle \mathbf{N}, +, \cdot, S, 0\rangle$. One such list is called elementary Peano Arithmetic. Here is a version:

$$\forall y[\neg Sy \approx 0]$$
$$\forall x, y[Sx \approx Sy \to x \approx y]$$
$$\forall x\,[x + 0 \approx x]$$
$$\forall x, y\,[S(x+y) \approx x + Sy]$$
$$\forall x[x \cdot 0 \approx 0]$$
$$\forall x, y(x \cdot Sy \approx x \cdot y + x)$$
$$\forall \bar{y}\,[(\varphi(0,\bar{y}) \wedge \forall x(\varphi(x,\bar{y}) \to \varphi(Sx,\bar{y}))) \to \forall x \varphi(x,\bar{y})]$$

where φ can be any formula in which x does not occur as a bound variable.

This list of axioms retains Peano's minimalist approach. Rather than taking, for example, the commutative law for $+$ as an axiom, one (that is, the eager graduate student) is expected to deduce it from the given axioms. In place of Peano's nonelementary induction axiom, we have taken all its elementary instances. Like the Axiom of Replacement in ZFC, our induction axiom is actually a schema of infinitely many sentences, each depending on a different choice of φ. Roughly speaking, we have asserted here that if P is any property *describable by an elementary formula* and if

- 0 has P, and
- if x has P, then Sx has P,

then every element as property P.

1.3 Problem Set 1

Problem Set on Galois Connections
8 September 2011

In Problem 4 to Problem 8 below, let A and B be two classes and let R be a binary relation with $R \subseteq A \times B$. For $X \subseteq A$ and $Y \subseteq B$ put

$$X^{\rightarrow} = \{b \mid x\ R\ b \text{ for all } x \in X\}$$
$$Y^{\leftarrow} = \{a \mid a\ R\ y \text{ for all } y \in Y\}$$

Problem 4.
Prove that if $W \subseteq X \subseteq A$, then $X^{\rightarrow} \subseteq W^{\rightarrow}$. (Likewise if $V \subseteq Y \subseteq B$, then $Y^{\leftarrow} \subseteq V^{\leftarrow}$.)

Problem 5.
Prove that if $X \subseteq A$, then $X \subseteq X^{\rightarrow\leftarrow}$. (Likewise if $Y \subseteq B$, then $Y \subseteq Y^{\leftarrow\rightarrow}$.)

Problem 6.
Prove that $X^{\rightarrow\leftarrow\rightarrow} = X^{\rightarrow}$ for all $X \subseteq A$ (and likewise $Y^{\leftarrow\rightarrow\leftarrow} = Y^{\leftarrow}$ for all $Y \subseteq B$).

Problem 7.
Prove that the collection of subclasses of A of the form $Y^{\leftarrow}$ is closed under the formation of arbitrary intersections. (As is the collection of subclasses of B of the form $X^{\rightarrow}$.) We call classes of the form $Y^{\leftarrow}$ and the form $X^{\rightarrow}$ closed.

Problem 8.
Let $A = B = \{q \mid 0 < q < 1 \text{ and } q \text{ is rational}\}$. Let R be the usual ordering on this set. Identify the system of closed sets. How are they ordered with respect to inclusion?

LECTURE 2

The Compactness Theorem

The Compactness Theorem, stated and proved in this Lecture, has a central position in model theory. Indeed, it is such a familiar tool among the practitioners of model theory that it is common to see the phrase "By a compactness argument, we see that ...." without any further explanation. This is an indication that there are so many applications of the Compactness Theorem in model theory that many of them become routine. This theorem also has many applications in other parts of mathematics.

2.1 The Compactness Theorem

The Compactness Theorem. *Let Γ be a set of sentences. If every finite subset of Γ has a model, then Γ has a model.*

Proof. A set of sentences is said to be **finitely consistent** provided each of its finite subsets has a model.

We suppose that Γ is finitely consistent.

Looking ahead to a successful conclusion of our proof, we find a structure $\mathbf{A}$ that is a model of Γ. Our ambition is to arrange matters so that every element of A will be named by a constant symbol. What can we say about the elementary theory $\operatorname{Th}\mathbf{A}$ of $\mathbf{A}$? It is evident that it will have each of the following properties:

(a) $\Gamma \subseteq \operatorname{Th}\mathbf{A}$.

(b) $\operatorname{Th}\mathbf{A}$ is finitely consistent.

(c) Either $\varphi \in \operatorname{Th}\mathbf{A}$ or $\neg\varphi \in \operatorname{Th}\mathbf{A}$, for every sentence φ.

(d) For every formula $\psi(x)$ with one free variable, there is a constant symbol d such that $\mathbf{A} \models \exists x\psi(x) \to \psi(d)$.

The last item on this list reflects our ambition that every element of $\mathbf{A}$ be named by a constant symbol.

Our proof will have two parts. In the first part, we will build an elementary theory T with all the properties listed above. In the second part we will construct a model of T.

The first stumbling block is that there might not be enough constant symbols in our signature to name all the elements we plan to name. Indeed, there might not be any constant symbols at all. We get over this block by expanding the signature with a whole flock of new constant symbols.

Let κ be the cardinality of the set of formulas of our original signature. We expand that signature by adjoining a list of κ new constant symbols:

$$c_0, c_1, c_2, \ldots, c_\alpha, \ldots, \text{where } \alpha \in \kappa.$$

Now constraints (c) and (d) listed above have to be made to hold. Each of these constraints has κ cases. Our idea is to construct T in κ stages and at appropriate stages in the construction to fulfill parts of constraints (c) and (d).

In our expanded signature there are still κ sentences as well as κ formulas. We list all the sentences:

$$\varphi_0, \varphi_1, \ldots, \varphi_\alpha, \ldots$$

and all the formulas with one free variable

$$\psi_0, \psi_1, \ldots, \psi_\alpha, \ldots$$

For each ordinal $\beta \in \kappa \cup \{\kappa\}$ we will define T_β as follows:

$$T_0 = \Gamma$$

$$T_{\alpha+1} = T'_\alpha \cup \{\exists x \psi_\alpha \to \psi_\alpha(d)\}$$

where x is the variable free in ψ and d is the first constant symbol not occurring in T'_α

and where $T'_\alpha = \begin{cases} T_\alpha \cup \{\varphi_\alpha\} & \text{if this set is finitely consistent and} \\ T_\alpha \cup \{\neg\varphi_\alpha\} & \text{otherwise} \end{cases}$

$$T_\beta = \bigcup_{\alpha \in \beta} T_\alpha, \text{ when } \beta \text{ is a limit ordinal.}$$

Notice that in the middle case $\beta = \alpha + 1$. This is the case when β is a successor ordinal.

Finally, we put $T = T_\kappa$. It should be clear that the constraints (a), (c), and (d) have been fulfilled. We have to verify condition (b). So we prove by transfinite induction that both T'_β and T_β are finitely consistent whenever $\beta \in \kappa \cup \{\kappa\}$.

Base Step: $\beta = 0$. We know that $T_0 = \Gamma$ and that Γ is finitley consistent. We need to show that T'_β is also finitely consistent. We have

$$T'_0 = \begin{cases} T_0 \cup \{\varphi_0\} & \text{if this set is finitely consistent and} \\ T_0 \cup \{\neg\varphi_0\} & \text{otherwise.} \end{cases}$$

Under the top alternative there is nothing left to prove. So suppose the bottom alternative applies. This means $T_0 \cup \{\varphi_0\}$ is not finitely consistent. So pick a finite subset $\Lambda \subseteq T_0$ so that $\Lambda \cup \{\varphi_0\}$ has no model. To prove that $T_0 \cup \{\neg\varphi\}$ is finitely consistent pick an arbitrary finite subset $\Delta \subseteq T_0$. We have to show that

$$\Delta \cup \{\neg\varphi\} \text{ has a model.}$$

Now $\Delta \cup \Lambda$ is a finite subset of T_0 so it has a model $\mathbf{B}$. Since $\Lambda \cup \{\varphi\}$ has no model, we conclude that $\mathbf{B}$ is not a model of φ. This means that $\mathbf{B} \models \neg\varphi$. Therefore

$$\mathbf{B} \models \Delta \cup \{\neg\varphi\} \text{ has a model}$$

as desired.

Inductive Steps: There are two subcases depending on whether β is a nonzero limit ordinal or a successor ordinal.

Suppose first that β is a limit ordinal. Then $T_\beta = \bigcup_{\alpha\in\beta} T_\alpha$. Let Δ be a finite subset of T_β. Since nonzero limit ordinals are infinite, we see that $\Delta \subseteq T_\alpha$ for some $\alpha \in \beta$. Appealing to the induction hypothesis, we see that Δ has a model, and so T_β is finitely consistent.

What about T'_β? Well, the argument used in the Base Step goes through if we replace 0 everywhere in it by β.

Next suppose that β is the successor of an ordinal, say α. That is, $\beta = \alpha + 1$. So $T_\beta = T'_\alpha \cup \{\exists x \psi_\alpha \to \psi(d)\}$, where d is a constant symbol not occurring in T'_α. Since α is smaller than β, we know, by the induction hypothesis, that T'_α is finitely consistent. Let Δ be any finite subset of T'_α. Let $\mathbf{B}$ be a model of Δ. We need to show that $\Delta \cup \{\exists x\psi_\alpha \to \psi_\alpha(d)\}$ has a model. We can obtain such a model by making, if needed, a small adjustment to $\mathbf{B}$. In the event that the hypothesis $\exists x\psi_\alpha$ fails in $\mathbf{B}$, then the implication $\exists x\psi_\alpha \to \psi(d)$ holds in $\mathbf{B}$ and no adjustment is needed. In the event that $\exists x\psi$ holds in $\mathbf{B}$, then there is an element $b \in B$ that witnesses this existential assertion. We adjust $\mathbf{B}$ to obtain $\mathbf{B}'$ by changing only the element denoted by the constant symbol d. We put $d^{\mathbf{B}'} = b$, otherwise $\mathbf{B}$ and $\mathbf{B}'$ are exactly the same. Evidently, $\mathbf{B}' \models \exists x\psi_\alpha \to \psi_\alpha(d)$. We see that $\mathbf{B}' \models \Delta$ has well since the constant symbol d occurs in no sentence belonging to Δ. In this way we see that T_β is finitely consistent. We leave the argument that T'_β is finitely consistent in the eager hands of the graduate students.

So the proof by transfinite induction is finished. Putting $\beta = \kappa$ we see that T is finitely consistent.

At this point we have a set T is sentences that has all the attributes (a), (b), (c), and (d) that we desire of the elementary theory $\mathrm{Th}\,\mathbf{A}'$ of the structure $\mathbf{A}'$ we hope to build. Let C be the set of constant symbols in our expanded signature. Our intention is to devise a model $\mathbf{A}'$ of T in which every element is named by a constant symbol. We could use C as the universe of $\mathbf{A}'$ except that T may compel several constant symbols to name the same element. (The annoying $'$ is there to remind us that we are building a model for a richer signature than the signature of our original set Γ of sentences.) So we define an equivalence relation $\sim$ on C by

$$c \sim d \text{ if and only if } c \approx d \in T.$$

The fact that $\sim$ is actually an equivalence relation follows since all the following sentences must belong to T (since they are true in every structure—their negations cannot belong to any finitely consistent set).

$$\begin{aligned} c &\approx c && \text{for each constant symbol } c \\ c \approx d &\to d \approx c && \text{for all constant symbols } c \text{ and } d \\ [c \approx d \wedge d \approx e] &\to c \approx e && \text{for all constant symbols } c, d, \text{ and } e. \end{aligned}$$

We take $A = C/\sim$, that is the elements of A are the equivalence classes modulo $\sim$. To make this set A into a structure $\mathbf{A}'$ we have to impose on it basic operations and relations. Here is how. Let Q be any operation symbol and take r to be its rank. Likewise, let R be any relation symbol and let r be its rank. Then put

$$\begin{aligned} Q^{\mathbf{A}'}(c_0/\sim, \dots, c_{r-1}/\sim) = e/\sim \quad &\text{if and only if} \quad Qc_0 \dots c_{r-1} \approx e \in T \\ (c_0/\sim, \dots, c_{r-1}/\sim) \in R^{\mathbf{A}'} \quad &\text{if and only if} \quad Rc_0 \dots c_{r-1} \in T. \end{aligned}$$

The trouble with definitions of this sort is that equivalence class may have more than one element, but the definition worked by picking representatives from equivalence classes. To secure the definition above we have to show that it is immaterial how the representatives are chosen. We deal only with the case of operation symbols and leave the case of relation symbols for the entertainment of the graduate students. So suppose $c_i \sim c'_i$ for all $i < r$. This means that $c_i \approx c'_i \in T$ for all $i < r$. Notice that $\exists x Q c_0 \dots c_{r-1} \approx x$ is a sentence that is true in every structure. So its negation cannot belong to any finitely consistent set, like T. That means that this existential assertion actually belongs to T. But then by attribute (d), there must be a constant symbol e so that $Qc_0 \dots c_{r-1} \approx e \in T$. But we also know that the implication

$$(c_0 \approx c'_0 \wedge \cdots \wedge c_{r-1} \approx c'_{r-1}) \rightarrow Qc_0 \dots c_{r-1} \approx Qc'_0 \dots c'_{r-1}$$

must hold in every structure. As we have seen, this means that this implication belongs to T. But the finite set

$$\begin{aligned} &\{c_0 \approx c'_0, \dots, c_{r-1} \approx c'_{r-1}\} \\ &\qquad \cup \{(c_0 \approx c'_0 \wedge \cdots \wedge c_{r-1} \approx c'_{r-1}) \rightarrow Qc_0 \dots c_{r-1} \approx Qc'_0 \dots c'_{r-1}\} \\ &\qquad\qquad \cup \{\neg Qc_0 \dots c_{r-1} \approx Qc'_0 \dots c'_{r-1}\} \end{aligned}$$

has no model. This means that $Qc_0 \dots c_{r-1} \approx Qc'_0 \dots c'_{r-1}$ belongs to T. Had we also $Qc'_0 \dots c'_{r-1} \approx e' \in T$, then we could conclude that $e \approx e' \in T$ as well, so that $e \sim e'$. In this way we see that our definition of $Q^{\mathbf{A}'}$ is sound.

We have in hand a structure $\mathbf{A}'$. The next step is to show for all sentences φ that

$$\mathbf{A}' \models \varphi \text{ if and only if } \varphi \in T.$$

We prove this by induction on the complexity of the sentence φ.

Base Steps: φ is atomic.
There are two cases. Either φ is $s \approx t$ for some terms s and t that have no variables, or else φ is $Rt_0 \dots t_{r-1}$ where R is a relation symbol, r is the rank of R, and $t_0, \dots, t_{r-1}$ are terms that have no variables. To handle these two cases, it helps to realize that for any term t with free variables say $y_0, \dots, y_{r-1}$, given any constant symbols $d_0, \dots, d_{r-1}$ there will be a constant symbol e so that

$$t(d_0, \dots, d_{r-1}) \approx e \in T.$$

This can be proved by induction on the complexity of the term t using propertis (b) and (d) of the set T. (I can almost hear the eager graduate students working their pencils on this.) By reasoning similar to what we saw just above, we find that it is enough to consider just the cases where φ is $Qc_0 \dots c_{r-1} \approx e$ and where φ is $Rc_0 \dots c_{r-1}$. But we have defined $Q^{\mathbf{A}'}$ and $R^{\mathbf{A}'}$ just so the base step of the induction works in these cases.

Inductive Steps: There are three cases

Case: φ is $\theta \vee \psi$.
Just observe

$$\begin{aligned} \mathbf{A}' \models \varphi \quad &\text{if and only if} \quad \mathbf{A}' \models \theta \vee \psi \\ &\text{if and only if} \quad \mathbf{A}' \models \theta \text{ or } \mathbf{A}' \models \psi \\ &\text{if and only if} \quad \theta \in T \text{ or } \psi \in T \\ &\text{if and only if} \quad \theta \vee \psi \in T \end{aligned}$$

The equivalence linking the third and fourth steps above results from an appeal to the induction hypothesis. The equivalence of the last two deserves a bit a thought, since it depends on the finite consistency of T and on attribute (c).

Case: φ is $\neg\psi$.
Just observe

$$\begin{array}{rll} \mathbf{A}' \models \varphi & \text{if and only if} & \mathbf{A}' \models \neg\psi \\ & \text{if and only if} & \mathbf{A}' \models \psi \text{ fails} \\ & \text{if and only if} & \psi \notin T \\ & \text{if and only if} & \neg\psi \in T \end{array}$$

The last step depends on attribute (c) and the finite consistency of T.

Case: φ is $\exists x\psi$.
Just observe

$$\begin{array}{rll} \mathbf{A}' \models \varphi & \text{if and only if} & \mathbf{A}' \models \exists x\psi \\ & \text{if and only if} & \mathbf{A}' \models \psi(d) \text{ for some constant symbol } d \\ & \text{if and only if} & \psi(d) \in T \text{ for some constant symbol } d \\ & \text{if and only if} & \exists x\psi \in T \end{array}$$

To defend the equivalence between the second and third step we appeal to the fact that every element of A is named by a constant. The equivalence of the third and fourth step follows by the induction hypothesis. Once again, the last step depends on attribute (c) and the finite consistency of T, but attribute (d) is also required. We give one of the details—the details for the upward implication. Suppose that $\exists x\psi \in T$. By (d) we know that $\exists x\psi \to \psi(d) \in T$ for some constant symbols d. On the other hand, the set

$$\{\exists x\psi, \exists x\psi \to \psi(d), \neg\psi(d)\}$$

has no models. This means that it is not a subset of T by the finite consistency of T. Hence, $\neg\psi(d)$ cannot belong to T. By attribute (c), we obtain $\psi(d) \in T$, as desired.

So we know in particular that $\mathbf{A}' \models T$. Since $\Gamma \subseteq T$, we find that $\mathbf{A}'$ is a model of Γ. We are almost done. The flaw in $\mathbf{A}'$ is that it is a structure for a richer signature than the one we started with. We obtain $\mathbf{A}$ from $\mathbf{A}'$ simply by ignoring the new constant symbols. So we throw away the names from our signature, being careful to keep in the structure the elements that were named. □

The Compactness Theorem and the proof of it given above have some immediate corollaries.

Corollary 2.1.1. *Let Γ be any set of elementary sentences and φ be any elementary sentence. If φ is a logical consequence of Γ, then φ is a logical consequence of some finite subset of Γ.*

Corollary 2.1.2. *Let fix a signature. Let κ be the cardinality of the set of all formulas of the signature and let Γ be a set of sentences of the signature. If Γ has a model, then Γ has a model of cardinality no larger than κ.*

There are some things to learn from this proof. First, it shows how to use transfinite recursion to build a structure satisfying an infinite list of constraints. While this proof, which appeared in the literature in 1949, is not the first construction that does this, it may be the first one that you have seen. Second, it is possible to abstract from this proof those features that make the proof as a whole hang together. This means other proofs that build structures with different kinds of features can be patterned after this one. It is even possible to formulate general theorems, whose proofs are recognizable variants of the one just given and which have the Compactness Theorem, as well as other theorems, as an easy corollary.

The Compactness Theorem has an interesting history. Kurt Gödel was a 23 year old graduate student in 1929 in Vienna working on his Ph.D. dissertation under the supervision of Hans Hahn (of Hahn-Banach fame) when he drew the Compactness Theorem for countable signatures as a corollary of the Completeness Theorem that was the central result of his dissertation and which he published in 1930. From Gödel's writings at the time, it is fairly clear that he did not realize how important this corollary would become. Another graduate student, the 26 year old Anatolĭi Mal'cev working at Moscow State on his Ph.D. under the direction of Andrei Kolmogorov, saw how to prove the Compactness Theorem for arbitrary signatures and also found many applications of it in traditional branches of algebra, like group theory. Mal'cev published the first of his findings in 1936. The proofs found by Gödel and Mal'cev relied heavily on the work in 1920 of another graduate student, Thoralf Skolem who was then working on his Ph.D. in Oslo under the supervision of the great number theorist Axel Thue. The proof given above, as Paul Erdős would say "the proof that belongs in the Book", was discovered by Leon Henkin in 1947. Henkin was 27 at the time and working toward is Ph.D. at Princeton under the supervision of Alonzo Church.

2.2 A Sampler of Applications of the Compactness Theorem

There are a number of striking statements that are immediate consequences of the Compactness Theorem. We conclude this lecture with a selection of these.

Theorem 2.2.1. *Any set of elementary sentence that has arbitrarily large finite models must also have infinite models of arbitrarily larger cardinality.*

Proof. Let Γ be any set of elementary sentences that has arbitrarily large finite models. Let κ be any infinite cardinal. Expand the signature of Γ by adjoining a list $c_0, c_1, \dots, c_\alpha, \dots$ where $\alpha \in \kappa$ of distinct new constant symbols. Observe that, since Γ has arbitrarily large finite models, that every finite subset of

$$\Gamma \cup \{\neg c_\alpha \approx c_\beta \mid \alpha, \beta \in \kappa \text{ and } \alpha \neq \beta\}$$

has a model. By the Compactness Theorem the whole set displayed above has a model. In this model, the new constant symbols name distinct elements, so that model must have cardinality at least κ. Of course, this model is in the expanded signature, but the reduct of the model obtained by ignoring the new constant symbols is also a model of Γ with the same universe. $\square$

This result can be tightened of obtain the conclusion that such a set of sentences must have models of every cardinality that is at least as large of the cardinality of the set of formulas of the signature. It is also evident that the hypothesis of having arbitrarily large finite models can be replaced by the hypothesis of having an infinite model. Later we shall see sharper, more powerful results like this one. This theorem is due to Leon Henkin and, independently, to Abraham Robinson, and come from their Ph.D. dissertations.

This theorem points up a feature of our elementary languages: they are unable to distinguish between infinite cardinalities, after some point. It is always possible to write down a set of sentences that has only infinite models. Likewise, one can easily express the fact that any model of the set has exactly n elements, for any single positive integer n. But no set of elementary sentences, for example, can axiomatize the class of finite groups.

Theorem 2.2.2. *Let* $\mathbf{R} = \langle \mathbb{R}, +, \cdot, -, 0, 1, \leq \rangle$ *be the ordered field of real numbers. Let* T *be the elementary theory of* $\mathbf{R}$*. Then* T *has a model* $\mathbf{R}^*$ *that extends* $\mathbf{R}$ *and has infinitesmal elements, that is elements* ϵ *so that* $0 < \epsilon < \frac{1}{n}$ *holds for all positive integers* n*.*

Proof. We have to provide a structure satisfying three constraints:

- it must be a model of T,
- it must have an element that is larger than any integer, and
- $\mathbf{R}$ must be a substructure.

To succeed, we express each of these constraints as sets of elementary sentences, in the hope that the Compactness Theorem will provide us with a model of the union of these sets of sentences.

The first constraint is already expressed as a set of sentences.

For the second, let us add a new constant symbol ∞ to our signature as a name for the desired element. Then the set

$$\Gamma = \{0 \leq \infty, 1 \leq \infty, 1 + 1 \leq \infty, (1 + 1) + 1 \leq \infty, \dots\}$$

of sentences expresses that ∞ is larger than any integer.

For the third, let us add a new constant c_r to name each real number r. Let us gather into a set Δ all the sentences that tell how the basic operations and relations of our structure $\mathbf{R}$ work. For example, we put into Δ all sentences of the form $c_7 + c_3 \approx c_{10}$ (this sentence reflects that $7+3 = 10$), all sentences similar to these to say how multiplication behaves, all sentences like $c_e \leq c_\pi$ to reflect how the ordering works, as well as all sentences like $\neg c_{100} \approx c_2$.

Now consider $T \cup \Gamma \cup \Delta$. Were we able to ignore Γ, then $\mathbf{R}$ could be made into a model of this set by simply declaring that c_r should name r, for each real number r. But we are unable to ignore Γ. With the help of the Compactness Theorem, we only have to pay attention to arbitrary finite subsets of Γ.

Consider a finite subset of Γ. This finite subset asserts that ∞ must name an element larger that some specific natural number. So we can make a model of any finite subset of $T \cup \Gamma \cup \Delta$ that expands $\mathbf{R}$ by letting ∞ name a sufficiently large real number and by letting each c_r name the real number r.

We find that every finite subset of $T \cup \Gamma \cup \Delta$ has a model. So the whole set has a model. The reduct, call it $\mathbf{R}^*$, of this model of $T \cup \Gamma \cup \Delta$ to the signature of $\mathbf{R}$ has all the properties we desire. (In $\mathbf{R}^*$ the element named by ∞ is positive and $n \leq \infty$ holds for every integer n. One of the sentences in T asserts that if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$. So the element $\frac{1}{\infty}$ is the promised infinitesmal.) $\square$

A stronger version of this theorem, where the notion of substructure is replaced by the notion of *elementary* substructure—which we will take up in a few weeks—was Abraham Robinson's starting point for the development of nonstandard analysis.

Here is another application of the Compactness Theorem due to Abraham Robinson. This result can be found in his 1949 Ph.D. dissertation written at the University of London under the direction of Paul Dienes.

Robinson's Principle. *Let Γ be any set of sentences in the signature of fields that includes the field axioms. Any sentence true in all models of Γ of characteristic 0 is true in all models of Γ of characteristic p for all large enough prime numbers p.*

Proof. We prove the contrapositive. So suppose that the sentence σ is a fails in some models of Γ of characteristic p for arbitrarily large primes p. This means that every finite subset of

$$\Gamma \cup \{\neg(1+1 \approx 0), \neg(1+1+1 \approx 0), \dots \neg(\underbrace{1+\cdots+1}_{n-\text{times}} \approx 0)\} \cup \{\neg\sigma\}$$

must have a model. By the Compactness Theorem the whole set above must have a model $\mathbf{K}$. But then $\mathbf{K}$ will be a field of characteristic 0 in which σ fails. □

This Principle indicates that the Compactness Theorem creates a link between the finite and the infinite. It is some times possible to use the Compactness Theorem to show that the truth of a statement on one of these levels leads to the truth of related statements on the other level.

As a last application of the Compactness Theorem we turn to algebraic geometry. Our proof relies on a theorem from 1949 of Alfred Tarski according to which any two algebraically closed fields of the same characteristic have the same elementary theory. We will prove this theorem later in the course. Tarski's theorem is sometimes called *the Lefschetz Principle*, since Solomon Lefschetz and André Weil had suggested earlier that any statement of algebraic geometry that is true over $\mathbb{C}$ should be true over every algebraically closed field of characteristic 0. Lefschetz had no proof of this, nor even any precise formulation of what a "statement of algebraic geometry" might be, but he was able to provide interesting and useful instances of this principle that could be proven. Variants of this result for formal languages more powerful than the ones we have developed here have been discovered.

Putting Tarski's theorem on algebraically closed fields together with Robinson's Principle we get

Robinson's Principle for Algebraically Closed Fields. *A sentence is true in algebraically closed fields of characteristic 0 if and only if it is true in algebraically closed fields of characteristic p, for all large enough primes p.*

Proof. Let Γ be the theory of algebraically closed fields. Suppose the sentence φ is true in an algebraically closed field of characteristic 0. According to Tarski, φ is true in every algebraically closed field of characteristic 0. By Robinson's Principle, φ is true to all algebraically closed fields of characteristic p, for all large enough primes p. For the converse, suppose φ is not true in some algebraically closed field of characteristic 0. The $\neg\varphi$ is true in some algebraically closed field of characteristic 0. Again according to Tarski $\neg\varphi$ is true in every algebraically closed field of characteristic 0. By Robinson's Principle $\neg\varphi$ is true in every algebraically closed field of characteristic p for all large enough primes p. Therefore φ is not true in any algebraically closed field of prime characteristic, except for a finite set of primes. □

An **affine variety** over the field $\mathbb{C}$ of complex numbers is just the set of all solutions to some system of polynomial equations in, say n variables, where the coefficients of the polynomials are complex numbers. So V is an affine variety provided

$$V = \{(z_0, \ldots, z_{n-1}) \mid z_0, \ldots, z_{n-1} \in \mathbb{C} \text{ and } p_i(z_0, \ldots, z_{n-1}) = 0 \text{ for all } i \in I\}$$

for some system $\langle p_i \mid i \in I\rangle$ of polynomials with complex coefficients. It is a consequence of Hilbert's Finite Basis Theorem that I can be taken to be finite. Observe that $V \subseteq \mathbb{C}^n$, giving it a profoundly geometric character.

A map $G : V \to W$ between affine varieties V and W contained in $\mathbb{C}^n$ is called a **polynomial map** provided there are polynomials $g_0, \ldots, g_{n-1}$ with complex coefficients such that

$$G(z_0, \ldots, z_{n-1}) = (g_0(z_0, \ldots, z_{n-1}), \ldots, g_{n-1}(z_0, \ldots, z_{n-1}))$$

for all points $(z_0, \ldots, z_{n-1}) \in V$.

The following theorem from 1968 is due to James Ax.

The Ax Polynomial Map Theorem. *Let V be an affine variety and $G : V \to V$ be a polynomial map. If G is one-to-one, then G is onto V.*

Proof. Actually, we prove something stronger, namely that this result holds when the field $\mathbb{C}$ of complex numbers is replaced by any algebraically closed field $\mathbf{K}$. Consider any variety $V \subseteq K^n$ and any polynomial map $G : V \to V$ that is one-to-one.

First consider the case when $\mathbf{K}$ is the algebraic closure of the field $\mathbb{Z}_p$ with p elements, where p is a prime number. $\mathbf{K}$ has characteristic p. Any finite set of elements of K belong to a finite extension of $\mathbb{Z}_p$, since each element of K is algebraic over $\mathbb{Z}_p$. But finite extensions of finite fields are finite (they are certain finite dimensional vectors spaces over the finite field. . .). We want to see that any point $\bar{a} = (a_0, \ldots, a_{n-1}) \in V$ arises as an image under G of some point in V. Let $\mathbf{L}$ be the smallest subfield of $\mathbf{K}$ that includes the elements $a_0, \ldots, a_{n-1}$. This field is finite and we see that $\bar{a} \in V \cap L^n$. Now because G is a polynomial map we see that G must send points of $V \cap L^n$ to points of $V \cap L^n$. The restriction of G to $V \cap L^n$ is a one-to-one function from the *finite* set $V \cap L^n$ into itself. The finiteness entails that this restriction of G is also onto $V \cap L^n$. So $\bar{a}$ must have a preimage under G, as desired. Hence, the theorem holds if the underlying field is the algebraic closure of some $\mathbb{Z}_p$.

We would like to take two further steps. First, we would like to say that the theorem holds for any algebraically closed field of prime characteristic. Second, we would like to transfer this knowledge from the case of prime characteristic to the case of algebraically closed fields of characteristic 0.

Now according to the theorem of Tarski mentioned above any two algebraically closed fields of the same characteristic have the same elementary theories. So if we could find some way to express our theorem with a set of elementary sentences, then we could complete the first of the remaining two steps. For the second step, we can just apply Abraham Robinson's Transfer Principle, with Γ a set of sentences axiomatizing the class of algebraically closed fields.

Consider a particular variety V over the field $\mathbf{K}$ and a particular polynomial map $G : V \to V$. Let $p_0, \ldots, p_{k-1}$ be polynomials in the n variables $x_0, \ldots, x_{n-1}$ that determine this variety and $g_0, \ldots, g_{n-1}$ be the polynomials that determine G. These polynomials have coefficients drawn from K. Since we envision talking about varieties over arbitrary algebraically closed fields, this use of elements of a particular field is inconvenient. Replace each coefficient by a distinct new variable. Since we have only finitely many polynomials and each has only finitely many coefficients, we end up with a finite list $y_0, \ldots, y_{\ell-1}$ of variables associated with coefficients. To make for easier reading, we let $\bar{x}$ abbreviate $x_0, \ldots, x_{n-1}$ and $\bar{y}$ abbreviate $y_0, \ldots, y_{\ell-1}$. Now adjust each p_i to make $p_i(\bar{y}, \bar{x})$ by systematically replacing each coefficient by the associated y_j. We can handle the polynomials g_i that specify the polynomial map G in the same way.

$$\text{Let } \varphi(\bar{y}, \bar{x}) \text{ be the formula}$$
$$p_0(\bar{y}, \bar{x}) \approx 0 \wedge \cdots \wedge p_{k-1}(\bar{y}, \bar{x}) \approx 0$$

If we assign the coefficients in K to the y_i's, then the n-tuples assigning values in K to the variables $x_0, \ldots, x_{n-1}$ that satisfy φ in $\mathbf{K}$ are exactly to members of the affine variety V. Of course, assigning different values from K to the y_i's also results in an affine variety, although it might differ from V. Also observe, that following the same procedure over a different field will also result in an affine variety over that field.

$$\text{Let } \psi(\bar{y}, \bar{x}, \bar{x}') \text{ be the formula}$$
$$\varphi(\bar{y}, \bar{x}) \wedge \varphi(\bar{y}, \bar{x}') \wedge g_0(\bar{y}, \bar{x}) \approx g_0(\bar{y}, \bar{x}') \wedge \cdots \wedge g_{n-1}(\bar{y}, \bar{x}) \approx g_{n-1}(\bar{y}, \bar{x}')$$

Loosely speaking, ψ expresses that the n-tuple assigned to $\bar{x}$ and $\bar{x}'$ belong to the variety and that the polynomial map gives them the same value.

$$\text{Let } \theta(\bar{y}) \text{ be the formula}$$
$$\forall \bar{x} \forall \bar{x}' \Big[\psi(\bar{y}, \bar{x}, \bar{x}') \to (x_0 \approx x_0' \wedge \cdots \wedge x_{n-1} \approx x_{n-1}') \Big]$$

This formula is intended to express that the polynomial map is one-to-one.

$$\text{Let } \delta(\bar{y}) \text{ be the formula}$$
$$\forall \bar{x} \Big[\varphi(\bar{y}, \bar{x}) \to \exists \bar{x}' [\varphi(\bar{y}, \bar{x}') \wedge g_0(\bar{y}, \bar{x}') \approx x_0 \wedge \cdots \wedge g_{n-1}(\bar{y}, \bar{x}') \approx x_{n-1}] \Big]$$

This formula is intended to express that the polynomial map is onto.

So let σ be the sentence $\forall \bar{y} \Big[\theta(\bar{y}) \to \delta(\bar{y}) \Big]$. The sentence σ depends on the form and number of the polynomials p_i and g_j that determine the variety V and the polynomial map G, although the specific coefficient of those polynomials has been eliminated in favor of the variables $\bar{y}$. The truth of σ for the algebraically closed field $\mathbf{K}$ is equivalent to the statement of the theorem, but restricted to those affine varieties and those polynomial maps of the right form.

We know that our theorem holds over algebraically closed fields of prime characteristic. Suppose, for the sake of contradiction, that it fails over an algebraically closed field $\mathbf{K}$ of characteristic 0. Pick a variety V and a polynomial map G that witnesses this failure. Let *sigma* be the sentence, constructed as above, for this V and G. Then we see that $\mathbf{K} \models \neg\sigma$. By Robinson's Transfer Principle we have a prime p so that $\neg\sigma$ holds in all algebraically closed fields of characteristic p. On the other hand, we have proven that σ must be true in all algebraically closed fields of prime characteristic. This is our contradiction that establishes the theorem. $\square$

The proof of the theorem above highlights the central role that is played by discerning whether a notion can be expressed in an elementary fashion. The length of this proof just reflects an expository elaboration of this point. A practicing model theorist might have rendered the proof as follows:

> One-to-one polynomial maps on an affine variety over a finite field must be onto. It follows that the same must hold over the algebraic closure of any finite prime field. Since our theorem can be expressed by a schema of elementary sentences, it must hold over every algebraically closed field of prime characteristic. By Robinson's Transfer Principle, it must hold over every algebraically closed field of characteristic 0. In particular, it holds over the field of complex numbers.

2.3 Problem Set 2

Problem Set About the Compactness Theorem
Due 22 September 2011

Problem 9.
Let L be the language for group theory with operation symbols $\cdot, ^{-1}$, and 1. Let T be a set of L-sentences which includes all the group axioms (so every model of T will be a group). Suppose that for each n, there is a model of T which has no elements, other than 1, of order smaller than n. Prove that there is a model of T such that 1 is the only element of finite order.

Problem 10.
Suppose that G is a group which has elements of arbitrarily large finite order. Prove that G is elementarily equivalent to a group with an element of infinite order.

Problem 11.
Let $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$ be the familiar structure consisting of the natural numbers equipped with addition, multiplication, the two distinguished elements 0 and 1, and the usual order relation. Let T consist of all the sentences true in $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$. Prove T has a model $\mathbf{M}$ with an element ω so that all the following are true in $\mathbf{M}$:

$$0 \leq \omega, 1 \leq \omega, 2 \leq \omega, \ldots.$$

Problem 12.
Let L be the language of rings. Find a set Σ of L-sentences such that $\operatorname{Mod} \Sigma$ is the class of algebraically closed fields. Then prove that there is no finite set of L-sentences which will serve the same purpose.

Problem 13.
Let L be the language of ordered sets. Prove that there is no set Σ of L-sentences such that $\operatorname{Mod} \Sigma$ is the class of all well-ordered sets.

LECTURE 3

Putting Structures Together with Ultraproducts

The Henkin-style proof we gave for the Compactness Theorem has the key feature of a good proof, it goes beyond demonstration to illuminate the theorem. However, the structure that emerged in the course of that proof was made from syntactical elements. After all, the Compactness Theorem asserts that if each finite subset of a set Γ of sentences has a model, then Γ should have a model. What could be more natural than to start with a system of structures, each a model of some finite subset of Γ, and fabricate in some loosely algebraic fashion a model of all of Γ from them?

In this lecture we will describe how this might be done. Along the way—perhaps more importantly, since we already have such a nice proof of the Compactness Theorem—we will introduce a useful method of producing structures with useful elementary properties.

The basic plan is to take a system $\langle \mathbf{A}_i \mid i \in I \rangle$ of structures, all of the same signature, and form their direct product

$$\prod_{i \in I} \mathbf{A}_i.$$

After that we will identify a suitable notion of largeness and define an equivalence relation on the direct product by setting

$$\bar{a} \sim \bar{a}' \text{ if and only if the set } \{i \mid a_i = a'_i\} \text{ is a large subset of } I$$

for all tuples $\bar{a}$ and $\bar{a}'$ in the direct product. We will be able to demonstrate that $\sim$ is, in fact, a congruence relation and this will allow us the form the quotient structure of the direct product modulo this congruence relation.

The details of how to define the direct product and how to form quotient structure will surprise no one who has seen these notions in algebra, say in group theory. But what a notion of largeness might be may be new. We begin with it.

3.1 Filters and Ultrafilters

Let I be any nonempty set. By a **filter** on I we mean a collection $\mathcal{F}$ of subsets of I satisfying the following constraints:

(a) $I \in \mathcal{F}$,

(b) If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$, and

(c) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.

Any filter is one of our candidates for a notion of largeness of subsets of I. Perhaps, constraint (c) seems worth a few words of explanation. On might consider that a subset of I is large when its complement in I is negligible. Then (c) is the assertion that the union of two negligible sets is negligible.

The collection of all subsets of I is evidently a filter. It is called the **improper** filter on I. Every other filter on I is said to be a **proper filter**. There is also the **trivial filter** $\{I\}$. Let Z be a subset of I. The **principal filter** based on Z is the collection $\{X \mid Z \subseteq X \subseteq I\}$. Suppose that I is infinite (as is nearly always the case below). The **Fréchet filter** is the collection $\{X \mid X \subseteq I \text{ and } I \setminus X \text{ is finite}\}$. When I is infinite the Fréchet filter on I is not principal. Filters arose from the topological investigation of convergence. The Fréchet filter was used by Maurice Fréchet, who was also one of the originators of the topological notion of compactness.

An informative example of a filter is to let I be the unit interval and take $\mathcal{F}$ to be the collect of all subsets of the unit interval that have Lebesgue measure 1.

An **ultrafilter** on I is a maximal proper filter on I. It follows easily from Zorn's Lemma that every proper filter can be extended to an ultrafilter. This was first proven in 1930 by Alfred Tarski, some years prior to Zorn's work. The hard-working graduate students will also prove that a filter $\mathcal{F}$ is an ultrafilter on the nonempty set I if and only if for all $X \subseteq I$ exactly one of X and $I \setminus X$ belongs to $\mathcal{F}$. Those same graduate students can find out why an ultrafilter $\mathcal{U}$ is principal if and only if there is an element $a \in I$ so that $\mathcal{U}$ is just the collection of all subsets of I that have a as an element.

Again let I be a nonempty set. It is convenient to know when a collection $\mathcal{C}$ of subsets of I can be extended to an ultrafilter. As the empty set cannot belong to any proper filter and as filters are closed under taking intersections of finitely many sets in the filter, it is necessary that $\mathcal{C}$ have the **finite intersection property**—that is, the intersection on any finite nonempty subcollection of $\mathcal{C}$ should be nonempty. This condition is also sufficient. For let

$$\mathcal{F} = \{X \mid \bigcap \mathcal{D} \subseteq X \subseteq I \text{ for some } \mathcal{D} \subseteq \mathcal{C} \text{ with } \mathcal{D} \text{ finite}\}.$$

It is easy to check that $\mathcal{F}$ is a proper filter and this proper filter can be extended to an ultrafilter.

3.2 Direct Products and Reduced Products

Fix a signature.

Let $\langle \mathbf{A}_i \mid i \in I\rangle$ be a system of structures of our signature. We form the direct product of this system by taking the universe of the direct product to be the direct product of the universes and imposing the basic operations on this universe coordinatewise. Here are the details.

The set $\prod_I A_i$ is the set of all functions $\bar{a} : I \to \bigcup_I A_i$ such that $\bar{a}(i) \in A_i$ for all $i \in I$. We construe $\bar{a}$ as an I-tuple $\langle a_i \mid i \in I\rangle$. We take $A = \prod_I A_i$ to be the universe of the structure $\mathbf{A} = \prod_I \mathbf{A}_i$. To complete the specification of $\mathbf{A}$ for each operation symbol Q and for each relation symbol R (we let r be the rank of either) we define

$$Q^{\mathbf{A}}(\bar{a}_0, \dots, \bar{a}_{r-1}) = \langle Q^{\mathbf{A}_i}(\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \mid i \in I\rangle$$

$$(\bar{a}_0, \dots, \bar{a}_{r-1}) \in R^{\mathbf{A}} \text{ if and only if } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i} \text{ for all } i \in I$$

Now let $\mathcal{F}$ be a filter on I. Define the relation $\sim_{\mathcal{F}}$ on $\prod_I A_i$ by

$$\bar{a} \sim_{\mathcal{F}} \bar{a}' \text{ if and only if } \{i \mid \bar{a}(i) = \bar{a}'(i)\} \in \mathcal{F}$$

for all $\bar{a}, \bar{a}' \in \prod_I A_i$. That is, two I-tuples are related if and only if they agree on a large set of coordinates—where the large sets are the sets belonging to the filter.

The binary relation $\sim_{\mathcal{F}}$ is evidently reflexive and symmetric. That it is also transitive relies on

$$\{i \mid \bar{a}(i) = \bar{a}'(i)\} \cap \{i \mid \bar{a}'(i) = \bar{a}''(i)\} \subseteq \{i \mid \bar{a}(i) = \bar{a}''(i)\}$$

and the closure of filters under the formation of finite intersections and supersets.

Furthermore, the relation $\sim_{\mathcal{F}}$ is also a congruence relation on the structure $\mathbf{A} = \prod_I \mathbf{A}_i$. This means that for all operation symbols Q and all $\bar{a}_0, \bar{a}'_0, \dots, \bar{a}_{r-1}, \bar{a}'_{r-1}$, where r is the rank of Q,

$$\text{If } \bar{a}_j \sim_{\mathcal{F}} \bar{a}'_j \text{ for all } j < r, \text{ then } Q^{\mathbf{A}}(\bar{a}_0, \dots, \bar{a}_{r-1}) = Q^{\mathbf{A}}(\bar{a}'_0, \dots, \bar{a}'_{r-1}).$$

The verification of this, left in the hands of eager graduate students, resembles the proof of transitivity given just above, and it relies on the fact that our operation symbols have finite rank.

In order to simplify notation we will denote the congruence class of $\bar{a}$ by $\bar{a}/\mathcal{F}$. So

$$\begin{aligned} \bar{a}/\mathcal{F} &= \{\bar{a}' \mid \bar{a}' \in A \text{ and } \bar{a} \sim_{\mathcal{F}} \bar{a}'\} \\ &= \{\bar{a}' \mid \bar{a}' \in A \text{ and } \{i \mid \bar{a}(i) = \bar{a}'(i)\} \in \mathcal{F}\}. \end{aligned}$$

We use $A/\mathcal{F}$ to denote the set of all these congruence classes, that is $A/\mathcal{F}$ is the partition associated with $\sim_{\mathcal{F}}$. We take $B = A/\mathcal{F}$ to the universe of a structure $\mathbf{B}$. To complete the specification of $\mathbf{B}$ for each operation symbol Q and each relation symbol R (we let r be the rank of either) we define

$$Q^{\mathbf{B}}(\bar{a}_0/\mathcal{F}, \dots, \bar{a}_{r-1}/\mathcal{F}) = Q^{\mathbf{A}}(\bar{a}_0, \dots, \bar{a}_{r-1})/\mathcal{F}$$
$$(\bar{a}_0/\mathcal{F}, \dots, \bar{a}_{r-1}/\mathcal{F}) \in R^{\mathbf{B}} \text{ if and only if } \{i \mid i \in I \text{ and } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i}\} \in \mathcal{F}$$

These definitions rely on selecting representative elements from the various congruences classes. To see that our definitions are definite, we have to see that any particular choice of representatives is immaterial. The part of this task concerning operation symbols works just like similar tasks for defining quotient groups and quotient rings: the additional properties that distinguish congruence relations from the wider class of equivalence relations are exactly what is needed. We leave the details to the graduate students. Here is how to carry out the task with the r-ary relation symbol R.

Suppose $\bar{a}_j \sim_{\mathcal{F}} \bar{a}'_j$ for all $j < r$. What we need to verify is

$$\begin{gathered} \{i \mid i \in I \text{ and } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i}\} \in \mathcal{F} \\ \text{if and only if} \\ \{i \mid i \in I \text{ and } (\bar{a}'_0(i), \dots, \bar{a}'_{r-1}(i)) \in R^{\mathbf{A}_i}\} \in \mathcal{F}. \end{gathered}$$

Due to the symmetry of this situation, it is enough to establish one direction of this if-and-only-if assertion. Let us consider the downward direction. In this case all the following sets belong to the filter $\mathcal{F}$:

$$\begin{gathered} \{i \mid \bar{a}_j(i) = \bar{a}'_j(i)\} \text{ for all } j < r \\ \{i \mid i \in I \text{ and } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i}\} \end{gathered}$$

As this is a finite list (the rank of the relation symbol R being finite) we see that the intersection of these sets also belongs to $\mathcal{F}$. To wit

$$\{i \mid i \in I \text{ and } (\bar{a}_0(i), \dots, \bar{a}_{r-1}(i)) \in R^{\mathbf{A}_i}\} \cap \bigcap_{j<r} \{i \mid \bar{a}_j(i) = \bar{a}'_j(i)\} \in \mathcal{F}.$$

But it is easy to check that the set displayed above is a subset of

$$\{i \mid i \in I \text{ and } (\bar{a}'_0(i), \dots, \bar{a}'_{r-1}(i)) \in R^{\mathbf{A}_i}\}.$$

Because filters are closed under the formation of supersets, we draw the desired conclusion that

$$\{i \mid i \in I \text{ and } (\bar{a}'_0(i), \dots, \bar{a}'_{r-1}(i)) \in R^{\mathbf{A}_i}\} \in \mathcal{F}.$$

So we see that the definiton of the structure $\mathbf{B}$ is sound. We denote this structure by

$$\prod_I \mathbf{A}_i / \mathcal{F}$$

and refer to it as the **reduced product** of the system $\langle A_i \mid i \in I \rangle$ **modulo the filter** $\mathcal{F}$. The quotient map η that send each $\bar{a}$ to its congruence class $\bar{a}/\mathcal{F}$ is a homomorphism, making the reduced product a rather special kind of homomorphic image of the direct product. In the event that $\mathcal{F}$ happens to be an ultrafilter we say that $\mathbf{B}$ is an **ultraproduct**.

3.3 The Fundamental Theorem for Ultraproducts

It turns out that the satisfaction of elementary formulas in ultraproducts in closely linked to their satisfaction in the factor structures. In 1955, Jerzy Łoś published the following theorem.

The Fundamental Theorem for Ultraproducts. *Let $\langle \mathbf{A}_i \mid i \in I \rangle$ be a nonempty system of structures, all of the same signature. Let $\mathcal{U}$ be a ultrafilter on I and let $\langle \bar{a}_j \mid j \in \mathbb{N} \rangle$ be an ω-tuple of elements of $\prod_I A_i$. Then for any elementary formula φ*

$$\langle \bar{a}_j/\mathcal{U} \mid j \in \mathbb{N} \rangle \textit{ satisfies } \varphi \textit{ in } \prod_I \mathbf{A}_i/\mathcal{U}$$
$$\textit{if and only if}$$
$$\{i \mid \langle \bar{a}_j(i) \mid j \in \mathbb{N} \rangle \textit{ satisfies } \varphi \textit{ in } \mathbf{A}_i\} \in \mathcal{U}.$$

Proof. Before launching into the proof proper, we develop a sharper view of how terms behave in ultraproducts. Let $\mathbf{B} = \prod_I \mathbf{A}_i/\mathcal{U}$.

Contention. For every term t and any ω-tuple $\langle \bar{a}_0, \bar{a}_1, \dots \rangle$ of elements of the direct product $\prod_I A_i$ and any $\bar{b} \in \prod_I A_i$, we have

$$t^{\mathbf{B}}(\bar{a}_0/\mathcal{U}, \dots) = \bar{b}/\mathcal{U} \text{ if and only if } \{i \mid t^{\mathbf{A}_i}(\bar{a}_0(i), \dots) = \bar{b}(i)\} \in \mathcal{U}.$$

This contention can be proved by induction on the complexity of the term t. We safely leave this in the hands of the worthy graduate students. We note that the contention holds even if $\mathcal{U}$ is merely a filter, rather than an ultrafilter.

We prove the Fundamental Theorem by induction on the complexity of the formula φ.

Base Steps:
There are two kinds of base step depending on whether φ is of the form $s \approx t$ or of the form $Rt_0, \dots, t_{r-1}$. These two cases are similar, so we restrict our attention to the latter. For each $j < r$ pick $\bar{b}_J \in \prod_I A_i$ so that $t_j^{\mathbf{B}}(\bar{a}_0/\mathcal{U}, \dots) = \bar{b}_j/\mathcal{U}$. Then

$$\begin{aligned}
\langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{B} & \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } Rt_0 \dots t_{r-1} \text{ in } \mathbf{B} \\
& \text{ if and only if } (\bar{b}_0/\mathcal{U}, \dots, \bar{b}_{r-1}/\mathcal{U}) \in R^{\mathbf{B}} \\
& \text{ if and only if } \{i \mid (\bar{b}_0(i), \dots, \bar{b}_{r-1}(i) \in R^{\mathbf{A}_i}\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid (t_0^{\mathbf{A}_i}(a_0(i), \dots), \dots, t_{r-1}^{\mathbf{A}_i}(a_0(i), \dots)) \in R^{\mathbf{A}_i}\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } Rt_0 \dots t_{r-1} \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{A}_i\} \in \mathcal{U}
\end{aligned}$$

Only the fourth of these if-and-only-if assertions merits a further word. We know that for each $j < r$ the set $\{i \mid t_j^{\mathbf{A}_i}(a_0(i), \dots) = b(i)\}$ belongs to the ultrafilter. Let K the intersection of these finitely many sets. So we see that $K \in \mathcal{U}$. But then

$$\begin{aligned}
K \cup \{i \mid (\bar{b}_0(i), \dots, \bar{b}_{r-1}(i) \in R^{\mathbf{A}_i}\} \subseteq \{i \mid (t_0^{\mathbf{A}_i}(a_0(i), \dots), \dots, t_{r-1}^{\mathbf{A}_i}(a_0(i), \dots)) \in R^{\mathbf{A}_i}\}, \text{ and} \\
K \cup \{i \mid (t_0^{\mathbf{A}_i}(a_0(i), \dots), \dots, t_{r-1}^{\mathbf{A}_i}(a_0(i), \dots)) \in R^{\mathbf{A}_i}\} \subseteq \{i \mid (\bar{b}_0(i), \dots, \bar{b}_{r-1}(i) \in R^{\mathbf{A}_i}\}
\end{aligned}$$

Therefore, if one of the two sets involved in the fourth if-and-only-if belongs to $\mathcal{U}$ then so must the other.
Inductive Steps:
The inductive hypothesis is that the theorem holds for formulas less complex that φ. There are three cases to consider, depending on the structure of φ.

Case: φ is $\neg\psi$
In this case we have

$$\begin{aligned}
\langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{B} & \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \neg\psi \text{ in } \mathbf{B} \\
& \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ does not satisfy } \psi \text{ in } \mathbf{B} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i\} \notin \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ does not satisfy } \psi \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \neg\psi \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{A}_i\} \in \mathcal{U}
\end{aligned}$$

The third if-and-only-if assertion invokes the induction hypothesis. The downward direction of the fourth if-and-only-if assertion follows since $\mathcal{U}$ is an emphultrafilter—for every subset of I either it or its complement must belong to the ultrafilter.

Case: φ **is** $\psi \lor \theta$

In this case we have

$$\begin{aligned}
\langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{B} & \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \psi \lor \theta \text{ in } \mathbf{B} \\
& \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{B} \\
& \qquad\qquad \text{ or } \langle \bar{a}_0/\mathcal{U}, \dots \rangle \text{ satisfies } \theta \text{ in } \mathbf{B} \\
& \text{ if and only if } \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \qquad\qquad \text{ or } \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \theta \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i\} \cup \\
& \qquad\qquad \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \theta \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \psi \text{ or it satisfies } \theta \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle \bar{a}_0(i), \dots \rangle \text{ satisfies } \psi \lor \theta \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{A}_i\} \in \mathcal{U}
\end{aligned}$$

Here, again, it is the fourth if-and-only-if assertion that merits further explanation. What is needed is to see

$$X \in \mathcal{U} \text{ or } Y \in \mathcal{U} \text{ if and only if } X \cup Y \in \mathcal{U}.$$

The left-to-right direction holds for any filter since filters are closed under the formation of supersets. For the reverse direction, suppose $X \cup Y \in \mathcal{U}$ but that $Y \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, we have $I \setminus Y \in \mathcal{U}$. It follows that $(X \cup Y) \cap (I \setminus Y) \in \mathcal{U}$. But $(X \cup Y) \cap (I \setminus Y) \subseteq X$. So we conclude that $X \in \mathcal{U}$, as desired.

Case: φ **is** $\exists x \psi$

It is harmless, but convenient, to suppose that $x = x_0$. In this case we have

$$\begin{aligned}
\langle \bar{a}_0/\mathcal{U}, \bar{a}_1/\mathcal{U}, \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{B} & \text{ if and only if } \langle \bar{a}_0/\mathcal{U}, \bar{a}_1/\mathcal{U}, \dots \rangle \text{ satisfies } \exists x_0 \psi \text{ in } \mathbf{B} \\
& \text{ if and only if } \langle \bar{a}'_0/\mathcal{U}, \bar{a}_1/\mathcal{U}, \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{B} \\
& \qquad\qquad \text{ for some } \bar{a}'_0 \in \prod_I A_i \\
& \text{ if and only if } \{i \mid \langle \bar{a}'_0(i), \bar{a}_1(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i\} \in \mathcal{U} \\
& \qquad\qquad \text{ for some } \bar{a}'_0 \in \prod_I A_i \\
& \text{ if and only if } \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{A}_i\} \in \mathcal{U}
\end{aligned}$$

To see the downward direction of the last if-and-only-if assertion, just observe that

$$\text{If } \langle \bar{a}'_0(i), \bar{a}_1(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i, \text{ then } \langle \bar{a}_0(i), \bar{a}_1(i), \dots \rangle \text{ satisfies } \exists x_0 \psi \text{ in } \mathbf{A}_i.$$

This means

$$\{i \mid \langle \bar{a}'_0(i), \bar{a}_1(i), \dots \rangle \text{ satisfies } \psi \text{ in } \mathbf{A}_i\} \subseteq \{i \mid \langle a_0(i), \dots \rangle \text{ satisfies } \varphi \text{ in } \mathbf{A}_i\}.$$

So if the first set in this inclusion belongs to the ultrafilter, then so does the second. The upward direction follows directly from the definition of satisfaction in the case of existential formulas.

This completes all cases of the induction step. So the Fundamental Theorem is established. □

Corollary 3.3.1. *Let $\langle \mathbf{A}_i \mid i \in I\rangle$ be a nonempty system of structures, all of the same signature. Let $\mathcal{U}$ be a ultrafilter on I. Then for any elementary sentence φ*

$$\prod_I \mathbf{A}_i/\mathcal{U} \models \varphi \text{ if and only if } \{i \mid \mathbf{A}_i \models \varphi\} \in \mathcal{U}.$$

This corollary is almost as useful as the Fundamental Theorem for Ultraproducts.

3.4 An Ultraproduct Proof of the Compactness Theorem

The Compactness Theorem. *If every finite subset of a set of elementary sentences as a model, then the whole set has a model.*

Proof. Let Γ be a set of elementary sentences. Let I be the set of all finite subsets of Γ. For each $i \in I$ pick a model $\mathbf{A}_i$ of the set i. For each sentence $\varphi \in \Gamma$ let $E_\varphi = \{i \mid \varphi \in i\}$ and put $\mathcal{C} = \{E_\varphi \mid \varphi \in \Gamma\}$. Then $\mathcal{C}$ is a collection of subsets of I.

We contend that $\mathcal{C}$ has the finite intersection property. To see this, let $\varphi_0, \dots, \varphi_{n-1} \in \Gamma$. Then

$$\{\varphi_0, \dots, \varphi_{n-1}\} \in E_{\varphi_0} \cap \dots \cap E_{\varphi_{n-1}}$$

revealing that the intersection above is nonempty.

Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{C}$.

For each $\varphi \in \Gamma$ we have that $\mathbf{A}_i \models \varphi$ for all $i \in E_\varphi$. This means that

$$E_\varphi \subseteq \{i \mid \mathbf{A}_i \models \varphi\}.$$

But $E_\varphi \in \mathcal{C} \subseteq \mathcal{U}$ and so it follows that

$$\{i \mid \mathbf{A}_i \models \varphi\} \in \mathcal{U}.$$

In view of the corollary to the Fundamental Theorem for Ultraproducts, we conclude that

$$\prod_I A_i/\mathcal{U} \models \varphi,$$

for every $\varphi \in \Gamma$. That is

$$\prod_I A_i/\mathcal{U} \models \Gamma.$$

□

This proof has a certain appeal that Henkin's proof, for example, lacks. This proof actually explains how to devise a model of the whole set Γ of sentences from the models of each of its finite subsets. Soon after Łos's Fundamental Theorem appeared, Alfred Tarski realized how to construct such a proof, when the sentences involved in Γ where *Horn sentences*. Horn sentences, named after Alfred Horn, have a certain syntactic form and where known to be preserved under the formation of direct products. On this basis, Tarski suggested to his former student Anne Morel and his then current student Thomas Frayne that a proof of the Compactness Theorem, in its full generality, could be constructed along the same lines. Frayne and Morel discovered the proof above and announced their result in 1958—the proof itself was published in 1962 in a paper written jointly by Frayne, Morel, and Dana Scott, which included other significant results concerning reduced products.

3.5 Problem Set 3

Second Problem Set About the Compactness Theorem
Due Tuesday 18 October 2011

Problem 14.
Let L be a signature and $\mathcal{K}$ be a class of L-structures. We say that $\mathcal{K}$ is *axiomatizable* provided $\mathcal{K} = \operatorname{Mod}\Sigma$ for some set Σ and L-sentences. $\mathcal{K}$ is *finitely axiomatizable* provided there is a finite such Σ. Prove that $\mathcal{K}$ is finitely axiomatizable if and only if both $\mathcal{K}$ and $\{A \mid A \text{ is an } L\text{-structure and } A \notin \mathcal{K}\}$ are axiomatizable.

Problem 15.
Show that the class of fields of finite characteristic is not axiomatizable.

Problem 16.
Show that the class of fields of characteristic 0 is not finitely axiomatizable.

Problem 17.
Let φ be any sentence in the signature of fields. Prove that if φ is true in every field of characteristic 0, then there is a natural number n so that φ is true in every field of characteristic p for all primes $p > n$.

Problem 18.
Let L be a signature and for each natural number n suppose that T_n is a set of L-sentences closed with respect to logical consequence. Further, suppose that $T_0 \subset T_1 \subset T_2 \subset \dots$ is strictly increasing. Let $T = \bigcup_{n\in\omega} T_n$. Prove that

(a) T has a model.

(b) T is closed under logical consequence.

(c) T is not finitely axiomatizable.

LECTURE 4

Elementary Embeddings

4.1 The Downward Löwenheim-Skolem-Tarski Theorem

One of the lessons of twentieth century mathematics has been that a significant advance in understanding a domain of mathematics can often be obtained by developing the theory of those maps between the objects of interest which arise naturally from the fundamental notions of the domain. So the advance of group theory was greatly helped by the understanding of homomorphisms between groups and the development of topology grew substantially with the understanding of continuous functions. Model theory, like other branches of of mathematics, profits from understanding the maps which arise from its fundamental notions.

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature and that $f : A \to B$. We call such a function an **elementary** map provided for all assignment $\bar{a} \in A^\omega$ and all formulas φ

$$\text{If } \mathbf{A} \models \varphi[\bar{a}], \text{ then } \mathbf{B} \models \varphi[f(\bar{a})].$$

Here, of course we mean by $f(\bar{a})$ the assignment $\langle f(\bar{a}(0)), f(\bar{a}(1)), \ldots \rangle$. Because one of the formulas φ is $\neg x_0 \approx x_1$, we see that every elementary map is one-to-one. Moreover, if Q is an operation symbol, say of rank 3, then taking φ to be the formula $Qx_0x_1x_2 \approx x_3$ we find

$$\text{If } Q^{\mathbf{A}}(a, b, c) = d, \text{ then } Q^{\mathbf{B}}(h(a), h(b), h(c)) = h(d),$$

for all $a, b, c, d \in A$. So we see that h preserves all the basic operations. Likewise, if R is a relation symbol, say of rank 3, then taking φ to the formula $Rx_0x_1x_2$ we find

$$\text{If } (a, b, c) \in R^{\mathbf{A}}, \text{ then } (h(a), h(b), h(c)) \in R^{\mathbf{B}}.$$

for all $a, b, c \in A$. But also, letting φ be $\neg Rx_0x_1x_2$, we find

$$\text{If } (a, b, c) \notin R^{\mathbf{A}}, \text{ then } (h(a), h(b), h(c)) \notin R^{\mathbf{B}},$$

for all $a, b, c \in A$. That is

$$(a, b, c) \in R^{\mathbf{A}} \text{ if and only if } (h(a), h(b), h(c)) \in R^{\mathbf{B}}$$

for all $a, b, c \in A$. Of course the same sort of thing holds for ranks other than 3. What this means is that every elementary map is really an embedding and that $\mathbf{A}$ will be isomorphic to a substructure of $\mathbf{B}$ via the map. For this reason elementary maps are called **elementary embeddings**. Generally speaking, elementary embeddings impose a much stronger connection between $\mathbf{A}$ and $\mathbf{B}$ than an ordinary embedding would. For one thing, in the event that there is an elementary embedding of $\mathbf{A}$ into $\mathbf{B}$, it will follow that any sentence true in one of these structures, must be true in the other—that is, $\mathbf{A} \equiv \mathbf{B}$.

In practice, the most frequently encountered case of elementary embeddings occurs when $\mathbf{A}$ is actually a substructure of $\mathbf{B}$ and the map involved is the inclusion map (sending each element of A to itself). Usually, the inclusion map is not an elementary embedding. When it is, we say that $\mathbf{A}$ is an **elementary substructure** of $\mathbf{B}$ and write $\mathbf{A} \preccurlyeq \mathbf{B}$. We also say that $\mathbf{B}$ is an **elementary extension** of $\mathbf{A}$.

The ordered set $\langle \mathbb{Z}, \leq \rangle$ of integers is a substructure of the ordered set $\langle \mathbb{Q}, \leq \rangle$ of rationals, but is *not* an elementary substructure. For one thing, the formula $\forall z[x \leq z \leq y \rightarrow (x \approx z \vee z \approx y)]$ is satisfied in $\langle \mathbb{Z}, \leq \rangle$ by putting $x = 0$ and $y = 1$, but this fails in $\langle \mathbb{Q}, \leq \rangle$.

On the other hand, the ordered set of rationals properly between 0 and 1 *is* an elementary substructure of the ordered set of all rationals. To see this will take a bit of work.

We develop first a better understanding of elementary substructures, in general, and some of the significant results that grow out of that understanding.

In 1915, Leopold Löwenheim published the following theorem, generally regarded as the earliest result belonging model theory proper.

Löwenheim's Theorem. *Every elementary sentence that has a model must have a countable model.*

Since a sentence contains only finitely many symbols we can always construe it to be associated with a finite signature. So we see that Löwenheim's Theorem is a consequence of Henkin's proof of the Compactness Theorem—a proof found 33 years later. Löwenheim's proof is rather elaborate and the paper itself is difficult to read (primarily because the elegant expository equipment for expressing ideas and results in model theory was not available to Löwenheim). Thoralf Skolem, beginning in 1920 and revisiting the matter at least three more times over the ensuing decade, devised first a more transparent proof, filled some of the gaps in Löwenheim's reasoning and eventually obtaining the following generalization.

The Löwenheim-Skolem Theorem, Version I. *If Γ is a set of sentence of a countable signature and $\mathbf{A}$ is a model of Γ, then $\mathbf{A}$ has a countable substructure $\mathbf{B}$ that is also a model of Γ.*

A slightly different statement easily seen as equivalent to this is

The Löwenheim-Skolem Theorem, Version II. *Let $\mathbf{A}$ be an infinite structure of countable signature. $\mathbf{A}$ has a countable substructure $\mathbf{B}$ such that $\mathbf{A}$ and $\mathbf{B}$ are elementarily equivalent.*

Skolem's method of proof, called the method of *Skolem functions*, has found many applications and we will see it later. Another method is at hand.

In 1928, Alfred Tarski had some form of the notion of elementary substructure in hand, and was able to obtain strong extensions to this result as well as a result that asserted the existence of (elementary) extensions of large cardinalities. With the Second World War intervening, Tarski did not put these results into final form until 1952. They finally appeared in print in 1958 in a joint paper of Tarski and Robert Vaught.

The Downward Löwenheim-Skolem-Tarski Theorem. *Let $\mathbf{A}$ be a structure, let $X \subseteq A$, and let κ be the cardinality of the set of formulas of the signature. Let λ be a cardinal so that*

$$|X| + \kappa \leq \lambda \leq |A|.$$

Then $\mathbf{A}$ has an elementary substructure $\mathbf{B}$ of cardinality λ such that $X \subseteq B$.

Proof. Let $B_0 \subseteq A$ so that $X \subseteq B_0$ and $|B_0| = \lambda$. Now B_0 is probably far from being a substructure of $\mathbf{A}$, much less being an elementary substructure of $\mathbf{A}$. It is likely to be deficient in elements, but at least it is the right size. So we aim to repair its deficiencies by adding to it certain well-chosen elements of A. Our construction will proceed through denumerably many stages, at each stage adding elements to correct deficiencies arising from elements available at earlier stages.

To help to chose these elements we begin by well-ordering A. When we refer to the least element satisfying some property we mean it in reference to this well-ordering.

The basic deficiency that we must remedy arises because of some formula $\varphi(x, y_0, \ldots, y_{n-1})$ and some n-tuple $\langle b_0, \ldots, b_{n-1}\rangle$ of elements we have already put into the structure we are building so that

$$\mathbf{A} \models \exists x \varphi(x, y_0, \ldots, y_{n-1})[b_0, \ldots, b_{n-1}].$$

If we are to succeed this very same formula must be satisfied in the structure we are building by the very same n-tuple appearing above. We will call the least element $b \in A$ such that $\mathbf{A} \models \varphi(x, y_0 \ldots, y_{n-1})[b, b_0, \ldots, b_{n-1}]$ the *principal remedy* for the deficiency posed by the formula $\exists x \varphi(x, y_0, \ldots, y_{n-1})$ and the tuple $\langle b_0, \ldots, b_{n-1}\rangle$.

Then define, for each natural number k,

$$B_{k+1} := B_k \cup \{b \mid b \text{ is the principal remedy for some formula and some tuple from } B_k\}.$$

Let $B = \bigcup_{k \in \omega} B_k$.

First let us note that in the process of adding all those principal remedies, B did not become too larger. We proceed by induction. At the base step of the induction, we find B_0 which has cardinality λ. As our inductive hypothesis we assert that B_k has cardinality λ. Now at stage $k+1$, the number of formulas that might require remedies is no larger than $\kappa \leq \lambda$, since κ is the cardinality of the set of all formulas. Also at stage $k+1$, each formula needing remedy has only finitely many (say n) free variables. The number of n-tuples of elements of B_k is also bounded above by λ. So the number of remedies added at stage $k+1$ is no more than λ. Since $B_0 \subseteq B_{k+1}$ we see that the cardinality of B_{k+1} is λ, as desired. Since λ is an infinite cardinal and B is the union of a countable collection of sets, each of cardinality λ, we conclude that $|B| = \lambda$.

Now we contend that B is closed under all the basic operations of $\mathbf{A}$. Let Q be any operation symbol and, for convenience, suppose its rank is 3. Let $b_0, b_1, b_2 \in B$. Pick k large enough so that $b_0, b_1, b_2 \in B_k$. Now φ be the formula $Q y_0 y_1 y_2 \approx x$. Plainly

$$\mathbf{A} \models \exists x \varphi[b_0, b_1, b_2].$$

So we put in $B_{k+1} \subseteq B$ a principal remedy b. That is

$$Q^{\mathbf{A}}(b_0, b_1, b_2) = b \in B,$$

demonstrating that B is closed under $Q^{\mathbf{A}}$.

Now let $\mathbf{B}$ be the substructure of $\mathbf{A}$ with universe B.

Contention. $\mathbf{B}$ is an elementary substructure of $\mathbf{A}$.

To establish this contention what we must show is that for any formula θ and any n-tuple $\bar{b}$ of elements of B, where n is the number of free variables of θ we have

$$\mathbf{B} \models \theta[\bar{b}] \text{ if and only if } \mathbf{A} \models \theta[\bar{b}].$$

We prove this equivalence by induction on the complexity of θ.
Base Step: θ is atomic
This holds since $\mathbf{B}$ is a substructure of $\mathbf{A}$.
Inductive Step: θ is $\neg\varphi$

$$\begin{array}{rlll}
\mathbf{B} \models \theta[\bar{b}] & \text{if and only if } \mathbf{B} \models \neg\varphi[\bar{b}] & & \text{by the definition of } \theta \\
& \text{if and only if } \mathbf{B} \not\models \varphi[\bar{b}] & & \text{by the definition of satisfaction} \\
& \text{if and only if } \mathbf{A} \not\models \varphi[\bar{b}] & & \text{by the inductive hypothesis} \\
& \text{if and only if } \mathbf{A} \models \neg\varphi[\bar{b}] & & \text{by the definiton of satisfaction} \\
& \text{if and only if } \mathbf{A} \models \theta[\bar{b}] & & \text{by the definition of } \theta.
\end{array}$$

Inductive Step: θ is $\varphi \vee \psi$
The proof in this case resembles the one above and depends only of the inductive hypothesis and the definition of satisfaction.
Inductive Step: θ is $\exists x\varphi$

$$\begin{array}{rll}
\mathbf{B} \models \theta[\bar{b}] & \text{if and only if } \mathbf{B} \models \exists x\varphi[\bar{b}] & \text{by the definition of } \theta \\
& \text{if and only if } \mathbf{B} \models \varphi[d, \bar{b}] \text{ for some } d \in B & \text{by the definition of satisfaction} \\
& \text{if and only if } \mathbf{A} \models \varphi[d, \bar{b}] \text{ for some } d \in B & \text{by the inductive hypothesis} \\
& \text{implies } \mathbf{A} \models \exists x\varphi[\bar{b}] & \text{by the definite of satisfaction,} \\
& & \text{since } B \subseteq A \\
& \text{implies } \mathbf{A} \models \varphi[d, \bar{b}] \text{ for some } d \in B & \text{since every formula like this has} \\
& & \text{a remedy in } B
\end{array}$$

□

4.2 Necessary and Sufficient Conditions for Elementary Embeddings

It is interesting to note that it is only in the very last step of the proof of the concluding contention of the proof we just gave for the Downward Löwenheim-Skolem-Tarski Theorem that the full weight of the construction of B by stages came into play. In fact, this concluding contention a can be reframed as a criteria for the notion of elementary substructure.

The Tarski's Criterion for Elementary Substructures. *Let* $\mathbf{A}$ *be a structure and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$*. The following are equivalent:*

- $\mathbf{B}$ *is an elementary substructure of* $\mathbf{A}$*.*
- *For any formula* $\varphi(x, y_0, \ldots, y_{n-1})$ *and any* n*-tuple* $\bar{b}$ *of elements of* B*, if* $\mathbf{A} \models \exists x\varphi[\bar{b}]$*, then there is* $d \in B$ *so that* $\mathbf{A} \models \varphi[d, \bar{b}]$*.*

The argument we gave for the contention inside the proof of the Downward Löwenheim-Skolem-Tarski Theorem, in fact, is a proof of this Criterion. As early as 1928, Tarski had some version of this proof but the conclusion he drew at that time was

The Downward Löwenheim-Skolem-Tarski Theorem (Weak Version). *Let* $\mathbf{A}$ *be a structure and let* κ *be the cardinality of the set of formulas of the signature. Let* λ *be a cardinal so that*

$$|X| + \kappa \leq \lambda \leq |A|.$$

Then $\mathbf{A}$ *has an substructure* $\mathbf{B}$ *of cardinality* λ *such that* $X \subseteq B$ *and* $\mathbf{A} \equiv \mathbf{B}$*.*

Thoralf Skolem's proof of Version II can be easily adapted to prove the full Downward Löwenheim-Skolem-Tarski Theorem. It took a long time for the significance of the notion of elementary maps or of elementary substructures to emerge.

We want other characterizations of the elementary substructure relation. Let $\mathbf{B}$ be a structure. The **diagram language** of $\mathbf{B}$ is obtained by expanding the signature of $\mathbf{B}$ by adding a new constant symbol c_b for each element of $b \in B$. We expand the structure $\mathbf{B}$ to the structure $\langle \mathbf{B}, b\rangle_{b\in B}$ by letting each new constant c_b denote the corresponding element b. When the structure $\mathbf{A}$ is an extension of $\mathbf{B}$ we expand $\mathbf{A}$ to $\langle \mathbf{A}, b\rangle_{b\in B}$ is the same way. The elementary theory $\mathrm{Th}\langle \mathbf{B}, b\rangle_{b\in B}$ is called the **elementary diagram** of $\mathbf{B}$. We can make this work using only a subset of B. To simplify notation, when $b_0, \ldots, b_{n-1} \in B$ we use $\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle$ to denote the expansion of $\mathbf{B}$ by n-new constant symbols that name the elements $b_0, \ldots, b_{n-1}$. Of course, $\langle \mathbf{A}, b_0, \ldots, b_{n-1}\rangle$ has the obvious meaning, when $\mathbf{A}$ is an extension of $\mathbf{B}$. Also, to keep the notation from proliferating, for a formula φ with free variables from the list $y_0, \ldots, y_{n-1}$, we use $\varphi(b_0, \ldots, b_{n-1})$ to denote the formula obtained by substituting, the new constant symbol c_{b_i} for each free occurrence of the variable y_i in φ, for each $i < n$. A more fastidious notation would use $\varphi(c_{b_0}, \ldots, c_{b_{n-1}})$ instead.

The Tarski-Vaught Criteria for Elementary Substructures. *Let* $\mathbf{A}$ *be a structure and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$*. The following are equivalent:*

(a) $\mathbf{B}$ *is an elementary substructure of* $\mathbf{A}$*.*

(b) $\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle \equiv \langle \mathbf{A}, b_0, \ldots, b_{n-1}\rangle$*, for any finite sequence* $\langle b_0, \ldots, b_{n-1}\rangle$ *of elements of* B*.*

(c) $\langle \mathbf{B}, b\rangle_{b\in B} \equiv \langle \mathbf{A}, b\rangle_{b\in B}$*.*

Proof.
(a) $\Longrightarrow$ (b)
Suppose $\mathbf{B} \preccurlyeq \mathbf{A}$ and let $b_0 \ldots, b_{n-1} \in B$. Every sentence of the expanded language is of the form $\varphi(b_0, \ldots, b_{n-1})$ where $\varphi(y_0, \ldots, y_{n-1})$ is a formula of the original signature with free variables among $y_0, \ldots, y_{n-1}$. Since $\mathbf{B} \preccurlyeq \mathbf{A}$ we have that if $\mathbf{B} \models \varphi(y_0, \ldots, y_{n-1})[b_0, \ldots, b_{n-1}]$, then $\mathbf{A} \models \varphi(y_0, \ldots, y_{n-1})[b_0, \ldots, b_{n-1}]$, and also that if $\mathbf{B} \models \neg\varphi(y_0, \ldots, y_{n-1})[b_0, \ldots, b_{n-1}]$, then $\mathbf{A} \models \neg\varphi(y_0, \ldots, y_{n-1})[b_0, \ldots, b_{n-1}]$. By invoking the definition of satisfaction we conclude

$$\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle \models \varphi(b_0, \ldots, b_{n-1}) \text{ if and only if } \langle \mathbf{A}, b_0, \ldots, b_{n-1}\rangle \models \varphi(b_0, \ldots, b_{n-1}).$$

All sentences of the expanded language are addressed in this way, so we conclude that $\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle \equiv \langle \mathbf{A}, b_0, \ldots, b_{n-1}\rangle$.

(b) $\Longrightarrow$ (c)
This implication is immediate since each sentence of the diagram language can only involve finitely many of the new constants.

(c) $\Longrightarrow$ (a)
Let φ be any formula of the original signature, where the list $y_0, \dots, y_{n-1}$ includes all the free variables of φ. Let $b_0, \dots, b_{n-1}$ be any elements of B such that $\mathbf{B} \models \varphi[b_0, \dots, b_{n-1}]$. Then $\varphi(b_0, \dots, b_{n-1})$ is a sentence of the diagram language that holds in $\langle \mathbf{B}, b \rangle_{b \in B}$. Since $\langle \mathbf{B}, b \rangle_{b \in B} \equiv \langle \mathbf{A}, b \rangle_{n-1}$, we deduce that this sentence holds also in $\langle \mathbf{A}, b \rangle_{b \in B}$. But this means $\mathbf{A} \models \varphi[b_0, \dots, b_{n-1}]$. Consequently, $\mathbf{B} \preccurlyeq \mathbf{A}$. □

Here is a variant of the Tarski-Vaught Criteria, framed for elementary embeddings rather than substructures.

The Elementary Diagram Lemma. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures of the same signature. The following are equivalent:*

(a) $\mathbf{B}$ *can be elementarily embedded into* $\mathbf{A}$*.*

(b) $\mathbf{A}$ *can be expanded to a model of the elementary diagram of* $\mathbf{B}$*.*

Finally, here is a useful sufficient condition for the elementary substructure relation.

Vaught's Condition for Elementary Substructure. *Let* $\mathbf{A}$ *be a structure and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$*.*
If

> *for every finite* $D \subseteq B$ *and every* $a \in A$*, there is an automorphism* α *of* $\mathbf{A}$ *such that* $\alpha(d) = d$ *for all* $d \in D$ *and such that* $\alpha(a) \in B$*.*

then $\mathbf{B}$ *is an elementary substructure of* $\mathbf{A}$*.*

Proof.

We apply Tarski's Criterion. So let $\varphi(x, y_0, \dots, y_{n-1})$ be a formula and $b_0 \dots, b_{n-1}$ be elements of B so that $\mathbf{A} \models \exists x \varphi(x, y_0, \dots, y_{n-1})[b_0, \dots, b_{n-1}]$. Pick $a \in A$ such that $\mathbf{A} \models \varphi(x, y_0, \dots, y_{n-1})[a, b_0, \dots, b_{n-1}]$. Let α be an automorphism of $\mathbf{A}$ so that $\alpha(b_i) = b_i$ for all $i < n$ and also such that $\alpha(a) \in B$. Since the satisfaction of arbitrary formulas is preserved under isomorphisms (a delight for the graduate students who enjoy proofs by induction on the complexity of formulas), we find $\mathbf{A} \models \varphi(x, y_0, \dots, y_{n-1})[\alpha(a), b_0, \dots, b_{n-1}]$. Since $\alpha(a) \in B$, we find that Tarski's Criterion is fulfilled. Hence, $\mathbf{B} \preccurlyeq \mathbf{A}$. □

Here another way to obtain elementary extensions.

Let I be a set and $\mathcal{U}$ be an ultrafilter on I. Let $\langle \mathbf{A}_i \mid i \in I \rangle$ be a system of structures such that $\mathbf{A} = \mathbf{A}_i$ for all $i \in A$. That is, all the structures in the system are the same. In this case, we say that ultraproduct $\prod_I \mathbf{A}_i / \mathcal{U}$ is an **ultrapower** of $\mathbf{A}$ and we denote it by $\mathbf{A}^I / \mathcal{U}$. The map $\delta : A \to A^I / \mathcal{U}$ defined by

$$\delta(a) = \langle a \mid i \in I \rangle / \mathcal{U},$$

for all $a \in A$ is called the **natural** or **diagonal** embedding of $\mathbf{A}$ into the ultrapower. The following result is a corollary of Łos's Fundamental Theorem for Ultraproducts.

Corollary: The Natural Embedding into Ultrapowers is Elementary. *Let I be a nonempty set and let $\mathcal{U}$ be an ultrafilter on I. Let $\mathbf{A}$ be a structure. The natural embedding of $\mathbf{A}$ into the ultrapower $\mathbf{A}^I/\mathcal{U}$ is an elementary embedding.*

Proof. Let φ be any formula. Let n be the number of free variables of φ and let $a_0, \dots, a_{n-1}$ be any n-tuple of elements of A. By the underlying definitions and the Fundamental Theorem for Ultraproducts

$$\begin{aligned} \mathbf{A} \models \varphi[a_0, \dots, a_{n-1}] &\text{ if and only if } \{i \mid i \in I \text{ and } \mathbf{A} \models \varphi[a_0, \dots, a_{n-1}]\} \in \mathcal{U} \\ &\text{ if and only if } \mathbf{A}^I/\mathcal{U} \models \varphi[\delta(a_0), \dots, \delta(a_{n-1})]. \end{aligned}$$

But this means the natural embedding is an elementary embedding. □

4.3 The Upward Löwenheim-Skolem-Tarski Theorem

The first theorem in our Sampler of Applications of the Compactness Theorem in Lecture 2.2 above asserted that any set of sentences that either had arbitrarily large finite models or else an infinite model, must also have models of every cardinality at least as large as the cardinality of the set of formulas of the signature. This, of course, suggests that there is an upward as well as a downward "Löwenhiem-Skolem-Tarski" theorem. Here it is.

The Upward Löwenheim-Skolem-Tarski Theorem. *Let $\mathbf{A}$ be an infinite structure and let κ be the cardinality of the set of formulas of the signature. Let λ be a cardinal so that*

$$|A| + \kappa \leq \lambda.$$

Then $\mathbf{A}$ has a proper elementary extension $\mathbf{B}$ of cardinality λ.

Proof. Let Δ be the elementary diagram of $\mathbf{A}$. Expand the diagram language by adjoining λ new constant symbols d_α for each ordinal $\alpha < \lambda$. Let Γ be the set of sentences that assert that the d_α's denote distinct elements and that they are also distinct from the elements denoted by the constant symbols used to expand from the original signature to the signature of the diagram language. Because $\mathbf{A}$ is infinite we see that $\mathbf{A}$ can be expanded to a model of any finite subset of $\Delta \cup \Gamma$. By the Compactness Theorem, $\Delta \cup \Gamma$ has a model $\mathbf{B}^*$ of cardinality no larger than λ. Since Γ holds in this model, we find that $\mathbf{B}^*$ has cardinality exactly λ. If we know reduce back to the original signature we find a structure $\mathbf{B}'$ that has a substructure $\mathbf{A}'$ (with universe the set of elements named by the new constant symbols from the diagram language) isomorphic to $\mathbf{A}$. Notice that the elementary diagram of $\mathbf{A}'$ and of $\mathbf{A}$ are the same. By the Tarski-Vaught Criteria, we see that $\mathbf{A}' \preccurlyeq \mathbf{B}'$ and that $\mathbf{B}'$ must be a proper extension of $\mathbf{A}'$, since the d_α's name elements that lie outside of A'. If is an exercise left to the graduate students be find some set-theoretic reasons for why the primes can be erased to get the conclusion that $\mathbf{A} \preccurlyeq \mathbf{B}$, where $\mathbf{B}$ is a proper extension of $\mathbf{A}$ and has cardinality λ. □

Alfred Tarski had, already in 1928, some version of this upward theorem. With both the notion of elementary substructure and the method of elementary chains (one of the topics in the next sequence of lectures) in hand, Tarski devised a proof without the help of the Compactness Theorem. Having other research programs to pursue, Tarski set these results aside. When he returned to this topic at mid-century, after the intervention of World War II, which isolated Tarski in the United States (where is was visiting when the Nazi's invaded Warsaw), he could no longer recall his proof. It seems probable that the lost proof depended on proving

Every infinite structure has a proper elementary extension.

without the help of the Compactness Theorem, as the remainder of the proof can be accomplished easily with the help of Tarski's Elementary Chain Theorem—this theorem will be discussed in the next lecture. It would be interesting, at least from a historical viewpoint, to know such a proof.

The Upward-Löwenheim-Skolem-Tarski Theorem was actually proved by Tarski and Vaught, but it is traditional to give this theorem the name I used. Some authors even refer to this theorem as the Upward Löwenheim-Skolem Theorem. It has, certainly, rather distant but identifiable connections to Löwenheim's 1915 result. Thoralf Skolem, perhaps, would have regarded the attachment of his name as ironic since he was no champion of the uncountable.

Let us take another look at the ordered set $\langle \mathbb{Q}, \leq \rangle$. It is easy to list some properties of this structure that are elementary on their faces:

(a) The ordering is a linear ordering.

(b) There is no first element.

(c) There is no last element.

(d) For any two distinct elements there must be a third element properly between the two.

It is easy to find other properties expressible by elementary sentences that are true in $\langle \mathbb{Q}, \leq \rangle$, but these all seem to be consequences of the sentences listed above. We called the set of all elementary sentences that are consequences of these listed sentences the **theory of dense linear orderings without endpoints**.

In 1895, Georg Cantor published the remarkable fact that any two countable dense linear orderings without endpoints are isomorphic. We offer here a proof, by tradition referred to as *Cantor's Back-and-Forth Method*, that is actually due to E. V. A. Huntington in 1904.

Cantor's Theorem on Countable Dense Linear Orders. *Any two countable dense linear orders without endpoints are isomorphic.*

Proof. Let $\langle A, \leq \rangle$ and $\langle B, \sqsubseteq \rangle$ be countable dense linearly ordered sets. We start by listing the elements of $A : a_0, a_1, \dots$ and $B : b_0, b_1, \dots$. Below, when we refer to the *earliest* element of A or of B with a certain property, we mean it is the sense of these lists.

We will build the desired isomorphism F in countably many stages. Of course, our desired F will be a subset of $A \times B$. At the k^{th} stage we will put the ordered pair (c_k, d_k) into the set were are building.

Here is how we do it.

To begin

$$\text{Put } F_0 = \{(a_0, b_0)\}, A_0 = \{a_0\}, \text{ and } B_0 = \{b_0\}/$$

To keep the notation uniform, we put $c_0 = a_0$ and $d_0 = b_0$. **At stage $k+1$, when k is even**

$$F_{k+1} = F_k \cup \{(c_{k+1}, d_{k+1})\}, A_{k+1} = A_k \cup \{c_{k+1}\}, \text{ and } B_{k+1} = B_k \cup \{d_{k+1}\},$$

where c_{k+1} is the earliest element of A not yet used and d_{k+1} is the earliest element of B so that $F_k \cup \{(c_{k+1}, d_{k+1})\}$ is an isomorphism between the substructures with universes A_{k+1} and B_{k+1}.

At stage $k+1$, when k is odd

$$F_{k+1} = F_k \cup \{(c_{k+1}, d_{k+1})\}, A_{k+1} = A_k \cup \{c_{k+1}\}, \text{ and } B_{k+1} = B_k \cup \{d_{k+1}\},$$

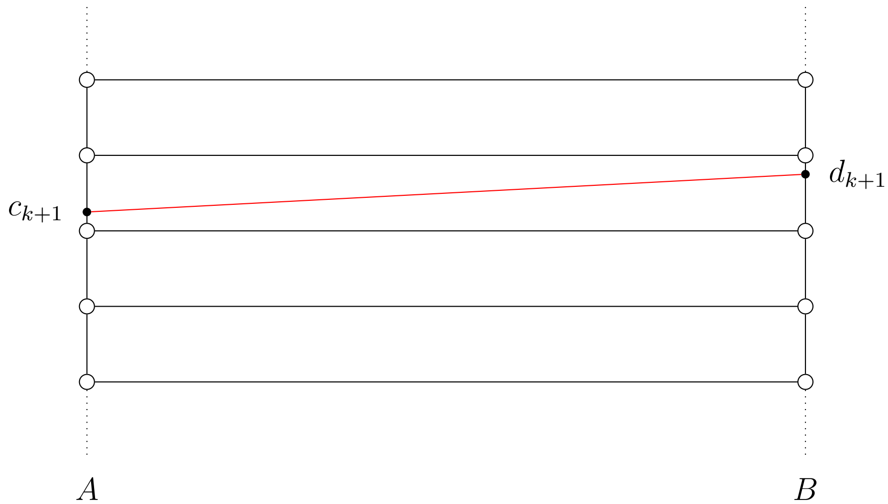
where d_{k+1} is the earliest element of B not yet used and c_{k+1} is the earliest element of A so that $F_k \cup \{(c_{k+1}, d_{k+1})\}$ is an isomorphism between the substructures with universes A_{k+1} and B_{k+1}.

Then we let $F = \bigcup_{k\in\omega} F_k$, $A_\omega = \bigcup_{k\in\omega} A_k$, and $B_\omega = \bigcup_{k\in\omega} B_k$.

Of course, there is a little touchy point about this construction: at any given stage, beyond Stage 0, how can we be sure to find the needed ordered pair (c_{k+1}, d_{k+1})? The even and odd cases pose logically symmetric difficulties, so let's just look at one and say that k is even. Since A is infinite (quick, why?) and up through stage k we have used just $k+1$ elements of A, there are plenty left and so we can have the desired c_{k+1}. Now the $k+1$ elements $c_0, c_1, \ldots, c_k$ divide the set A into $k+2$ intervals. Likewise, $d_0, \ldots, d_k$ divides the set B into $k+2$ intervals. Moreover, F_k induces a matching of the intervals of A with the intervals of B. Notice that each of the individual intervals is infinite, by denseness and lack of endpoints. Now c_{k+1} lies properly inside one of the intervals of A. The corresponding interval of B is nonempty. Let d_{k+1} be the earliest element of B properly inside this corresponding interval.

It is easy to argue, to the delight of graduate students, that the union of a nested chain of isomorphisms (like the F_k's) will always be an isomorphism F. In our, F will be an isomorphism from $\langle A_\omega$ onto B_ω. So our proof is done, once we show that $A = A_\omega$ and $B = B_\omega$. But a simple-minded induction shows that $a_n \in A_{2n}$ and for $n > 1$, that $b_n \in B_{2n-1}$. This is the back-and-forth part of the argument: We use the even steps to ensure that we gather up all the elements of A and the odd steps to ensure that we gather up all the elements of B. $\square$

The proof given in words above is much more transparent as an animated drawing. Here is just one frame of the animation.



Putting the next pair into the back-and-forth isomorphism

Here is one more interesting fact about the elementary theory of dense linear orderings without endpoints.

Theorem 4.3.1. *Let* $\mathbf{B}$ *be a substruture of* $\langle \mathbb{Q}, \leq \rangle$ *that is also a model of the theory of dense linear orderings without endpoints. Then* $\mathbf{B}$ *is an elementary substructure of* $\langle \mathbb{Q}, \leq \rangle$*.*

Proof. We invoke Vaught's Condition. To do this, let $b_0, \ldots, b_{n-1}$ be n distinct elements of B and let $a \in \mathbb{Q}$. The b_i's break $\mathbb{Q}$ into $n+1$ open intervals. The element a must be in one of the intervals or be one of the b_i's. Leaving aside, for the attention of the graduate students, the case when a is actually one the b_i's, we consider the case when a is inside the interval I. By denseness holding in $\mathbf{B}$, there must be an element $b \in B$ lying in the same interval with a. We need an automorphism α of $\langle \mathbb{Q}, \leq \rangle$ that fixes all the elements not inside the interval I, but when α is restricted to I we want it to take a to b. But we can just repeat the back-and-forth construction to make α. So Vaught's Condition gets us the conclusion that $\mathbf{B} \preccurlyeq \langle \mathbb{Q}, \leq \rangle$. □

Now let $\mathbf{A}$ be any dense linear order without endpoints. Since the theory of dense linear orders without endpoints has no finite models, we see that $\mathbf{A}$ is infinite. By the Upward Löwenheim-Skolem-Tarksi Theorem, $\mathbf{A}$ has an elementary extension $\mathbf{A}'$ of an arbitrarily large cardinality. Then by the Downward Löwenheim-Skolem-Tarski Theorem, $\mathbf{A}'$ has a countable elementary substructure $\mathbf{B}$. According to Cantor, $\mathbf{B}$ is isomorphic with $\langle \mathbb{Q}, \leq \rangle$. So $\mathbf{A} \equiv \langle \mathbb{Q}, \leq \rangle$. It follows that the theory of dense linear orderings without endpoints is a complete elementary theory.

This simple line of reasoning can apply to other elementary theories. To state it, we need another notion. Let T be an elementary theory and κ be a cardinal. The theory T is **κ-categorical** provided T has a model of cardinality κ and all models of T of cardinality κ are isomorphic to each other. So Cantor proved that the theory of dense linear orderings without endpoints is ω-categorical.

The Łoś-Vaught Test. *Suppose that* T *is an elementary theory and* κ *is a cardinal at least as large as the cardinality of the set of all formulas of the signature of* T*. If* T *has no finite models and* T *is* κ*-categorical, then* T *is a complete theory.*

As an application of the Łoś-Vaught Test we can deduce a theorem of Tarski, provided we are willing to import a bit of algebra. In 1910 Ernst Steinitz published an influential treatise on algebraically closed fields. Among other things he introduced a cardinal invariant of algebraically closed fields, the *transcendence degree*, that is analogous to the notion of dimension for vector spaces. Every field has a smallest subfield, called its *prime field*. This subfield is isomorphic to $\mathbb{Z}_p$, in case the field is of prime characteristic p, and is otherwise isomorphic to $\mathbb{Q}$. One of the theorems of Steinitz is that any two algebraically closed fields of the same characteristic and the same transcendence degree over their prime fields are isomorphic. From the definitions involved (not given here) it follows that if the transcendence degree is uncountable, then the transcendence degree is that same as the cardinality of the field. This gives

The Categoricity Theorem of Steinitz. *The elementary theory of algebraically closed fields of a given characteristic is κ-categorical for each uncountable cardinal κ.*

It is an exercise and sometimes a Qualifying Examination problem to show that every algebraically closed field in infinite. So from the Łoś-Vaught Test we deduce the following theorem.

Tarski's Completeness Theorem for Algebraically Closed Fields. *The elementary theory of algebraically closed fields of a given characteristic is complete.*

Tarski's proof of this theorem was achieved by the method of elimination of quantifiers, which, while more involved, gives a deeper insight into the elementary theory of algebraically closed fields.

LECTURE 5

Elementary Chains and Amalgamation of Structures

5.1 Elementary Chains and Amalgamation

At this point, many of the basic concepts of of elementary model theory are in our hands. The syntactical apparatus has been described, the key notions of satisfaction and truth have been laid out, the first theorem rich in consequences, namely the Compactness Theorem, has been established, the crucial idea of elementary embedding has been introduced and given several characterizations. We have seen, in the Löwenheim-Skolem-Tarski Theorems how prevalent models of arbitrary infinite cardinalities of an elementary theory turn out to be.

The purpose of this lecture in to put all these things into play to fill out the development of model theory. In succeeding lectures we will be pushing the beyond these beginnings.

First, let us address the task of putting structures together to make larger more complicated structures. Of course, we would like to know how the elementary properties of the less complicated structure relates to the elementary properties of the more involved structures.

Let $\langle \mathbf{A}_i \mid i \in I \rangle$ be a system of structures, all of the same signature. We say this system is **up-directed** by $\preccurlyeq$ provided for all $i, j \in I$ there is $k \in I$ so that

$$\mathbf{A}_i \preccurlyeq \mathbf{A}_k \text{ and } \mathbf{A}_j \preccurlyeq \mathbf{A}_k.$$

Of course, in a similar way we could make sense of what it means for the system to be up-directed with respect to some other binary relation in place of $\preccurlyeq$. The substructure relation and the relation of elementary embeddability are just two other interesting possibilities.

Let $\langle \mathbf{A}_i \mid i \in I \rangle$ be up-directed by $\preccurlyeq$. There is a transparent way to arrive at a limiting structure $\mathbf{A}$ of this system. Just let

$$\begin{aligned} A &= \bigcup_{i \in I} A_i \\ Q^{\mathbf{A}} &= \bigcup_{i \in I} Q^{\mathbf{A}_i} \text{ for each operation symbol } Q \\ R^{\mathbf{A}} &= \bigcup_{i \in I} R^{\mathbf{A}_i} \text{ for each relations symbol } R \end{aligned}$$

While it is evident that the union of a collection of r-place relations is itself an r-relation, we might anticipate that the union of a collection of r-place operations could well fail to be an operation, unless a certain amount of compatibility is present. The up-directed character of the system ensures enough compatibility. Indeed, suppose Q is a 3-place operation symbol and $a, b, c \in A = \bigcup_{i \in I} A_i$. Using the fact that the system is up-directed, that must be a $k \in I$ so that $a, b, c \in A_k$. Let $d = Q^{\mathbf{A}_k}(a, b, c)$. Suppose $\ell \in I$ so that $a, b, c \in A_\ell$ and that $e = Q^{\mathbf{A}_\ell}(a, b, c)$. Using the up-directedness again, pick $m \in I$ so that $\mathbf{A}_k \preccurlyeq \mathbf{A}_m$ and $\mathbf{A}_\ell \preccurlyeq \mathbf{A}_m$.

Then it follows that

$$d = Q^{\mathbf{A}_k}(a,b,c) = Q^{\mathbf{A}_m}(a,b,c) = Q^{\mathbf{A}_\ell}(a,b,c) = e.$$

This means $Q^{\mathbf{A}}$ really is a 3-place operation on A. A little reflection should convince you that $\mathbf{A}_k$ is a substructure of $\mathbf{A}$, for all $k \in I$. More is true.

Tarski's Elementary Chain Theorem. *Let $\langle \mathbf{A}_i \mid i \in I\rangle$ be a system of structures, all of the same signature, that is up-directed by $\preccurlyeq$ and let $\mathbf{A}$ be the limiting structure of this system. Then $\mathbf{A}_k \preccurlyeq \mathbf{A}$ for all $k \in I$.*

Proof. What we have to prove is that for every formula $\varphi(\bar{y})$ with, say, n free variables, for all $k \in I$, and all n-tuples $\bar{a}$ from A_k and all $k \in I$ we have

$$\mathbf{A}_k \models \varphi[\bar{a}] \text{ if and only if } \mathbf{A} \models \varphi[\bar{a}].$$

We do this by Sam's technique,namely induction on the complexity of formulas. The proof is, except at one crucial point, almost identical to the proof of Tarski's Criterion. By now, we are able to see that the only step that is troublesome is

Inductive Step: φ is $\exists x\theta(x,\bar{y})$

Even here the implication

$$\mathbf{A}_k \models \varphi[\bar{a}] \text{ implies } \mathbf{A} \models \varphi[\bar{a}].$$

is entirely straightforward.

Here is the implication in the other direction.

$$\begin{aligned}
\mathbf{A} \models \varphi[\bar{a}] &\text{ implies } \mathbf{A} \models \exists x\theta(x,\bar{y})[\bar{a}] \\
&\text{ implies } \mathbf{A} \models \theta(x,\bar{y})[b,\bar{a}] \text{ for some } b \in A \\
&\text{ implies } \mathbf{A} \models \theta(x,\bar{y})[b,\bar{a}] \text{ for some } b \in A_\ell \text{ for some } \ell \in I \\
&\text{ implies } \mathbf{A} \models \theta(x,\bar{y})[b,\bar{a}] \text{ for some } b \in A_m \text{ for some } m \in I \text{ so that } \mathbf{A}_k \preccurlyeq \mathbf{A}_m \\
&\text{ implies } \mathbf{A}_m \models \theta(x,\bar{y})[b,\bar{a}] \text{ for some } b \in A_m \text{ for some } m \in I \text{ so that } \mathbf{A}_k \preccurlyeq \mathbf{A}_m \\
&\text{ implies } \mathbf{A}_m \models \exists x\theta(x,\bar{y})[\bar{a}] \text{ for some } m \in I \text{ so that } \mathbf{A}_k \preccurlyeq \mathbf{A}_m \\
&\text{ implies } \mathbf{A}_m \models \varphi[\bar{a}] \text{ for some } m \in I \text{ so that } \mathbf{A}_k \preccurlyeq \mathbf{A}_m \\
&\text{ implies } \mathbf{A}_k \models \varphi[\bar{a}]
\end{aligned}$$

The fourth implication in this sequence relies on the up-directedness, the fifth invokes the induction hypothesis, and the last uses the elementary substructure relation. □

Tarski's Elementary Chain Theorem turns out to have a host of consequences. To use it, we need a method for assembling elementary chains or more elaborate systems of structures up-directed the elementary substructure relation. For this purpose theorems like the next are well-suited. Roughly speaking, the next theorem specifies circumstances under which two structures that have some common overlap can be seen as part of a single more comprehensive structure—when this is possible the larger structure shows how to amalgamate the given structures over their common part. Essentially a concept from category theory, amalgamation can be framed by way of commutative diagrams. We have framed the next theorem using elementary equivalence, elementary embeddings, and the elementary substructure relation. The applications we have in mind actually use small variations of this theorem that, for example, might involves structures of several signatures or be concerned only elementary formulas of a particular form—leading to looser notions of elementary equivalence, elementary embeddings, and elementary substructure. The consequent adjustments to the statement and proof of the next theorem will require almost no effort.

The Elementary Amalgamation Theorem. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures of the same signature and let* $\bar{a}$ *and* $\bar{b}$ *be* I*-tuples of distinct elements of* A *and* B*, respectively. Let* $\mathbf{D}$ *be the substructure of* $\mathbf{A}$ *generated by the elements listed in the tuple* $\bar{a}$*. If* $\langle \mathbf{A}, \bar{a}\rangle \equiv \langle \mathbf{B}, \bar{b}\rangle$*, then there is* $\mathbf{C}$ *and some elementary embedding* g *so that*

- $\mathbf{A} \preccurlyeq \mathbf{C}$*,*
- $g : \mathbf{B} \to \mathbf{C}$ *such that* $g(\bar{b}) = \bar{a}$*, and*
- *There exists a unique embedding* f *so that* $f : \mathbf{D} \to \mathbf{B}$ *where* $f\bar{a} = \bar{b}$*.*

The situation described in this theorem is illustrated below.

$$\begin{tikzcd}
& \mathbf{C} & \\
\mathbf{A} \arrow[ur, phantom, "\preccurlyeq" sloped] & & \mathbf{B} \arrow[ul, "g"'] \\
& \mathbf{D} \arrow[ul, phantom, "\leq" sloped] \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

Proof. We are going to use the elementary diagrams of $\mathbf{A}$ and $\mathbf{B}$, but in order to keep things in good order we adopt the convention that the new constant symbols involved in these diagrams shall satisfy the following constraints:

- The same new constant symbol is associate with both a_i and b_i for all $i \in I$.
- Apart from the constant symbols in the line above, the constant symbols associated with the elements of A and B are entirely different.

Let $\Delta_{\mathbf{A}}$ be the elementary diagram of $\mathbf{A}$ and $\Delta_{\mathbf{B}}$ be the elementary diagram of $\mathbf{B}$. Notice that the structure $\mathbf{D}$ is fully described by sentences that belong to both $\mathrm{Th}\langle \mathbf{A}, \bar{a}\rangle$ and $\mathrm{Th}\langle \mathbf{B}, \bar{b}\rangle$. That justifies the bottom of the diagram. To simplify things, we assume that $\bar{a} = \bar{b}$. We want a model of $\Delta_{\mathbf{A}} \cup \Delta_{\mathbf{B}}$. We'll use the Compactness Theorem.

Suppose no such model exists. Then there is some finite subset $\Gamma \subseteq \Delta_{\mathbf{B}}$ so that $\Delta_{\mathbf{A}} \cup \Gamma$ has no model. Γ is a finite set of sentences true in $\langle \mathbf{B}, B\rangle$. We can make the conjunction of those sentences. Let φ be this conjunction; that is, $\varphi(\bar{a}, \bar{d})$ is the conjunction of Γ, where $\bar{d}$ lists constant symbols for elements of B not listed in $\bar{a}$. Notice $\varphi(\bar{a}, \bar{d}) \in \Delta_{\mathbf{B}}$. So $\Delta_{\mathbf{A}} \cup \{\varphi(\bar{a}, \bar{d})\}$ has no model. Thus

$$\Delta_{\mathbf{A}} \models \neg\varphi(\bar{a}, \bar{d}) \text{ and so } \Delta_{\mathbf{A}} \models \forall \bar{y} \neg\varphi(\bar{a}, \bar{y})$$

since $\Delta_{\mathbf{A}}$ places no constraints on the constant symbols listed in $\bar{d}$. But $\forall \bar{y} \neg\varphi(\bar{a}, \bar{y})$ is just something written down in the language of $\langle \mathbf{A}, A\rangle$. So it is true in $\langle \mathbf{A}, A\rangle$. That is, it is true in $\langle \mathbf{A}, \bar{a}\rangle$. Since $\langle \mathbf{A}, \bar{a}\rangle$ and $\langle \mathbf{B}, \bar{a}\rangle$ are elementarily equivalent, it has to be true in $\langle \mathbf{B}, \bar{a}\rangle$:

$$\langle \mathbf{B}, \bar{a}\rangle \models \forall \bar{y} \neg\varphi(\bar{a}, \bar{y})$$

But we know that

$$\langle \mathbf{B}, \bar{a} \rangle \models \varphi(\bar{a}, \bar{d})$$

This is a contradiction. So our original supposition is wrong; hence there is a model of $\Delta_{\mathbf{A}} \cup \Delta_{\mathbf{B}}$. Let $\mathbf{C}'$ be a model of $\Delta_{\mathbf{A}} \cup \Delta_{\mathbf{B}}$ and let $\mathbf{C}$ be the reduct of $\mathbf{C}'$ back to the original signature. Now we know

- There is an elementary embedding g so that $g : \mathbf{B} \to \mathbf{C}$
- There is an elementary embedding h so that $h : \mathbf{A} \to \mathbf{C}$.

We can take h to be the inclusion map, so then $\mathbf{A} \preccurlyeq \mathbf{C}$. □

There is a kind of hidden assumption in the statement of this theorem. In the event that I is empty, that is no elements are listed in $\bar{a}$ and $\bar{b}$, and that the signature provides no constant symbols, then we find that the empty set would be the universe of $\mathbf{D}$. But we have insisted that all our structures have nonempty universes. So the formulation of the Elementary Amalgamation Theorem excludes this possibility. Nevertheless, the proof still works. What it proves is

The Elementary Joint Embedding Theorem. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures of the same signature. If* $\mathbf{A} \equiv \mathbf{B}$*, then there is* $\mathbf{C}$ *and some elementary embedding* g *so that*

- $\mathbf{A} \preccurlyeq \mathbf{C}$,
- $g : \mathbf{B} \to \mathbf{C}$.

5.2 Multiple Signatures: Joint Consistency, Interpolation, and Definability

The next theorem has been extracted from a proof Abraham Robinson gave in the 1950's for his Joint Consistency Theorem. It is put here in a stand-alone form because it has become a paradigm for other similar constructions and because some of the consequences of the Joint Consistency Theorem are more readily seen as consequences of this theorem. While I gave it another name, reflecting its proof rather than its statement, you should see this as a sort of two signature amalgamation theorem.

Robinson's Two Signature Chain Construction Theorem. *Let* L_0, L_1*, and* L_2 *be signatures where* L_0 *is the signature of symbols common to both* L_1 *and* L_2*. Let* $\mathbf{A}$ *be an* L_1*-structure and let* $\mathbf{B}$ *be an* L_2*-structure. Let* $\bar{b}$ *be a list of elements in both* A *and* B*. If* $\langle \mathbf{A} \restriction_{L_0}, \bar{b} \rangle \equiv \langle \mathbf{B} \restriction_{L_0}, \bar{b} \rangle$ *, then there is some* $L_1 \cup L_2$*-structure* $\mathbf{C}$ *so that* $\mathbf{A} \preccurlyeq C \restriction_{L_1}$ *and there is an* L_2*-elementary embedding* g *so that*

$$g : \mathbf{B} \hookrightarrow \mathbf{C} \restriction_{L_2}$$

where $g\bar{b} = \bar{b}$*.*

Proof. The proof is almost fully displayed in this illustration:

$$
\begin{tikzcd}[column sep=small]
 & \mathbf{A}_0 & \preccurlyeq & \mathbf{A}_1 \arrow[dd, hook, "f_1"'] & \preccurlyeq & \mathbf{A}_2 \arrow[dd, hook, "f_2"'] & \preccurlyeq & \cdots & \preccurlyeq & \mathbf{A}_\omega \arrow[dd, hook, shift right, "f_\omega"'] \\
\mathbf{D} \arrow[ur, phantom, sloped, "\preccurlyeq"] \arrow[dr, phantom, sloped, "\preccurlyeq"] & & & & & & & & & \\
 & \mathbf{B}_0 \arrow[uurr, "g_0"] & \preccurlyeq & \mathbf{B}_1 \arrow[uurr, "g_1"] & \preccurlyeq & \mathbf{B}_2 \arrow[uurr, "g_2"] & \preccurlyeq & \cdots & \preccurlyeq & \mathbf{B}_\omega \arrow[uu, hook, shift right, "g_\omega"']
\end{tikzcd}
$$

To start, we take $\mathbf{D}$ to be the L_0-substructure of $\mathbf{A}\restriction_{L_0}$ generated by the elements listed in the tuple $\bar{b}$. Notice that $\mathbf{D}$ is also an L_0 substructure of $\mathbf{B}\restriction_{L_0}$, since $\mathbf{A}\restriction_{L_0}\equiv\mathbf{B}\restriction_{L_0}$. We also put

$$\mathbf{A}_0 = \mathbf{A} \text{ and } \mathbf{B}_0 = \mathbf{B}.$$

To get $\mathbf{A}_1$ and g_0, we invoke a version of the Elementary Amalgamation Theorem. In this version, we use L_1 for the part involving $\mathbf{A}_0$ and $L_0 \cup B_0$ for the part involving $\mathbf{B}_0$. This means that $\mathbf{A}_0 \preccurlyeq \mathbf{A}_1$ (in the appropriate, namely L_1, sense) but that g_0 is an elementary embedding of $\langle \mathbf{B}_0\restriction_{L_0}, B_0\rangle$ into $\langle \mathbf{A}_1\restriction_{L_0}, g_0(B_0)\rangle$.

To get $\mathbf{B}_1$ and f_1, we invoke a version of the Elementary Amalgamation Theorem again. But in this version we use L_2 for the part involving $\mathbf{B}_0$ and L_0 for the part involving $\mathbf{A}_1$.

We continue in this way, taking a step for each natural number and alternating the use of L_1 and L_2. In the course of this construction, we build the L_1 elementary chain along the top of our illustration and the L_2 elementary chain along the bottom. The maps going up and down across the middle are elementary embeddings for L_0 enhanced by a growing supply of new constant symbols.

A simple but important property of our illustration is that it is a commutative diagram in the sense of category theory. That is that $f_{k+1}(g_k(b)) = b$ for all k and all $b \in B_k$ and likewise that $g_k(f_k(a)) = a$ for all $k > 0$ and all $a \in A_k$. The point is that these composite maps are elementary embeddings with respect to L_0 enhanced by enough new constants. For example, the formula $x \approx a$ is satisfied in $\mathbf{A}_k$ by a so the formula $x \approx a$ must be satisfied in $\mathbf{A}_k$ by $g_k(f_k(a))$. That is $g_k(f_k(a)) = a$.

At the limit, we just take unions, even for the functions. Ignoring all the new constants symbols accumulated in this construction, we find that f_ω is an L_0 isomorphism from $\mathbf{A}_\omega$ onto $\mathbf{B}_\omega$ and that g_ω is its inverse. To make the desired structure $\mathbf{C}$ we use these isomorphisms to impose on A_ω all the L_2 relations and operations not in L_0. For example, if R is a 3-place relation symbol of L_2 and $a, b, c \in A_\omega$ we put

$$(a, b, c) \in R^{\mathbf{C}} \text{ if and only if } (f_\omega(a), f_\omega(b), f_\omega(c)0 \in R^{\mathbf{B}_\omega}.$$

In this way, g_ω will become an L_2 elementary embedding of $\mathbf{B}_\omega$ into $\mathbf{C}\restriction_{L_2}$ and $\mathbf{A}_\omega = \mathbf{C}\restriction_{L_1}$. □

Robinson's Joint Consistency Theorem. *Let $L_0, L_1,$ and L_2 be signatures where L_0 is the signature of symbols common to both L_1 and L_2. Let T be a complete L_0 theory, let T_1 be an L_1 theory with $T_0 \subseteq T_1$, and let T_2 be an L_2 theory with $T_0 \subseteq T_2$. If both T_1 and T_2 have models, then $T_1 \cup T_2$ has a model.*

Proof. Let $\mathbf{A} \models T_1$ and let $\mathbf{B} \models T_2$. We will use Robinson's Two Signature Chain Construction Theorem. We take $\bar{b}$ of that theorem to be the empty tuple. We see that $\mathbf{A}\restriction_{L_0} \models T$ and that $\mathbf{B}\restriction_{L_0} \models T$. Because T is a complete L_0 theory, we deduce that $\mathbf{A}\restriction_{L_0}\equiv\mathbf{B}\restriction_{L_0}$. By the Chain Construction Theorem there is an $L_1 \cup L_2$ structure $\mathbf{C}$ so that $\mathbf{A} \preccurlyeq \mathbf{C}\restriction_{L_1}$ and an L_2-elementary embedding $g : \mathbf{B} \hookrightarrow \mathbf{C}\restriction_{L_2}$. But this means $\mathbf{C} \models T_1 \cup T_2$. □

The Two Signature Interpolation Theorem. *Let L_0, L_1, and L_2 be signatures where L_0 is the signature of symbols common to both L_1 and L_2. Let T_1 be an L_1 theory and let T_2 be an L_2 theory. If $T_1 \cup T_2$ has no model, then there is an L_0 sentence φ so that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$.*

Proof. Let $\Phi = T_1 \restriction_{L_0}$. By the Compactness Theorem, we only need to prove that $\Phi \cup T_2$ has no model. Because $T_1 \cup T_2$ has no models, there cannot be $\mathbf{A} \models T_1$ and $\mathbf{B} \models T_2$ so that $\mathbf{A} \restriction_{L_0} \equiv \mathbf{B} \restriction_{L_1}$, for otherwise the Chain Construction Theorem would provide a model of $T_1 \cup T_2$.

Now suppose, for contradiction, that $\mathbf{B} \models \Phi \cup T_2$. We know that $\mathrm{Th}\, \mathbf{B} \restriction_{L_0} \cup T_1$ has no model $\mathbf{A}$, as noted above. So by the Compactness Theorem there must be $\theta \in \mathrm{Th}\, \mathbf{B} \restriction_{L_0}$ so that $\{\theta\} \cup T_1$ has no model. But this means that $T_1 \models \neg\theta$ and so that $\neg\theta \in \Phi$. In this way, we see that $\mathbf{B}$ is a model of both θ and $\neg\theta$, which cannot be. $\square$

Craig's Interpolation Theorem. *Let ψ and θ be sentences so that $\psi \models \theta$. There is a sentence φ such that $\psi \models \varphi$ and $\varphi \models \theta$ and each relation symbol and each operation symbol that occurs in φ occurs also in both ψ and θ.*

This theorem, which illuminates the use of the word "interpolation", is just an instance of the Two Signature Interpolation Theorem in which T_1 is the set of all the logical consequences of ψ and T_2 is the set of all logical consequences of $\neg\theta$. William Craig gave the first demonstration of this theorem by means of proof theory.

The Preservation of Symbols Theorem. *Let L_0 and L_1 be signatures so that $L_0 \subseteq L_1$. Let T be an L_1 theory and let $\varphi(\bar{y})$ be an L_1 formula. The following statements are equivalent.*

(a) *For any models $\mathbf{A}$ and $\mathbf{B}$ of T such that $\mathbf{A} \restriction_{L_0} = \mathbf{B} \restriction_{L_0}$ and all tuples $\bar{a}$ of A we have*

$$\mathbf{A} \models \varphi[\bar{a}] \text{ if and only if } \mathbf{B} \models \varphi[\bar{a}]$$

(b) *There is an L_0 formula $\psi(\bar{y})$ so that*

$$T \models \forall \bar{y}\big(\varphi(\bar{y}) \leftrightarrow \psi(\bar{y})\big).$$

Proof. That (b) implies (a) is clear.

So let us suppose that (a) holds. Obtain L_1^+ by adding to L_1 a new constant symbol for each entry in $\bar{y}$ and take $\bar{c}$ to be the tuple of these new constant symbols. L_0^+ is obtained by adding the same constant symbols of L_0. Let $\Psi(\bar{c})$ be the set of all logical consequences of $T \cup \{\varphi(\bar{c})\}$ that are L_0^+-sentences.

Contention. $T \cup \Psi(\bar{c}) \models \varphi(\bar{c})$.

Indeed, let $\mathbf{A} \models T \cup \Psi(\bar{c})$. Consider the set $T \cup \mathrm{Th}\, \mathbf{A} \restriction_{L_0^+} \cup \{\varphi(\bar{c})\}$. This set must have a model $\mathbf{B}$, since otherwise we can invoke the Compactness Theorem to find a sentence $\gamma(\bar{c}) \in \mathrm{Th}\, \mathbf{A} \restriction_{L_0^+}$ such that $T \cup \{\varphi(\bar{c})\} \models \neg\gamma(\bar{c})$—this would put $\neg\gamma(\bar{c}) \in \Psi(\bar{c})$ and we would be confronted with $\mathbf{A} \models \gamma(\bar{c})$ as well as $\mathbf{A} \models \neg\gamma(\bar{c})$. Now we have $\mathbf{A} \restriction_{L_0^+} \equiv \mathbf{B} \restriction_{L_0^+}$. By the Chain Construction Theorem there are L_1^+ elementary extensions $\mathbf{A}_\omega$ of $\mathbf{A}$ and $\mathbf{B}_\omega$ of $\mathbf{B}$ that are L_0^+-isomorphic. By a bit a set-theoretic fiddling, we can have that $\mathbf{A}_\omega \restriction_{L_0^+} = \mathbf{B}_\omega \restriction_{L_0^+}$. In this way we discover that $\mathbf{A}_\omega \models \varphi(\bar{c})$ by the condition in (a). But this means $\mathbf{A} \models \varphi(\bar{c})$. So the contention is established.

By the Compactness Theorem there is a finite subset $\Psi'(\bar{c}) \subseteq \Psi(\bar{c})$ so that $T \cup \Psi'(\bar{c}) \models \varphi(\bar{c})$. Let $\psi(\bar{a})$ be the conjunction of the finitely many sentences in $\Psi'(\bar{c})$. Then we have

$$T \cup \{\varphi(\bar{c})\} \models \psi(\bar{c}) \text{ and } T \cup \{\psi(\bar{c})\} \models \varphi(\bar{c}).$$

But this gives

$$T \models \varphi(\bar{c}) \to \psi(\bar{c}) \text{ and } T \models \psi(\bar{c}) \to \varphi(\bar{c}).$$

This is the same as

$$T \models \varphi(\bar{c}) \leftrightarrow \psi(\bar{c}).$$

But T makes no mention of the constant symbols listed in the tuple $\bar{c}$ and so cannot constrain them in any way. This gives

$$T \models \forall \bar{y}\big(\varphi(\bar{y}) \leftrightarrow \psi(\bar{y})\big)$$

which is the desired conclusion (b). □

What condition (a) in this theorem says about the formula φ is, roughly, that in models of T whether φ is satified by a tuple depends only on how the symbols in L_0 are interpreted in the model. What condition (b) says is that φ is equivalent, in models of T to an L_0-formula.

To see this more sharply, we turn to another key idea in model theory, that of definability. Let us first consider the particular structure $\mathbf{Z} = \langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$, namely the ring of integers. It is a famous theorem of Lagrange that every nonnegative integer can be expressed as the sum of four squares. Another way to say this is

$$\{a \mid a \in \mathbb{Z} \text{ and } a \text{ is nonnegative}\} = \{a \mid \mathbf{Z} \models \exists y_0, \ldots, y_3(x \approx y_0^2 + \cdots + y_3^2)[a]\}.$$

In this case we say that the formula $\exists y_0, \ldots, y_3(x \approx y_0^2 + \cdots + y_3^2)$ defines the set of nonnegative integers in the structure $\mathbf{Z}$. The definable subsets of a structure are those that can be defined by some formula. In an entirely similar way, we arrive at the notion of definable relations (of whatever rank) and of definable operations.

Observe that given a formula, say with one free variable, then in any structure of the signature involved that formula determines a definite subset. The formula provides an explicit definition of the subset (relation, etc.) and we refer to such subsets (relations, etc.) as **explicitly definable**.

Let us restrict our attention to the models of some elementary theory T that has among its relation symbols the symbol R. Let L_0 be some signature that is included in the signature L_1 of T. We will say that T **implicitly defines**

R with respect to L_0 provide whenever $\mathbf{A} \models T$ and $\mathbf{B} \models T$ so that $\mathbf{A}\restriction_{L_0} = \mathbf{B}\restriction_{L_0}$, then $R^{\mathbf{A}} = R^{\mathbf{B}}$.

Now let $\varphi(\bar{y})$ be the formula $Ry_0y_1 \ldots y_{n-1}$. In this instance, the Preservation of Symbols Theorem asserts that T implicitly defines R with respect to L_0 if and only if some L_0-formula explicitly defines R in all models of T. The same conclusion holds for relation symbols. In this way, we have

Beth's Definability Theorem. *Let L_0 be a signature contained in the signature L_1. Let T be an L_1 theory. Any relation symbol or operation symbol of L_1 is implicitly definable by T with respect to L_0 if and only if there is an L_0-formula that defines it explicitly in every model of T.*

Evert Beth published his definability theorem in 1953. Beth spent the year 1951-52 in Berkeley as a research associate of Alfred Tarski. Abraham Robinson published his Joint Consistency Theorem in 1956 and William Craig published his Interpolation Theorem in 1957, the same year that saw the publication of the Tarski-Vaught paper. Both Robinson and Craig were motivated by Beth's Definability Theorem, giving new proofs of it.

Our approach follows closely an 1997 exposition of Wilfrid Hodges.

LECTURE 6

Sentences Preserved under the Formation of Substructures

6.1 Classes That Are Relativized Reducts of Elementary Classes

Suppose $\mathcal{K}$ is a class of structures of the same signature. Under what circumstances can we be sure that $\mathcal{K}$ is an elementary class? We know some necessary features of such classes. If they have infinite structures, or even arbitrarily large finite structures, then they must of structures of all sufficiently large infinite cardinalities—it follows from the Upward Löwenheim-Skolem-Tarski Theorem. Elementary classes must also be closed not only under isomorphism but under the more generous relation of elementary equivalence. Elementary classes must be closed under the ultraproduct construction as well. All these are necessary conditions.

What about sufficient conditions? This proves to be a difficult question, particularly if we insist on necessary and sufficient conditions. We will touch on this question later, but for now we want to present a useful sufficient condition.

The first step is to introduce a notion wider than that of elementary class.

Let us look first at an example. A group $\mathbf{G}$ has a **faithful n-dimensional representation** provided $\mathbf{G}$ can be embedded into the group of $n \times n$ invertible matrices over some field. Let $\mathcal{K}_n$ be the class of all groups with faithful n-dimensional representations. We would like to know whether there is some set Γ of sentences in the signature of group theory that axiomatizes this class $\mathcal{K}$. The basic difficulty is that the definition we gave talks about the existence of things like embeddings and some nice vector space over some field... none of the things are elements of the groups at hand. But we can deal with them if we are willing to contend with a richer signature. Here is what we could use:

- A one-place relation symbol G to name the universe of our group.
- A one-place relation symbol F to name the universe of a field.
- A one-place relation symbol V to name the universe of a vector space.
- Operation symbols to stand for the group operations on G.
- Operation symbols to stand for the ring operations on F.
- Operation symbols to stand for the vector space operations on V.
- A two-place operation symbol $\star$ to stand for the action of the group on the vector space.

Now we would have to write down in detail a whole bunch of elementary sentences to try to capture $\mathcal{K}_n$. I will only indicate what has to be said and leave the engaging task of actually writing out the elementary sentences to the graduate students.

- "G is closed under the group operations".
- "Under the group operations, G becomes a group" (This is basically a handful of equations.)
- "F is closed under the ring operations".
- "Under the ring operations, F becomes a field".
- "V is closed under the vector space operations." (Careful about scalar multiplication!)
- "Under the vector space operations, V becomes a vector space over the field."
- "The dimension of the vector space is n". (Remember enough about linear independence?)
- "For all $g \in G$ it turns out that $g \star v$ is a function from V to V".
- "For all $g \in G$ the function $g \star v$ is a linear operator on V".
- "The assignment $g \mapsto g \star v$ is a group embedding."

Let Λ be the set of all these elementary sentences.

Now the idea is that $\mathbf{G} \in \mathcal{K}_n$ if and only if there is some model $\mathbf{M}$ of Λ so that $\mathbf{G}$ is just the "group part" of $\mathbf{M}$. This not enough to say that $\mathcal{K}_n$ is an elementary class. On the other hand, it is closely tied to one. Classes like $\mathcal{K}_n$ that admit such descriptions are known as **relativized reducts** of elementary classes. That is, roughly speaking, $\mathcal{K}$ is such class provided in some richer signature there is an elementary class $\mathcal{L}$ so that $\mathcal{K}$ consists precisely of the "$\mathcal{K}$" parts of structures belonging to $\mathcal{L}$. More precisely a class $\mathcal{K}$ of structures of signature L is a **relativized reduct of an elementary class** provided there is a signature L^+ expanding L and a one-place relation symbol U of L^+ but not of L and a set Λ of L^+ sentences such that

- The set Λ includes sentences that assert the U is nonempty and is closed under all the operations symbolized in L.
- The class $\mathcal{K}$ consists precisely of those L-structures $\mathbf{A}$ obtained from L^+-structures $\mathbf{B}$ that are models of Λ by letting $U^{\mathbf{B}}$ be the universe of $\mathbf{A}$ and for each operation or relation symbol of L by letting its interpretation in $\mathbf{A}$ be the restriction of its interpretation in $\mathbf{B}$ to $U^{\mathbf{B}}$.

In the last constraint above, we denote by $\mathbf{B}_U$ the L-structure obtained from the L^+-structure $\mathbf{B} \models \Lambda$. We call this the **relativized reduct over U of $\mathbf{B}$ to L.**

6.2 The Łoś-Tarski Theorem

In the mid-1950's Jerzy Łoś and Alfred Tarski, working independently, found versions the following theorem.

The Łoś-Tarski Theorem. *Let $\mathcal{K}$ be a class of structures, all of the same signature. If $\mathcal{K}$ is a relativized reduct of an elementary class and $\mathcal{K}$ is closed under the formation of substructures, then $\mathcal{K}$ is an elementary class that can be axiomatized by some set of universal sentences.*

Proof. Let L be the signature of $\mathcal{K}$ and let Ψ be the set of all universal L-sentences true in $\mathcal{K}$. Let L^+ be an expansion of L and let Λ be a set of L^+-sentences that witness that $\mathcal{K}$ is a relativized reduct of an elementary class. Further let U be the new one-place relation symbol of L^+ that names the universes of structures in $\mathcal{K}$.

What we need is to prove that if $\mathbf{A} \models \Psi$, then $\mathbf{A} \in \mathcal{K}$. So suppose $\mathbf{A} \models \Psi$. Expand L^+ by adding new constant symbols c_a to name the elements of $a \in A$. Let Σ be the set of atomic and negated atomic sentences of the signature L expanded by the new constants that hold in $\langle \mathbf{A}, a\rangle_{a\in A}$. (Sets like Σ are referred to as diagrams. Elementary diagrams are richer sets of sentences.) Observe that $\mathbf{C} \models \Sigma$ if and only if $\mathbf{A}$ is embeddable into $\mathbf{C} \restriction_L$.

Claim. The set $\Sigma \cup \{Uc_a \mid a \in A\} \cup \Lambda$ has a model.

Let us suppose our claim is false. Then there is a finite subset $\Phi \subseteq \Sigma$ and a finite $F \subseteq A$ so that $\Phi \cup \{Uc_a \mid a \in F\} \cup \Lambda$ that has no model. Let φ be the conjunction of the finitely many sentences in Φ. Notice that no quantifiers occur in φ. By enlarging F if necessary we can suppose that if c_a occurs in φ then $a \in F$. Let $F = \{a_0, a_1, \ldots, a_{m-1}\}$. So we see

$$\Lambda \cup \{Uc_{a_0} \wedge Uc_{a_1} \wedge \cdots \wedge Uc_{a_{m-1}}\} \models \neg\varphi(c_{a_0}, \ldots, c_{a_{m-1}}).$$

But then

$$\Lambda \models (Uc_{a_0} \wedge Uc_{a_1} \wedge \cdots \wedge Uc_{a_{m-1}}) \to \neg\varphi(c_{a_0}, \ldots, c_{a_{m-1}}).$$

Since the new constants do not occur in Λ we have

$$\Lambda \models \forall \bar{y}\Big((Uy_0 \wedge \cdots \wedge Uy_{m-1}) \to \neg\varphi(y_0, \ldots, y_{m-1})\Big).$$

Now suppose $\mathbf{B} \models \Lambda$. Then we see that $\mathbf{B} \models \forall \bar{y}\Big((Uy_0 \wedge \cdots \wedge Uy_{m-1}) \to \neg\varphi(y_0, \ldots, y_{m-1})\Big)$. But this means $\mathbf{B}_U \models \forall \bar{y} \neg\varphi(\bar{y})$. Since every structure in $\mathcal{K}$ is some such $\mathbf{B}_U$, we find that $\mathcal{K} \models \forall \bar{y} \neg\varphi(\bar{y})$. This means $\forall \bar{y}\neg\varphi(\bar{y}) \in \Psi$. But $\mathbf{A} \models \Psi$. This means

$$\mathbf{A} \models \forall \bar{y}\neg\varphi(\bar{y}), \text{ and}$$
$$\mathbf{A} \models \varphi(\bar{y})[a_0, \ldots, a_{m-1}],$$

which is a contradiction. This establishes the claim.

So let $\mathbf{B} \models \Sigma \cup \{Uc_a \mid a \in A\} \cup \Lambda$. Now $\mathbf{B}$ is a structure for the signature L^+ expanded by the extra constant symbols. Let $\mathbf{B}_U^*$ be the relativized reduct of $\mathbf{B}$ to the signature L expanded by the new constant symbols and let $\mathbf{B}_U$ be the relativized reduct of $\mathbf{B}$ to L. Since $\mathbf{B}_U^* \models \Sigma$, we see that $\mathbf{A}$ is isomorphic to a substructure of $\mathbf{B}_U$. Since $\mathbf{B}_U \in \mathcal{K}$, we finally obtain $\mathbf{A} \in \mathcal{K}$. This is what we wanted. It means that $\mathcal{K} = \operatorname{Mod}\Psi$. $\square$

A sentence φ is said to be **preserved under the formation of substructures** provided that if $\mathbf{A} \models \varphi$ and $\mathbf{B}$ is a substructure of $\mathbf{A}$, then $\mathbf{B} \models B$.

Corollary 6.2.1 (Sometimes also called the Łoś-Tarski Theorem)**.** *A sentence is preserved under the formation of substructures if and only if it is logically equivalent for a universal sentence.*

Proof. It is pretty evident that universal sentences are preserved under the formation of substructures. For the converse, suppose φ is preserved under substrutures. Let $\mathcal{K} = \operatorname{Mod}\varphi$. This is an elementary class, so it is a relativized reduct of an elementary class. It is closed under the formation of substrutures. By the Łos-Tarski Theorem it is axiomatized by some set Ψ of universal sentences. This means, in particular, that $\Psi \models \varphi$. By the Compactness Theorem there a finite $\Psi' \subseteq \Psi$ so that $\Psi' \models \varphi$. Let ψ be the conjunction of the finitely many universal sentences in Ψ'. Since a conjunction of universal sentences is logically equivalent to a universal sentence, we may suppose that ψ is itself universal. Also notice $\psi \in \Psi$. Altogether, this means $\psi \models \varphi$ and $\varphi \models \psi$. So φ is logically equivalent to the universal sentence ψ. □

Let us return to our example. Let n be a positive natural number and let $\mathcal{K}_n$ be class of all groups that have faithful n-dimensional representations. We already convinced ourselves that $\mathcal{K}_n$ is a relativized reduct of an elementary class. But it is easy to see that any subgroup of a group with a faithful n-dimensional representation has itself a faithful n-dimensional representation. This means that $\mathcal{K}_n$ is actually an elementary class that can be axiomatized by a set of universal sentences.

Now a universal sentence has the form $\forall \bar{y}\theta(\bar{y})$ where no quantifiers occur in θ. Another way to view this, in fact our official view, is as $\neg\exists\bar{y}\neg\theta(\bar{y})$. That is, universal sentences are the negations of existential sentences: they assert that there are no such elements that satisfy some kind of quantifier-free condition. Roughly speaking, each universal sentence can be understood to forbid some kind of finite configuration from occurring in any model of the sentence. So one way to view the Łoś-Tarski theorem is as asserting that certain kinds of classes of structures can be characterized by forbidding certain finite configurations, perhaps infinitely many such configurations.

6.3 Characterizations of Universal Classes

Shortly after he had published his version of the Łoś-Tarski theorem, Łoś found the following strengthening of it.

The Łoś-Tarski Theorem, Ultraproduct Form. *Let $\mathcal{K}$ be a class of structures of the same signature. The class $\mathcal{K}$ is the class of all models of some set of universal sentences if and only if $\mathcal{K}$ is closed under the formation of ultraproducts and the formation of substructures.*

Proof. We already know that any elementary class is closed under the formation of ultraproducts and that any class axiomatized by a set of universal sentences must be closed under the formation of substructures. So we only concern ourselves with the converse. So we suppose that $\mathcal{K}$ is closed with respect to the formation of ultraproducts and substructures.

Let Ψ be the set of all universal sentences true in $\mathcal{K}$. All we need to do is show that if $\mathbf{A} \models \Psi$, then $\mathbf{A} \in \mathcal{K}$. So let $\mathbf{A} \models \Psi$. Expand the signature by adding a new constant c_a for every element of $a \in A$. Let $\Sigma_{\mathbf{A}}$ the diagram of $\mathbf{A}$.

For each finite subset $\Theta(\bar{c}) \subseteq \Sigma_{\mathbf{A}}$ we let $\theta(\bar{c})$ be its conjunction. Let φ_Θ be the sentence $\forall\bar{y}\neg\theta(\bar{y})$. We see $\mathbf{A} \not\models \varphi_\Theta$. This means $\varphi_\Theta \notin \Psi$. So pick $\mathbf{B}_\Theta \in \mathcal{K}$ so that $\mathbf{B}_\Theta \models \neg\varphi_\Theta$.

Untangling things a bit, we find that $\neg\varphi_\Theta$ is just another way to write $\exists\bar{y}\theta(\bar{y})$. This means that for each $a \in A$ so that the new constant symbol c_a occurs in $\Theta(\bar{c})$ we can pick $b_\Theta(a) \in B_\Theta$ so that expanding $\mathbf{B}_\Theta$ by letting the new constant symbols name these elements leads to a model of $\Theta(\bar{c})$. Let us pick, arbitrarily, a default element $\infty_\Theta \in B_\Theta$. We can then extend b_Θ to be defined on A by setting $b_\Theta(a) = \infty_\Theta$ in case c_a does not occur in $\Theta(\bar{c})$.

Let $I = \{\Theta(\bar{c}) \mid \Theta(\bar{c})$ is a finite subset of $\Sigma_\mathbf{A}\}$. For each $\Theta(\bar{c}) \in I$ let

$$E_\Theta = \{\Phi(\bar{c}) \mid \Phi(\bar{c}) \in I \text{ and } \Theta(\bar{c}) \subseteq \Phi(\bar{c})\}.$$

Just as in the ultraproduct proof of the Compactness Theorem (see Lecture 3.4), the collection $\mathcal{C} = \{E_\Theta \mid \Theta(\bar{c}) \in I\}$ has the finite intersection property. Let $\mathcal{U}$ be an ultrafilter on I that extends $\mathcal{C}$.

Put $\mathbf{B} = \prod_I \mathbf{B}_\Theta/\mathcal{U}$. We see that $\mathbf{B} \in \mathcal{K}$, since $\mathcal{K}$ is closed under the formation of ultraproducts. It remains to prove that $\mathbf{A}$ can be embedded into $\mathbf{B}$.

Define $h : A \to \prod_I B_\Theta/\mathcal{U}$ by

$$h(a) := \langle b_\Theta(a) \mid \Theta(\bar{c}) \in T\rangle/\mathcal{U},$$

for all $a \in A$. The function h is our desired embedding. We need to show that it is one-to-one, that it preserves the basic operations, and that it preserves the basic relations as well as the failures of the basic relations. That is,

$$\begin{aligned}
\mathbf{A} \models (\neg x_0 \approx x_1)[a, d] &\text{ implies } \mathbf{B} \models (\neg x_0 \approx x_1)[h(a), h(d)]\\
\mathbf{A} \models (Qx_0 \dots x_{r-1} \approx x_r)[a_0, \dots, a_r] &\text{ implies } \mathbf{B} \models (Qx_0 \dots x_{r-1} \approx x_r)[h(a_0), \dots, h(a_r)]\\
\mathbf{A} \models (Rx_0 \dots x_{r-1})[a_0, \dots, a_{r-1}] &\text{ implies } \mathbf{B} \models (Rx_0 \dots x_{r-1})[h(a_0), \dots, h(a_{r-1})]\\
\mathbf{A} \models (\neg Rx_0 \dots x_{r-1})[a_0, \dots, a_{r-1}] &\text{ implies } \mathbf{B} \models (\neg Rx_0 \dots x_{r-1})[h(a_0), \dots, h(a_{r-1})]
\end{aligned}$$

To say this another way, we need to show that h preserves the satisfaction of atomic formulas and their negations.

So let $\sigma(\bar{y})$ be any atomic formula. Taking into account the Fundamental Theorem of Ultraproducts, what we need to show is that for any assignment $\bar{a}$ from A,

$$\mathbf{A} \models \sigma(\bar{x})[\bar{a}] \text{ implies } \{\Theta(\bar{c}) \mid \mathbf{B}_\Theta \models \sigma(\bar{x})[b_\Theta(\bar{a})]\} \in \mathcal{U}$$

and

$$\mathbf{A} \models \neg\sigma(\bar{x})[\bar{a}] \text{ implies } \{\Theta(\bar{c}) \mid \mathbf{B}_\Theta \models \neg\sigma(\bar{x})[b_\Theta(\bar{a})]\} \in \mathcal{U}$$

Taking $\bar{a} = \langle a_0, a_1, a_2, \dots\rangle$, we see that $\mathbf{A} \models \sigma(\bar{x})[\bar{a}]$ means exactly the same as

$$\sigma(c_{a_0}, c_{a_1}, \dots) \in \Sigma_\mathbf{A}$$

and that $\mathbf{A} \models \neg\sigma(\bar{y})[\bar{a}]$ means $\neg\sigma(c_{a_0}, c_{a_1}, \dots) \in \Sigma_\mathbf{A}$. But recall that

$$E_{\{\sigma(c_{a_0}, c_{a_1}, \dots)\}} = \{\Theta(\bar{c}) \mid \Theta(\bar{c}) \in I \text{ and } \sigma(c_{a_0}, c_{a_1}, \dots) \in \Theta(\bar{c})\} \in \mathcal{U}.$$

and a similar statement holds with $\neg\sigma(c_{a_0}, c_{a_1}, \dots)$ in place of $\sigma(c_{a_0}, c_{a_1}, \dots)$.

We have taken particular care in the definition of b_Θ so that if $\sigma(c_{a_0}, c_{a_1}, \dots) \in \Theta(\bar{c})$, then $\mathbf{B}_\Theta \models \sigma(\bar{x})[b_\Theta(\bar{a})]$. This means that

$$\text{If } \sigma(c_{a_0}, c_{a_1}, \dots) \in \Sigma_\mathbf{A}, \text{ then } E_{\{\sigma(c_{a_0}, c_{a_1}, \dots)\}} \subseteq \{\Theta(\bar{c}) \mid \mathbf{B}_\Theta \models \sigma(\bar{x})[b_\Theta(\bar{a})]\}.$$

Of course, the same applies with $\neg\sigma$ in place of σ. This completes the proof. □

It is a straightforward exercise to establish that every relativized reduct of an elementary class is closed under the formation of ultraproducts. So the Łoś-Tarski Theorem is an immediate consequence of this ultraproduct version. Also, the ultraproduct version characterizes the elementary classes that can be axiomatized by sets of universal sentences. On the other hand, the version with relativized reducts of elementary classes is, in most cases, easier to apply.

Here is another characterization of universal classes. We call a class $\mathcal{K}$ of structures **locally finite** provided for all $\mathbf{A} \in \mathcal{K}$ we have that every finitely generated substructure of $\mathbf{A}$ is finite. The class $\mathcal{K}$ is **uniformly locally finite** provided there is a bounding function $g : \omega \to \omega$ such that whenever $\mathbf{A} \in \mathcal{K}$ and n is a natural number, then every substructure of $\mathbf{A}$ generated by n or fewer elements has cardinality less than $g(n)$.

Tarski's Characterization of Uniformly Locally Finite Universal Classes. *Let $\mathcal{K}$ be a uniformly locally finite class of structures all of the same finite signature. The following are equivalent:*

(a) $\mathcal{K}$ *is a universal class.*

(b) $\mathcal{K}$ *is closed under the formation of isomorphic images and under the formation of substructures and the limit of any system of structures belonging to $\mathcal{K}$ that is up-directed by the substructure relation must itself belong to $\mathcal{K}$.*

(c) $\mathcal{K}$ *is closed under the formation of isomorphic images and under the formation of substructures and for any structure $\mathbf{A}$ if every finitely generated substructure of $\mathbf{A}$ belongs to $\mathcal{K}$, then $\mathbf{A} \in \mathcal{K}$.*

Proof. **(a) implies (b)**
Let $\mathcal{K} = \operatorname{Mod}\Gamma$, where Γ is a set of universal sentences . That $\mathcal{K}$ is closed with under the formation of isomorphic images and substructures is evident. So let $\langle \mathbf{A}_i \mid i \in I \rangle$ be a system of structures, each belonging to $\mathcal{K}$, that is up-directed by the substructure relation. Let $\mathbf{A}$ be the limit of this system. To see that $\mathbf{A} \in \mathcal{K}$, let $\varphi \in \Gamma$ be chosen aribitrarily. So φ is $\forall \bar{y}\psi(\bar{y})$, where no quantifiers occur in ψ. We need to see that $\mathbf{A} \models \varphi$. To this end, let $\bar{a}$ be any assignment from A to the variables occurring in ψ. There are finitely many such variables, so using the up-directedness, we can pick $i \in I$ so that all the entries in $\bar{a}$ belong to A_i. Since $\varphi \in \Gamma$ and $\mathbf{A}_i \in \mathcal{K}$, we see that $\mathbf{A}_i \models \psi(\bar{y})[\bar{a}]$. But this means that $\mathbf{A} \models \psi(\bar{y})[\bar{a}]$ since ψ is quantifier-free and $\mathbf{A}_i$ is a substructure of $\mathbf{A}$. This means $\mathbf{A} \models \varphi$. So $\mathbf{A} \models \Gamma$, as desired.

(b) implies (c)
This follows immediately, since every structure is the limit of its up-directed system of finitely generated substructures.

(c) implies (a)
Let $\Gamma = \{\varphi \mid \varphi \text{ is a universal sentence and } \mathcal{K} \models \varphi\}$. It only remains to prove that every model of Γ belongs to $\mathcal{K}$. So let $\mathbf{A} \models \Gamma$. We will argue that every finitely generated substructure of $\mathbf{A}$ belongs to $\mathcal{K}$ and then appeal to (c). We need the following contention.

Contention. The class $\operatorname{Mod}\Gamma$ is uniformly locally finite.

Proof of the contention. To see this, first observe that, up to isomorphism, for each n there are only finitely many structures generated by n or fewer elements that are substructures of structures belonging to $\mathcal{K}$. This follows since $\mathcal{K}$ is uniformly locally finite and since the signature is finite. Let F_n be a set of distinct representatives of these isomorphism classes. Consider such a substructure $\mathbf{B} \in F_n$ and let $\{b_0, \dots, b_{n-1}\}$ be a generating set of $\mathbf{B}$. Now B has no more than $g(n)$ elements, say $c_0, \dots, c_{g(n)-1}$, repeating elements on this list as needed. For each $j < g(n)$ pick a term $t_j(x_0, \dots, x_{n-1})$ so that

$$t_j^{\mathbf{B}}(b_0, \dots, b_{n-1}) = c_j,$$

taking care to select the term x_i to be associated with the element b_i for all $i < n$. For each operation symbol Q, let its rank be r, we put into the set Φ all equations of the form

$$Qt_{j_0} \dots t_{j_{r-1}} \approx t_{j_r}$$

whenever $Q^{\mathbf{B}}(c_{j_0}, \dots, c_{j_{r-1}}) = c_{j_r}$. Also put into Φ all inequations $\neg t_j \approx t_k$ in the event that $c_j \neq c_k$, for $j, k < g(n)$. So Φ is a finite set of equations and inequations. Once an assignment of elements of some algebra has been made to the variables $x_0, \dots, x_{n-1}$, what Φ does is assure us that the set of values assigned to the representative terms is closed under the operations—as a consequence, the substructure generated by the assigned elements can have no more than $g(n)$ elements. Let $\varphi_{\mathbf{B},\bar{b}}$ denote the conjunction of the finite set Φ, where $\bar{b} = \langle b_0, \dots, b_{n-1} \rangle$. Now let ψ_n denote

$$\forall x_0, x_1, \dots, x_{n-1} \bigvee_{\substack{\mathbf{B} \in F_n \\ \bar{b} \text{ from } B}} \varphi_{\mathbf{B},\bar{b}}.$$

The sentence ψ_n asserts about a structure $\mathbf{A}$ that once any set of n or fewer elements of A is selected, then the substructure generated by these elements is isomorphic, as far as the basic operations are concerned, to one of the structures in F_n. Hence each ψ_n is true in $\mathcal{K}$. Since these sentences are universal, we conclude that $\psi_n \in \Gamma$ for each n. This, in turn, entails that $\operatorname{Mod}\Gamma$ also uniformly locally finite—the same bounding function will do. This finishes the proof of the contention. □

Returning to our proof that (c) implies (a), let $\mathbf{A} \models \Gamma$. We argue that every finitely generated substructure of $\mathbf{A}$ belongs to $\mathcal{K}$.

So let $\mathbf{B}$ be a substructure of $\mathbf{A}$ generated by a finite set. Because $\operatorname{Mod}\Gamma$ is locally finite, we know that $\mathbf{B}$ is finite. Expand the signature by adding a new constant symbol for each element of B. Let $\delta(\bar{c})$ be the sentence resulting from forming the conjunction of the diagram of $\mathbf{B}$, which is finite since $\mathbf{B}$ is finite of finite signature. Observe that $\mathbf{A} \models \exists \bar{y} \delta(\bar{y})$. This means that $\mathbf{A} \not\models \forall \bar{y} \neg \delta(\bar{y})$. So $\forall \bar{y} \neg \delta(\bar{y}) \notin \Gamma$. Since this sentence is universal, we can pick $\mathbf{C} \in \mathcal{K}$ so that $\mathbf{C} \models \neg \forall \bar{y} \neg \delta(\bar{y})$. This is the same as $\mathbf{C} \models \exists \bar{y} \delta(\bar{y})$. So we see that $\mathbf{C}$ can be expanded to a model of the diagram of $\mathbf{B}$. This means that $\mathbf{B}$ is isomorphic to a substructure of $\mathbf{C}$. According to (c), this entails that $\mathbf{B} \in \mathcal{K}$, just as desired. □

LECTURE 7

Denumerable Models of Complete Theories

7.1 Realizing and Omitting Types

Let a and b be distinct real numbers. Is there an elementary formula φ in the signature of ordered rings that distinguishes a from b in the ring of real numbers? That is, we are asking for a formula φ with one free variable so that

$$\langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle \models \varphi(x)[a] \text{ and } \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle \models \neg\varphi(x)[b].$$

Another way to frame this is to ask whether the set $\{\varphi(x) \mid \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle \models \varphi(x)[a]\}$ and the set $\{\varphi(x) \mid \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle \models \varphi(x)[b]\}$ are different. In this case, it is not so hard to come up with such a formula. It does no harm to suppose that $a < b$. Then there must be integers p and q, with q positive, so that $a < \frac{p}{q} < b$. This is the same as $aq < p < bq$. Now, the integers p and q can be represented by terms with no variables. (Recall $-3 = -(1+1+1)$.) So take $\varphi(x)$ to be $xq < p$. This formula does the job. In fact, this is essentially the device invented by Eudoxus, a contemporary of Plato, to rescue geometry from the challenge represented by the irrationality of lengths like $\sqrt{2}$. The set

$$\{\varphi(x) \mid \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle \models \varphi(x)[a]\}$$

consists of all those attributes of the real number a that can be expressed in our elementary language. It this case, we have enough expressive power to completely determine a. Taking a viewpoint from analysis, we see that that a is the greatest lower bound of all the rational number properly larger than a.

Sets like $\Gamma(x) = \{\varphi(x) \mid \langle \mathbb{R}, +, \cdot, -, 0, 1 \rangle \models \varphi(x)[a]\}$ are called complete 1-types. This set of formulas has the following properties:

(a) There is a structure $\mathbf{A}$ and an element $a \in A$ so that for each $\varphi(x) \in \Gamma(x)$ we have $\mathbf{A} \models \varphi(x)[a]$, and

(b) Given any formula $\psi(x)$ in at most 1 free variable either $\psi(x) \in \Gamma$ or $\neg\psi(x) \in \Gamma(x)$.

In our example, $\mathbf{A} = \langle \mathbb{R}, +, \cdot, -, 0, 1, < \rangle$. Any set of formulas of a given signature that has the properties itemized above is called a **complete 1-type**. For any natural number n, the notion of an **complete n-type** is like the notion of a complete 1-type, but the formulas are permitted to have at most n-free variables. Γ is called an **n-type** provided it has property (a) adjusted for n variables. Observe that any n-type is contained in a complete n-type.

Let $\Gamma(x_0, \ldots, x_{n-1})$ be an n-type (or more generally, just a set of formulas with free variables among $x_0, \ldots, x_{n-1}$). Let $\mathbf{A}$ be a structure of the same signature. We say that **$\mathbf{A}$ realizes** $\Gamma(x_0, \ldots, x_{n-1})$ provided there are $a_0, \ldots, a_{n-1} \in A$ so that for all $\varphi(\bar{x}) \in \Gamma(\bar{x})$ we have $\mathbf{A} \models \varphi(\bar{x})[\bar{a}]$. On the other hand, we say that **$\mathbf{A}$ omits** $\Gamma(\bar{x})$ if and only if $\mathbf{A}$ does not realize $\Gamma(\bar{x})$. To tell it in a fabulous way, one can imagine writing in English a very detailed description of a unicorn. This description amounts to an English language variant of a 1-type, the type of unicorns. The world we live in probably omits this type, but in some more fabulous world the type of unicorns may be realized.

Let T be an elementary theory. We say that $\Gamma(\bar{x})$ is an n**-type of** T if $\Gamma(\bar{x})$ is realized in some model of T. It follows from the Compactness Theorem that $\Gamma(\bar{x})$ is an n-type of T if and only if every finite subset of $\Gamma(\bar{x})$ is an n-type of T. There is one attribute that an n-type can have that ensures that it is realized in every model of T: namely, if there is a formula $\varphi(\bar{x})$ so that

$$\begin{aligned}&T \cup \{\exists \bar{x} \varphi(\bar{x})\} \text{ has a model and}\\ &T \models \forall \bar{x}\Big[\varphi(\bar{x}) \to \gamma(\bar{x})\Big] \text{ for all } \gamma(\bar{x}) \in \Gamma(\bar{x}).\end{aligned}$$

Such a formula $\varphi(\bar{x})$ is said to **support** $\Gamma(\bar{x})$ over T or to be a **generator** of $\Gamma(\bar{x})$ with respect to T, or sometimes to **isolate** $\Gamma(\bar{x})$ with respect to T. In the event that $\Gamma(\bar{x})$ is a complete n-type and T has a model, then $\varphi(\bar{x})$ will belong to $\Gamma(\bar{x})$. Realizing this one formula from the n-type is enough to realize the whole n-type. For countable signatures there is a converse.

The Omitting Types Theorem. *Let L be a countable signature. Let $\Phi_0, \Phi_1, \ldots$ be a countable list of types. Let T be a set of L-sentences so that each Φ_i is unsupported over T and so that T has a model. Then T has a countable model which omits all the types Φ_0, Φ_1, $\Phi_2, \ldots$.*

Proof. As with the proof of the Compactness Theorem, we need to construct a structure satisfying a list of constraints. Our proof will be a modification of Henkin's proof of the Compactness Theorem. We begin by expanding the signature with a countably infinite list $c_0, c_1, c_2, \ldots$ of new constant symbols. These symbols will name the elements of the structure we are going to assemble. As with the proof of the Compactness Theorem we will build a theory in our expanded signature that will turn out to be complete (this is one constraint we will have the fulfill) and have enough witnesses of existential sentences (this is a second constraint). But we have a countably infinite list of other constraints, namely for each natural number m we have to ensure that Φ_m is not realized.

Our construction will proceed by stages, one stage for each natural number. We will arrange matters in such a way that we have the opportunity to satisfy each of our constraints infinitely often. To this end, we partition the set of natural numbers into a countably infinite collection of countably infinite pieces. Let us call one piece C for "completeness", one piece E for "existential", and label the rest as $F_0, F_1, F_2, \ldots$ to correspond with the types $\Phi_0, \Phi_1, \Phi_2, \ldots$.

Our construction starts with T_0 being the empty set. At stage i of the construction we will produce a finite set T_{i+1} of sentences so that, among other things, $T_j \subseteq T_i$ for all $j < i$ and so that $T \cup T_{i+1}$ has a model. After all the stages have been completed we will form $T_\omega = T \cup \bigcup_{i \in \omega} T_i$. At that point we will produce a countable structure $\mathbf{A}$ just as we did in the proof of the Compactness Theorem.

We contract out this construction to countably many experts, each specialized in one of our constraints. Before our construction can get underway, each expert must do some preliminary work.

The Preliminary Work of the Completion Expert
She must arrange all the sentences of our expanded signature in a list $\theta_0, \theta_1, \theta_2, \ldots$.

The Preliminary Work of the Existential Expert
She must arrange all the formulas with one free variable in a list $\psi_0, \psi_1, \psi_2, \ldots$ taking care that each such formula appears on her list infinitely often.

The Preliminary Work of each Type Omitting Expert
Let us suppose that Φ_m is an n-type. The expert in charge of omitting Φ_m must arrange all the n-tuples of distinct new constant symbols in a list $\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots$.

What happens at stage i of the construction is that T_i is placed into the hands of one of the experts according to whether i belongs to her block of our partition of the natural numbers. So, for example, if $i \in C$, then the construction passes into the hands of the completion expert, whereas if $i \in F_m$, then it passes into the hands of the expert in charge of omitting Φ_m. In this way, each expert gets her hands on the construction infinitely often.

Here is what each expert does, each time the construction comes to her.

The Action of the Completion Expert
The completion expert takes the earliest sentence θ on her list that she has not yet marked off. If $T \cup T_i \cup \{\theta\}$ has a model then she sets $T_{i+1} = T_i \cup \{\theta\}$ and marks off θ on her list. Otherwise, she puts $T_{i+1} = T_i \cup \{\neg\theta\}$ and marks off θ from her list. At the end of her work at this stage, she knows that $T \cup T_{i+1}$ has a model.

The Action of the Existential Expert
The existential expert takes the earliest formula $\psi(y)$ on her list that she has not yet marked off. She puts $T_{i+1} = T_i \cup \{\exists y \psi(y) \to \psi(c)\}$, where c a new constant symbol that does not occur in T_i. She then marks off that one occurrence of $\psi(y)$ on her list. At the end of her work at this stage, she knows that $T \cup T_{i+1}$ has a model and that at some future stage she will be dealing with the same formula $\psi(y)$ again.

The Action of an Type Omitting Expert
When the work comes to the expert in charge of omitting Φ_m, she takes the earliest n-tuple $\bar{c}$ from her list that she has not already marked off. Some or all of the constant symbols in $\bar{c}$ may occur in the finite set T_i. In addition, there may be others of the new constant symbols that occur in T_i. Let $\bar{d}$ be a finite tuple listing these other new constant symbols. Let $\tau(\bar{c}, \bar{d})$ be the conjunction of T_i. Now $T \cup \{\exists\bar{x}\exists\bar{y}\tau(\bar{x}, \bar{y})\}$ has a model. But our expert knows that Φ_m is not supported over T. This means that there is $\varphi(\bar{x}) \in \Phi_m$ so that $T \not\models \forall\bar{x}[\exists\bar{y}\tau(\bar{x}, \bar{y}) \to \varphi(\bar{x})]$. By logical equivalence, this is the same as $T \not\models \forall\bar{x}\forall\bar{y}\Big[\tau(\bar{x}, \bar{y}) \to \varphi(\bar{x})$. The new constant symbols listed in $\bar{c}$ and $\bar{d}$ do not occur in T. This means that $T \not\models \tau(\bar{c}, \bar{d}) \to \varphi(\bar{c})$ or what is the same $T \cup \{\tau(\bar{c}, \bar{d})\} \not\models \varphi(\bar{c})$. So the expert in charge on omitting Φ_m puts $T_{i+1} = T_i \cup \{\neg\varphi(\bar{c})\}$. She then marks off $\bar{c}$ from her list. At the end of her work at this stage, she knows that $T \cup T_{i+1}$ has a model and that in any model of it the tuple of elements named by $\bar{c}$ cannot realize the type Φ_m.

So we now have T_ω. We see that it has the following properties

(a) $T \subseteq T_\omega$.

(b) T_ω is a complete theory in the expanded signature, due to the diligent work of the completion expert.

(c) Sentences of the form $\exists x \psi(x) \to \psi(d)$ belong to T_ω for all formulas ψ of one free variable and for infinitely many new constatnt symbols d, due the diligent work of the existential expert.

(d) For each appropriate natural number n, each n-tuple $\bar{c}$ of distinct new constant symbols and each m there is $\varphi(\bar{x}) \in \Phi_m$ so that $\neg\varphi(\bar{c}) \in T_\omega$, due to the diligent work of the type omitting experts.

Now we construct the structure $\mathbf{A}'$ exactly as we did in the proof of the Compactness Theorem. The elements of A' are equivalence classes of the new constant symbols. In this way every element of A' is named by a constant symbol, indeed by infinitely many distinct constant symbols since we made the existential expert work harder than she had to in the proof of the Compactness Theorem. We also see that A' is countable.

Just as in the proof of the Compactness Theorem we find

$$\mathbf{A}' \models \gamma \text{ if and only if } \gamma \in T_\omega$$

for all sentences γ of our expanded language. Obtain $\mathbf{A}$ as the reduct of $\mathbf{A}'$ resulting from ignoring the new constant symbols. We see that $\mathbf{A}$ is a countable model of T. To see that it omits the type Φ_m of formulas on n free variables, let $\bar{a}$ be any n-tuple of elements of A. Because the existential expert worked hard enough to ensure that every element of A is named by infinitely many of the new constant symbols, there is an n-tuple $\bar{c}$ of distinct new constant symbols so that $\bar{c}$ is a tuple of names for $\bar{a}$ in $\mathbf{A}'$. The type omitting expert has inserted into T_ω a sentence $\neg\varphi(\bar{c})$ so that $\varphi(\bar{x}) \in \Phi_m$. It follows that $\mathbf{A}' \models \neg\varphi(\bar{c})$. But this is the same as $\mathbf{A} \models \neg\varphi(\bar{x})[\bar{a}]$. So we see that Φ_m cannot be realized by $\bar{a}$. Since $\bar{a}$ was chosen arbtrarily, we find that Φ_m cannot be realized by any n-tuple from A. So $\mathbf{A}$ omits Φ_m, as desired. $\square$

The Omitting Types Theorem has its roots in the independent work of Leon Henkin and Steven Orey in the early 1950's on ω-logic. This line of work was advanced further by Andrzej Grzegorczyk, Andrzej Mostowski, and Czeslaw Ryll-Nardzewski, where a more explicit version of the theorem can be found. The form of the theorem we have used is apparently due to Andrzej Ehrenfeucht around 1955—Ehrenfeucht never published his version.

7.2 Meager Structures

Given a structure $\mathbf{A}$ and an n-type $\Gamma(\bar{x})$ it might be that $\Gamma(\bar{x})$ is omitted by $\mathbf{A}$ but is realized in some other model of The $\operatorname{Th}\mathbf{A}$. We can obtain a deeper understanding of the structure $\mathbf{A}$ by investigating which types are realized in $\mathbf{A}$. Some structures are meager in this regard—the only types they realize are the supported types, which must be realized in every model of $\operatorname{Th}\mathbf{A}$. We consider such meager structures in this section, while to next section deals with structures that realizes as many types as possible.

We say a structure $\mathbf{A}$ is **atomic** provided that every complete type that is realized in $\mathbf{A}$ is supported with respect to $\operatorname{Th}\mathbf{A}$.

The Uniqueness Theorem for Countable Atomic Structures. *Let $\mathbf{A}$ and $\mathbf{B}$ be countable atomic structures of the same signature. If $\mathbf{A} \equiv \mathbf{B}$, then $\mathbf{A} \cong \mathbf{B}$.*

Proof. We have already observed much earlier in these notes that $\mathbf{A} \equiv \mathbf{B}$ implies $\mathbf{A} \cong \mathbf{B}$, when $\mathbf{A}$ (and therefore $\mathbf{B}$) is finite. So here we consider only the case that A and B are countably infinite. Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be lists of the the elements of A and B respectively.

We use a back-and-forth construction to make the desired isomorphism, which will be a set $\{(e_k, d_k) \mid k \in \omega\}$ of ordered pairs. We do this in stages. At the even stages we let e_k be the next element of A [so of course $e_0 = a_0$] and go forth to B to find a proper element d_k. At the odd stages we let d_k be the next element of B and go back to A to find an appropriate e_k.

Expand the signature by adding new constant symbols $c_0, c_1, \ldots$.

So at stage k we have already $(e_0, d_0), \ldots, (e_{k-1}, d_{k-1})$ such that

$$\langle \mathbf{A}, e_0, \ldots, e_{k-1} \rangle \equiv \langle \mathbf{B}, d_0, \ldots, d_{k-1} \rangle.$$

Let us suppose that k is odd. Pick d_k to be the earliest element on the list of elements of B that does not occur among $d_0, \ldots, d_{k-1}$. Let $\Gamma(x_0, \ldots, x_k)$ be the complete $k+1$ type of all formulas realized by $\langle d_0, \ldots, d_k \rangle$. Because $\mathbf{B}$ is atomic, there is a formula $\theta(x_0, \ldots, x_k)$ that supports $\Gamma(x_0, \ldots, x_k)$ over $\operatorname{Th} \mathbf{B}$. Now

$$\langle B, d_0, \ldots, d_{k-1} \rangle \models \exists x_k \theta(c_0, \ldots, c_{k-1}, x_k).$$

Since $\langle \mathbf{A}, e_0, \ldots, e_{k-1} \rangle \equiv \langle B, d_0, \ldots, d_{k-1} \rangle$, we see

$$\langle A, e_0, \ldots, e_{k-1} \rangle \models \exists x_k \theta(c_0, \ldots, c_{k-1}, x_k).$$

So pick $e_k \in A$ so that $\mathbf{A} \models \theta(x_0, \ldots, x_k)[e_0, \ldots, e_k]$. Since $\theta(x_0, \ldots, x_k)$ supports $\Gamma(x_0, \ldots, x_k)$ over $\operatorname{Th} \mathbf{A} = \operatorname{Th} \mathbf{B}$, we see that $\langle e_0, \ldots, e_k \rangle$ realizes $\Gamma(x_0, \ldots, x_k)$. It follows that

$$\langle \mathbf{A}, e_0, \ldots, e_k \rangle \equiv \langle \mathbf{B}, d_0, \ldots, d_k \rangle.$$

So after countably steps we have $f = \{(e_k, d_k) \mid k \in \omega\}$ where $e_0, e_1, \ldots$ lists all the elements of A and $d_0, d_1, \ldots$ list all the elements of B. Moreover, since

$$\langle \mathbf{A}, e_0, e_1, \ldots \rangle \equiv \langle \mathbf{B}, d_0, d_1 \ldots \rangle,$$

we conclude that f is an isomorphism. □

Under what circumstances will a (complete) theory T have a countable atomic model? Consider first the kinds of formulas that can support an n-type over T. We call a formula $\theta(\bar{x})$ **complete with respect to** T provided

- $\theta(\bar{x})$ is realized in some model of T, and
- for every formula $\varphi(\bar{x})$ of the signature either

$$T \models \forall \bar{x}[\theta(\bar{x}) \to \varphi(\bar{x})] \text{ or } T \models \forall \bar{x}[\theta(\bar{x}) \to \neg\varphi(\bar{x})].$$

The complete formulas are the ones that can support types over T. Further, let us say a formula $\varphi(\bar{x})$ is **completeable with respect to** T provided

$$T \models \forall \bar{x}[\theta(\bar{x}) \to \varphi(\bar{x})]$$

for some formula $\theta(\bar{x})$ that is complete with respect to T.

The Existence Theorem for Atomic Models of Complete Theories. *Let T be a complete theory of a countable signature. T has a countable atomic model if and only if every $\varphi(\bar{x})$ that is realized in some model of T is completeable with respect to T.*

Proof. First suppose that T has a countable atomic model $\mathbf{A}$ and that $\varphi(\bar{x})$ is realized in some model of T. Then $\exists\bar{x}\varphi(\bar{x})$ is true in some model of T. Because T is complete, we see that $\exists\bar{x}\varphi(\bar{x}) \in T$ and so $\mathbf{A} \models \exists\bar{x}\varphi(\bar{x})$. So pick an n-tuple $\bar{a}$ of elements of A that realizes $\varphi(\bar{x})$. Let $\Gamma(\bar{x})$ be the complete n-type realized by $\bar{a}$ in $\mathbf{A}$. Since $\mathbf{A}$ is atomic, there is a formula $\theta(\bar{x})$ that supports $\Gamma(\bar{x})$. So $\theta(\bar{x})$ is complete with respect to T and $T \models \forall\bar{x}[\theta(\bar{x}) \to \varphi(\bar{x})]$ since $\varphi(\bar{x}) \in \Gamma(\bar{x})$. This means $\varphi(\bar{x})$ is completeable with respect to T.

For the converse, for each n take $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$ and let

$$\Gamma_n = \{\neg\theta(\bar{x}) \mid \theta(\bar{x}) \text{ is a complete formula with respect to } T\ \}.$$

Contention. Γ_n is unsupported with respect to T.

Let $\varphi(\bar{x})$ be any formula that is realized in some model of T. Then $\varphi(\bar{x})$ is completeable. So let $\theta(\bar{x})$ be complete with respect to T so that

$$T \models \forall\bar{x}[\theta(\bar{x}) \to \varphi(\bar{x})]/$$

We see that $\theta(\bar{x}) \wedge \varphi(\bar{x})$ is realized in some model of T—indeed, in any model in which $\theta(\bar{x})$ is realized. But this means that $T \not\models \forall\bar{x}[\varphi(\bar{x}) \to \neg\theta(\bar{x})]$. So $\varphi(\bar{x})$ cannot support Γ_n over T. This establishes the contention.

So by the Omitting Types Theorem, T has a countable model $\mathbf{A}$ that omits each Γ_n. Consider any n-tuple $\bar{a}$ of elements of A. Since $\bar{a}$ cannot realize Γ_n, we can pick $\neg\theta\bar{x}) \in \Gamma_n$ so that $\mathbf{A} \models \theta(\bar{x})[\bar{a}]$. Now $\theta(\bar{x})$ is complete with respect to T. So it must support a complete n-type and that n-type must be the complete n-type of all formulas realized by $\bar{a}$ in $\mathbf{A}$. This means that every complete type realized in $\mathbf{A}$ is supported. This means that $\mathbf{A}$ is atomic, as desired. □

There is another condition that ensures the existence of a countable atomic model.

Let T be an elementary theory and let n be a natural number. We denote that set of all complete n-type of T by $S_n(T)$. For any cardinal κ we let $\mu(\kappa, T)$ denote the number of models of T of cardinality κ, counted up to isomorphism. For countable signatures, $|S_n(T)| \leq \aleph_0 \cdot \mu(\aleph_0, T)$, since each complete n-type of T is the type of some n-tuple of elements of some model $\mathbf{B} \models T$. By the Downward-Löwenheim-Skolem-Tarski Theorem, we can assume that $\mathbf{B}$ is countable. So there are only countably many n-tuples of elements of B and only $\mu(\aleph_0, T)$ many choices of $\mathbf{B}$.

Second Existence Theorem for Atomic Models. *Let T be a complete theory in a countable signature. If $|S_n(T)| < 2^{\aleph_0}$ for all n, then T has a countable atomic model.*

Corollary 7.2.1. *Let T be a complete theory in a countable signature. If $\mu(\aleph_0, T) < 2^{\aleph_0}$, then T has a countable atomic model.* □

Proof. We prove the contrapostive of the theorem. That is, we assume that T has no countable atomic model, and deduce that the number of complete n-type is $2^{\aleph_0}$, for some n.

Since T has no countable atomic model by the Existence Theorem for Atomic Models, we have a formula $\varphi(x_0, \ldots, x_{n-1})$ that is realized in some model of T and that is not completeable. So we can pick a formula $\sigma_0(\bar{x})$ so that

$$T \not\models \forall \bar{x}[\varphi(\bar{x}) \to \sigma_0(barx)] \text{ and } T \not\models \forall \bar{x}[\varphi(\bar{x}) \to \neg\sigma_0(\bar{x})].$$

This is equivalent to

$\varphi(\bar{x}) \wedge \sigma_0(\bar{x})$ is realized in some model of T, and
$\varphi(\bar{x}) \wedge \neg\sigma_)(\bar{x})$ is realized in some model of T

Take $\varphi_0(\bar{x})$ to be $\varphi(\bar{x}) \wedge \sigma_0(\bar{x})$ and take $\varphi_1(\bar{x})$ to be $\varphi(\bar{x}) \wedge \neg\sigma_0(\bar{x})$. So we have

$$T \models \forall \bar{x}[\varphi_0(\bar{x}) \to \varphi(\bar{x}) \text{ and } T \models \forall \bar{x}[\varphi_1(\bar{x}) \to \varphi(\bar{x}),$$

and $\varphi_0(\bar{x}) \wedge \varphi_1(\bar{x})$ is not realized in any structure. Moreover, we see that both $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ are not completeable.

Now we can repeat this process using $\varphi_0(\bar{x})$ in place of $\varphi(\bar{x})$ to find further formulas $\varphi_{00}(\bar{x})$ and $\varphi_{01}(\bar{x})$ so that

$$T \models \forall \bar{x}[\varphi_{00}(\bar{x}) \to \varphi_0(\bar{x}) \text{ and } T \models \forall \bar{x}[\varphi_{01}(\bar{x}) \to \varphi_0(\bar{x}),$$

and $\varphi_{00}(\bar{x}) \wedge \varphi_{01}(\bar{x})$ is not realized in any structure. Moreover, we see that both $\varphi_{00}(\bar{x})$ and $\varphi_{01}(\bar{x})$ are not completeable.

Of course, in the same way we can repeat our process with $\varphi_1(\bar{x})$ in place of $\varphi(\bar{x})$ to obtain formulas $\varphi_{10}(\bar{x})$ and $\varphi_{11}(\bar{x})$ with entirely similar properties.

What we build in this way is a tree of formulas. The first three stages are illustrated below.

$$\begin{tikzcd}
& & \varphi_{00}(\bar{x}) \arrow[dl] \\
& \varphi_0(\bar{x}) \arrow[ddl] & \\
& & \varphi_{01}(\bar{x}) \arrow[ul] \\
\varphi(\bar{x}) & & \\
& & \varphi_{10}(\bar{x}) \arrow[dl] \\
& \varphi_1(\bar{x}) \arrow[uul] & \\
& & \varphi_{11}(\bar{x}) \arrow[ul]
\end{tikzcd}$$

The entire construction through denumerably many stages produces a binary tree of formulas. This tree has $2^{\aleph_0}$ branches (think of the real numbers between 0 and 1 represented in binary notation). Each branch gives us a countably infinite set of formulas and this set of formulas is realized in some model of T. This follows, by way of the Compactness Theorem, from the fact that each finite subset is realized. So each branch is contained in a complete n-type. On the other hand, distinct branches are incompatible: No structure can realize all the formulas on two different branches. So the complete n-types of different branches must be different. This means that there are $2^{\aleph_0}$ complete n-types. So $|S_n(T)| = 2^{\aleph_0}$, as desired. □

An atomic structure $\mathbf{A}$ is meager in the sense that the only n-types that are realized in $\mathbf{A}$ are the supported n-types, which must be realized in all models of $\operatorname{Th}\mathbf{A}$. There is another way to view the smallness of atomic structures. Let T be an elementary theory. A model $\mathbf{A} \models T$ is said to be **elementarily prime with respect to** T provided $\mathbf{A} \models T$ and $\mathbf{A}$ can be elementarily embedded into every model of T.

The Countable Atomic Equals Prime Theorem. *Let T be a complete theory of countable signature. For any structure* $\mathbf{A}$*, the following are equivalent.*

(a) $\mathbf{A}$ *is an elementarily prime model of T.*

(b) $\mathbf{A}$ *is an countable atomic model of T.*

Proof. Consider first the case when $\mathbf{A}$ is finite. Since T is complete, we know that $T = \mathrm{Th}\,\mathbf{A}$. So we know that, up to isomorphism, $\mathbf{A}$ is the only model of T. We leave it in the hands of the hungry graduate students to prove that every finite structure is both atomic and elementarily prime.

Now let us suppose that $\mathbf{A}$ is an elementarily prime model of T. According to the Downward Löwemheim-Skolem-Tarski Theorem, $\mathbf{A}$ is countable. We have to show that every complete type realized in $\mathbf{A}$ is supported. So let $\Gamma(\bar{x})$ be an complete n-type that is not supported with respect to $T = \mathrm{Th}\,\mathbf{A}$. By the Omitting Types Theorem there is a countable model $\mathbf{B}$ of T that omits $\Gamma(\bar{x})$. Since $\mathbf{A}$ is elementarily prime we can suppose without harm that $\mathbf{A} \preccurlyeq \mathbf{B}$. Let $\bar{a}$ be an arbitrary n-tuple of elements of A. Since $\bar{a}$ cannot realize $\Gamma(\bar{x})$ in $\mathbf{B}$, we can pick $\gamma(\bar{x}) \in \Gamma(\bar{x})$ so that $\mathbf{B} \models \neg\gamma(\bar{x})[\bar{a}]$. But since $\mathbf{A} \preccurlyeq \mathbf{B}$, se have that $\mathbf{A} \models \neg\gamma(\bar{x})[\bar{a}]$. So $\bar{a}$ does not realize $\Gamma(\bar{x})$ over $\mathbf{A}$. As $\bar{a}$ was an arbitrary n-tuple from A, we see that $\mathbf{A}$ omits $\Gamma(\bar{x})$. But $\Gamma(\bar{x})$ was an arbitrary complete n-type unsupported with respect to T, so $\mathbf{A}$ is a countable atomic model of T.

For the converse, we use the "forth" part of the "back-and-forth" construction from the proof of the Uniqueness Theorem for Countable Atomic Structures. Let $\mathbf{A}$ be a countable atomic model of T and let $\mathbf{B}$ be any model of T. The key observation here, as in that proof, is that the type of any n-tuple over $\mathbf{A}$ is supported and that supported types are realized in every model of T. We leave the details to the industry of the graduate students. □

7.3 Countably Saturated Structures

The meager models of a complete theory were the ones that realized only the complete types that had to be realized in any model of the theory. In contrast, the countably saturated models will realize has many complete types are possible.

Recall that for any structure $\mathbf{A}$ and any subset $D \subseteq A$, we used $\langle \mathbf{A}, D\rangle$ to denote the structure obtained by expanding $\mathbf{A}$ by taking each element of the set D as a new distinguished element—so the signature has been expanded by adding new constant symbols to name the elements of D. We will say that $\mathbf{A}$ is **ω-saturated** if and only if for every finite $D \subseteq A$ and every type $\Gamma(\bar{x})$ in the expanded signature that is realized in some model of $\mathrm{Th}\langle \mathbf{A}, D\rangle$ is realized in $\langle \mathbf{A}, D\rangle$. The structure $\mathbf{A}$ is **countably saturated** provided it is countable and ω-saturated.

Happily, this notion simplifies, as the following theorem asserts.

Theorem 7.3.1. *Let $\mathbf{A}$ be a structure such that for all finite $D \subseteq A$, we have that all 1-types types $\Gamma(x)$ of the expanded signature that are realized in some model of $\mathrm{Th}\langle \mathbf{A}, D\rangle$ are realized in $\langle \mathbf{A}, D\rangle$. Then $\mathbf{A}$ is ω-saturated.*

Proof. We have to prove for every positive natural number n that for all finite $D \subseteq A$ and all n-types $\Gamma(\bar{x})$ that are realized in some model are already realized in $\langle mathbf A, D\rangle$. We do this by induction on n.

The hypothesis of the theorem gives us the base step.

So let $D \subseteq A$ be finite and let $\Gamma(x_0, \dots, x_{n-1}, x_n)$ be an $n+1$-type realized in some model of $\mathrm{Th}\langle \mathbf{A}, D\rangle$. It is harmless to suppose that $\Gamma(x_0, \dots, x_n)$ is closed under finite conjunctions. Let $\Gamma'(x_0, \dots, x_{n-1}) = \{\exists x_n \gamma(x_0, \dots, x_{n-1}, x_n) \mid \gamma(x_0, \dots, x_n) \in \Gamma\}$. Observe that $\Gamma'(x_0, \dots, x_{n-1})$ is an n-type realized in some model of $\mathrm{Th}\langle \mathbf{A}, D\rangle$. By the inductive hypothesis, there is an n-tuple $\langle a_0, \dots, a_{n-1}\rangle$ of elements of A that realize $\Gamma'(x_0, \dots, x_{n-1})$ in $\langle \mathbf{A}, D\rangle$. Let $D' = D \cup \{a_0, \dots, a_{n-1}\}$. Let $c_0, \dots, c_{n-1}$ be the new constant symbols naming $a_0, \dots, a_{n-1}$. The set $\Gamma(c_0, \dots, c_{n-1}, x_n)$ is realized in some model of $\langle \mathbf{A}, D'\rangle$. But this is a 1-type and must be realized in $\langle \mathbf{A}, D'\rangle$. So pick $a_n \in A$ that realizes this 1-type. Then $\langle a_0, \dots, a_n\rangle$ realizes $\Gamma(x_0, \dots, x_n)$ in $\langle \mathbf{A}, D\rangle$. □

Even though the notions of countable atomic structures and of countably saturated structures are complementary, the basic facts about them are similar.

The Uniqueness Theorem for Countably Saturated Structures. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures of the same signature. If* $\mathbf{A}$ *and* $\mathbf{B}$ *are countably saturated and* $\mathbf{A} \equiv \mathbf{B}$*, then* $\mathbf{A} \cong \mathbf{B}$.

Proof. Once again we employ a back-and-forth construction. Let $a_0, a_1, a_2, \dots$ be a listing of the elements of A and $b_0, b_1, b_2, \dots$ be a listing of the elements of B. We expand the signature by countably many new constant symbols $c_0, c_1, c_2, dots$.

Our construction, which proceeds through countably many stages, produces a list of pairs $(d_0, e_0), (d_1, e_1), (d_2, e_2), \dots$ so that

$$\{(d_k, e_k) \mid k \in \omega\}$$

is the desired isomorphism. At the beginning of stage k we have in hand $(d_0, e_0), \dots, (d_{k-1}, e_{k-1})$ and we know $\langle \mathbf{A}, d_0, \dots, d_{k-1}\rangle \equiv \langle \mathbf{B}, e_0, \dots, e_{k-1}\rangle$. In particular, at stage 0 our list of pairs is empty and we know $\mathbf{A} \equiv \mathbf{B}$. What happens at stage k is that we find a way to add a pair to our list that extends our elementary equivalence.

Here is what we do at the stages when k is even. We let e_k be the earliest element on our list of elements of B that has not appeared among $e_0, \dots, e_{k-1}$. Let $\Gamma(x)$ be the complete 1-type of e_k over $\langle \mathbf{B}, e_0, \dots, e_{k-1}\rangle$. Because $\mathrm{Th}\langle \mathbf{A}, d_0, \dots, d_{k-1}\rangle = \mathrm{Th}\langle \mathbf{B}, e_0, \dots, e_{k-1}\rangle$, we see that $\Gamma(x)$ is a 1-type that is realized in some model of $\mathrm{Th}\langle \mathbf{A}, d_0, \dots, d_{k-1}\rangle$. Since $\mathbf{A}$ is ω-saturated, $\Gamma(x)$ is realized in $\langle \mathbf{A}, d_0, \dots, d_{k-1}\rangle$. Pick $d_k \in A$ that realizes $\Gamma(x)$. So we can add (d_k, e_k) to our list knowing that $\langle A, d_0, \dots, d_k\rangle \equiv \langle \mathbf{B}, e_0, \dots, e_k\rangle$.

At the stages where k is odd, we reverse field and pick the next element from A and use its complete 1-type to find an appropriate element of B.

So at the end of the construction, we have $\langle \mathbf{A}, d_0, d_1, d_2, \dots\rangle \equiv \langle \mathbf{B}, e_0, e_1, e_2, \dots\rangle$. This entails that our list of pairs is the desired isomorphism. □

The Existence Theorem for Countably Saturated Models. *Let* T *be a complete theory of countable signature.* T *has a countably saturated model if and only if* $|S_n(T)|$ *is countable for every natural number* n.

Proof. First suppose that $\mathbf{A}$ is a countably saturated model of T. Then every complete n-type that is realized in any model of T (that is, any arbitrary element of $S_n(T)$), must be realized in $\mathbf{A}$. Since A is countable, the number of n-tuples of elements of A must also be countable.

But the $S_n(T)$ must be countable, since distinct complete n-types cannot be realized by the same n-tuple.

For the converse, we want to give two proofs. The first will, like the proof of the Omitting Types Theorem, be a variation on Henkin's proof of the Compactness Theorem. The second proof uses the Compactness Theorem and Tarski's Elementary Chain Theorem.

We begin the first proof by expanding the signature with a countably infinite list $c_0, c_1, c_2, \dots$ of new constant symbols. For each finite subset D of these new constant symbols, the complete 1-types of T in the signature expanded by D is in one-to-one correspondence with the complete $|D| + 1$-types in the original signature. Since there are countably many finite subsets of new constant symbols, there are only countable many complete 1-types of T over these finite expansions of the original signature. Let $\Gamma_0(x), \Gamma_1(x), \Gamma_2(x), \dots$ be a list of all these 1-types.

We need to build a countable model $\mathbf{A}$ (with the original signature) of T so that for any finitely many $a_0, \dots, a_{m-1} \in A$ and any complete 1-type $\Gamma(c_0, \dots, c_{m-1}, x)$, if $\Gamma(c_0, \dots, c_{m-1}, x)$ is realized in any model of $\mathrm{Th}\langle \mathbf{A}, a_0, \dots, a_{m-1}\rangle$, then it is realized in $\langle \mathbf{A}, a_0, \dots, a_{m-1}\rangle$. As $\mathbf{A} \models T$, we see that $\Gamma(c_0, \dots, c_{m-1}, x)$ is one of the types listed in the previous paragraph.

To pull off this construction, we hire a team of experts: A Completion Expert, an Existential Expert, and an Expert Type Realizer. Each of these experts has to get her hands on the construction infinitely often. Together they build an increasing sequence $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of sets of sentences in our expanded signature, where T_{k+1} is obtained from T_k by adding some set of sentences using at most finitely many new constant symbols. As we already understand what the Completion Expert and the Existential Expert do, I will only describe the activities of the Expert Realizer. Once T_k comes into her hands, she takes the next complete type $\Gamma(x)$ from her list. If there is a model of T_k in which $\Gamma(x)$ is realized, our expert sets $T_{k+1} = T_k \cup \Gamma(d)$, where d is the first new constant symbol not occurring in T_k. If, on the other hand, $\Gamma(x)$ is not realized in any model of T_k, then she makes $T_{k+1} = T_k$.

At the end of this construction, we take $T_\omega = \bigcup_{k\in\omega} T_k$. As in Henkin's proof of the Compactness Theorem, T_ω will have countable model $\langle \mathbf{A}, a_0, a_1, \dots\rangle$. We have to see that $\mathbf{A}$ is ω-saturated. So let $a_{i_0}, \dots, a_{i_{m-1}}$ be finite list of elements of A. Let $\Sigma(x)$ be a complete 1-type over $\langle \mathbf{A}, a_{i_1}, \dots, a_{i_{m-1}}\rangle$. As $\mathbf{A} \models T$, we can pick k the type $\Sigma(x) = \Gamma_k(x)$. So our Expert Realizer must have included $\Gamma_k(d)$ in T_ω, for some constant symbol d. Therefore $\Sigma(x)$ must be realized in $\langle \mathbf{A}, a_0, a_1, \dots\rangle$ by the element named by d. This means that $\mathbf{A}$ is countably saturated. This completes the first proof.

For the second proof, let $\mathbf{A}$ be a countable model of T. To get the flavor of what we intend, let $\Gamma(x)$ be a complete 1-type over T. Let $\mathbf{B}$ be a model of T in which $\Gamma(x)$ is realized. By the Downward Löwenheim-Skolem-Tarski Theorem, we may as well suppose that $\mathbf{B}$ is countable. By the Elementary Joint Embedding Theorem there is a structure $\mathbf{C}$ into which both $\mathbf{A}$ and $\mathbf{B}$ can be embedded, since T is a complete theory. Since $\mathbf{B} \preccurlyeq \mathbf{C}$, an element of B that realizes $\Gamma(x)$ over $\mathbf{B}$ must also realize $\Gamma(x)$ over $\mathbf{C}$. An examination of the proof of the Elementary Joint Embedding Theorem shows that $\mathbf{C}$ can be taken to be countable. So we see that $\mathbf{C}$ is at least a bit more saturated than $\mathbf{A}$—at least we know that $\Gamma(x)$ is realized in $\mathbf{C}$. We have to enhance this just a little. In the first place, we have to worry about realizing 1-types in expansions of our signature by finitely many new constants to name elements of A and we have to manage this for not just a single 1-type but for countable list, in fact the sublist of the list $\Gamma_0(x), \Gamma_1(x), \dots$ that we used in the first proof that work over $\mathbf{A}$.

Consider the case of a complete 1-type $\Gamma_k(x) = \Gamma_k(c_0, \dots, c_{m-1}, x)$ so that there is a finite list $a_0, \ldots, a_{m-1} \in A$ so that $\Gamma_k(x)$ is realized in some model of $\mathrm{Th}\langle \mathbf{A}, a_0, \ldots, a_{m-1}\rangle$. By the observations above let $\langle \mathbf{C}_k, a_0, \ldots, a_{m-1}\rangle$ be a countable structure that realizes $\Gamma_k(x)$ where $\mathbf{A} \preccurlyeq \mathbf{C}_k$. We can at least do that with the Elementary Joint Embedding Theorem. Just to define $\mathbf{C}_k$ for all k, in the event that the case above does not apply to $\Gamma_k(x)$, take $\mathbf{C}_k = \mathbf{A}$. We can even arrange matters so that the sets $C_k \setminus A$ are pairwise disjoint.

Now expand the original signature by a new constant symbol for each element of each of the countably many countable sets. Let Δ be the union of the elementary diagrams of the $\mathbf{C}_k$ written in this expanded signature. Every finite subset of Δ has a model, by the obvious inductive extension of the Elementary Amalgamation Theorem. So by the Compactness Theorem, Δ has a countable model. Let $\mathbf{A}_1$ be the reduct of this model back to the original signature. So $\mathbf{A} \preccurlyeq \mathbf{C}_k \preccurlyeq \mathbf{A}_1$ for all k. This means that any 1-type that uses parameters from A that should be realized is, in fact, realized in $\mathbf{A}_1$. This would make $\mathbf{A}_1$ ω-saturated, except that in building $\mathbf{A}_1$ we added more elements and so there are more m-tuples of elements of A_1 to contend with. So repeat this step infinitely often to obtain an elementary chain

$$\mathbf{A} \preccurlyeq \mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \preccurlyeq \ldots \mathbf{A}_k \preccurlyeq \mathbf{A}_{k+1} \preccurlyeq \ldots$$

of countable structures so that every 1-type with parameters from A_k is realized in an appropriate expansion of $\mathbf{A}_{k+1}$. Let $\mathbf{A}_\omega$ be the union of this elementary chain. By Tarski's Elementary Chain Theorem we see that $\mathbf{A}_\omega$ is a countable elementary extension of $\mathbf{A}$ and of each $\mathbf{A}_k$. To see that $\mathbf{A}_\omega$ is ω-saturated let $a_0, \ldots, a_{m-1} \in A_\omega$ and let $\Gamma(c_0, \ldots, c_{m-1}, x)$ be a 1-type realized in some model of $\mathrm{Th}\langle \mathbf{A}_\omega, a_0, \ldots, a_{m-1}\rangle$. But notice $a_0, \ldots, a_{m-1} \in A_k$ for some k. But then $\mathrm{Th}\langle \mathbf{A}_\omega, a_0, \ldots, a_{m-1}\rangle = \mathrm{Th}\langle \mathbf{A}_k, a_0, \ldots, a_{m-1}\rangle$. Thus $\Gamma(c_0, \ldots, c_{m-1}, x)$ is realized in some model of $\mathrm{Th}\langle \mathbf{A}_k, a_0, \ldots, a_{m-1}\rangle$ and so it must be realized over $\langle \mathbf{A}_{k+1}, a_0, \ldots, a_{m-1}\rangle$ and hence over $\langle \mathbf{A}_\omega, a_0, \ldots, a_{m-1}\rangle$. So $\mathbf{A}_\omega$ is countably saturated, as desired. The completes the second proof. □

Just as with the Second Existence Theorem for Countable Atomic Models, we can draw the next corollary from the Existence Theorem for Countably Saturated Models.

Corollary 7.3.2. *Let T be a complete theory in a countable signature. If $\mu(\aleph_0, T)$ is countable, then T has a countably saturated model.* □

7.4 The Number of Denumerable Models of a Complete Theory

The elementary theory of dense linear orders without endpoints has, up to isomorphism, only one denumerable (=countably infinite) model. The elementary theory of nontrivial vector spaces over the rationals has a countable infinity of such models (one for each countable dimension, except dimension 0). On the other hand, the theory of discrete linear orderings without endpoints has $2^{\aleph_0}$ pairwise nonisomorphic denumerable models. [The discrete linear orderings without endpoints are those where every element has both an immediate predecessor and an immediate successor.] Each of these is a complete theory of countable signature. So we have examples of theories T so that $\mu(\aleph_0, T) = 1, \aleph_0$, and $2^{\aleph_0}$. What other values can $\mu(\aleph_0, T)$ have, under the constraint that T is a complete theory of countable signature? What kind of properties of T must hold under the constraint that $\mu(\aleph_0, T)$ has a particular value?

Our next objective is to characterize those complete theories of countable signature that are ω-categorical. These are the theories where $\mu(\aleph_0, T) = 1$.

The ω-Categoricity Theorem. *Let T be a complete theory with an infinite model in a countable signature. The following are equivalent:*

(a) *T is ω-categorical;*

(b) *T has a countable model that is both ω-saturated and atomic;*

(c) *For each natural number n, every complete n-type of T contains a complete formula;*

(d) *$|S_n(T)|$ is finite for all n;*

(e) *For each natural number n there are only finitely many T-equivalence classes of formulas $\varphi(x_0, \ldots, x_{n-1})$;*

(f) *All models of T are atomic.*

Proof. We will prove (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (f) $\Longrightarrow$ (a).

(a) implies (b)
By the Corollary to the Second Existence Theorem for Countable Atomic Models we know that T has a countable atomic model. By the Corollary to the Existence Theorem for Countably Saturated Models, we know that T has a countably saturated model. Since T cannot have finite models, these countable models of T must have cardinality ω. Since T is ω-categorical, they must be isomorphic. So T has a countable model that is both ω-saturated and atomic.

(b) implies (c)
Let $\mathbf{A}$ be a countable model of T that is both ω-saturated and atomic. Let n be a natural number of $\Gamma(x_0, \ldots, x_{n-1})$ be a complete n-type of T. This means that $\Gamma(x_0, \ldots, x_{n-1})$ is realized in some model of $T = \operatorname{Th} \mathbf{A}$. Because $\mathbf{A}$ is ω-saturated, this type must be realized in $\mathbf{A}$. But $\mathbf{A}$ is atomic, so only supported types are realized in $\mathbf{A}$. Thus there must be a complete formula that supports $\Gamma(x_0, \ldots, x_{n-1})$. Since this type is complete, the complete formula must belong to $\Gamma(x_0, \ldots, x_{n-1})$.

(c) implies (d)
Let n be a natural number and and let $\bar{x} = \langle x_0 \ldots, x_{n-1}$. Let

$$\Lambda(\bar{x}) = \{\neg\theta(\bar{x}) \mid \theta(\bar{x}) \text{ is a complete formula with respect to } T\}.$$

Since every complete n-type contains a complete formula, we see that $\Lambda(\bar{x})$ is not included in any complete n-type of T. This means that $\Lambda(\bar{x})$ cannot be realized in any model of T. By the Compactness Theorem, so some finite subset $\{\neg\theta_0(\bar{x}), \ldots, \theta_{m-1}(\bar{x})\}$ of $\Lambda(\bar{x})$ cannot be realized in any model of T. Let $\Gamma(\bar{x})$ be any complete n-type of T. Pick $\mathbf{B} \models T$ and $\bar{b}$ an n-tuple from B so that $\Gamma(\bar{x})$ is the complete n-type realized by $\bar{b}$ over $\mathbf{B}$. Then there must be $k < m$ so that $\mathbf{B} \models \theta_k(\bar{x})[\bar{b}]$. This means that $\Gamma(\bar{x})$ must be the complete n-type determined by $\theta_k(\bar{x})$ with respect to T. So T can have at most m complete n-types.

(d) implies (e)
Let n be a natural number and and let $\bar{x} = \langle x_0 \ldots, x_{n-1}$. Recall that the formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are T-equivalent provided

$$T \models \forall \bar{x}[\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})].$$

Another way to say this is that for any model $\mathbf{B}$ of T and any n-tuple $\bar{b}$ from B, we have that $\varphi(\bar{x})$ belongs to the n-type realized by $\bar{b}$ if and only if $\psi(\bar{x})$ belongs to the n-type realized by $\bar{b}$. That is, the formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are T-equivalent if and only if they belong to the same complete n-types of T. Since T has only finitely many complete n-types, there can be only finitely many T-equivalence classes of formulas whose free variables come from $\bar{x}$.

(e) implies (f)
Let n be a natural number and Γ be a complete n-type of T. Up to T-equivalence, there are only finitely many formulas in Γ; here is a list of them $\theta_0, \ldots, \theta_{m-1}$. Let $\varphi = \theta_0 \wedge \theta_1, \wedge \cdots \wedge \theta_{m-1}$. Since Γ is a complete n-type, we see that $\varphi \in \Gamma$ and also that φ is a complete formula with respect to T. So every n-type of T is supported. It follows that every model of T is atomic.

(f) implies (a)
By the Uniqueness Theorem for Countable Atomic Models, we know that any two countable atomic models of a complete theory must be isomorphic. Since all models of T are atomic, we find that all countable models are isomorphic. That is, T is ω-categorical. □

So what can be said about complete theories T so that $\mu(\aleph_0, T) = 2$?

Vaught's Two Countable Models Theorem. *There is no complete theory T of countable signature that has, up to isomorphism, precisely two countable models.*

Proof. Let T be a complete theory of countable signature with at least two nonisomorphic countable models but no more than countably many. We want to show that T must have at least three pairwise nonisomorphic countable model. Now T must have a countably saturated model $\mathbf{B}$ and a countable atomic model $\mathbf{A}$. Since T is not ω-categorical, by the ω-Categoricity Theorem we know that T cannot have a countable model that is both ω-saturated and atomic. This means that $\mathbf{A} \not\cong \mathbf{B}$. In particular, $\mathbf{A}$ is not countably saturated and $\mathbf{B}$ is not atomic. So for some natural number n there must be a complete n-type $\Gamma(x_0, \ldots, x_{n-1})$ realized in $\mathbf{B}$ that contains no complete formula. Suppose $b_0, \ldots, b_{n-1} \in B$ so that $\langle b_0, \ldots, b_{n-1}\rangle$ realizes $\Gamma(x_0, \ldots, x_{n-1})$. Now the expanded structure $\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle$ is still ω-saturated. So there must be a countable structure $\mathbf{D}$ and elements $d_0, \ldots, d_{n-1} \in D$ so that $\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle \equiv \langle \mathbf{D}, d_0, \ldots, d_{n-1}\rangle$ and $\langle \mathbf{D}, d_0, \ldots, d_{n-1}\rangle$ is atomic.

Of course, $\mathbf{D}$ is a countable model of T. We claim that it is isomorphic to neither $\mathbf{A}$ nor to $\mathbf{B}$.

To see this, notice that $\langle d_0, \ldots, d_{n-1}\rangle$ realizes $\Gamma(x_0, \ldots, x_{n-1})$ over $\mathbf{D}$. That is $\mathbf{D}$ realizes an unsupported n-type. So $\mathbf{D}$ is not atomic and, in particular, $\mathbf{A} \not\cong \mathbf{D}$.

We also contend that $\mathbf{D}$ is not ω-saturated (and hence not isomorphic to $\mathbf{B}$). Indeed, let $T' = \mathrm{Th}\langle \mathbf{B}, b_0, \ldots, b_{n-1}\rangle$. Notice that T' is a complete theory of countable signature. We know that $\langle \mathbf{D}, d_0, \ldots, d_{n-1}\rangle$ is a countable atomic model of T'. If we were able to argue that T' is not ω-categorical, then by the ω-Categoricity Theorem no countable atomic model of T' can be ω-saturated. This would do it. So does T' really fail to be ω-categorical?

Well, since T is not ω-categorical, for some m there are infinitely many pairwise T-inequivalent formulas with free variables all drawn from $x_0, x_1, \ldots, x_{m-1}$. Let's make a list of such formulas: $\varphi_0(\bar{x}), \varphi_1(\bar{x}), \ldots$. So we know

$$T \not\models \forall \bar{x}[\varphi_i(\bar{x}) \leftrightarrow \varphi_j(\bar{x})],$$

whenever $i \neq j$. But any model of T has an elementary extension that can be expanded to a model of T', according to the Elementary Joint Embedding Theorem. As a consequence

$$T' \nvDash \forall \bar{x}[\varphi_i(\bar{x}) \leftrightarrow \varphi_j(\bar{x})],$$

whenever $i \neq j$. So there are infinitely many pairwise T'-inequivalent formulas. By the ω-Categoricity Theorem we conclude that T' fails to be ω-categorical. So $\langle \mathbf{D}, d_0, \ldots, d_{n-1}\rangle$ is not ω-saturated. Therefore, $\mathbf{D}$ is not ω-saturated. So $\mathbf{D} \ncong \mathbf{B}$, and T has at least three pairwise nonisomorphic countable models, namely the atomic model $\mathbf{A}$, the ω-saturated model $\mathbf{B}$, and the model $\mathbf{D}$ that is neither atomic nor ω-saturated. □

The ω-Categoricity Theorem, in its enhanced form including Problem 19 and Problem 20 below, is due variously to the following people: Robert Vaught, Lars Svenonius, Erwin Engeler, and Czeslaw Ryll-Nardzewski. The balance of the results, including the notions of atomic and saturated models, can be found in Robert Vaught's 1954 Ph.D. dissertation written under the direction of Alfred Tarski.

7.5 Problem Set 4

Problem Set About Infinite Models of Complete Theories
1 December 2011

Suppose $\mathbf{A}$ is a structure. The group $\operatorname{Aut}\mathbf{A}$ of all automorphisms of $\mathbf{A}$ partitions A into orbits. [Elements $a, b \in A$ belong to the same orbit if and only if there is an automorphism f such that $f(a) = b$.] Notice that the same applies the n-tuples from A: the group $\operatorname{Aut}\mathbf{A}$ partitions A^n into orbits.

Problem 19.
Let T be a complete theory of countable signature that has infinite models. Prove that T is ω-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits for every natural number n, for every countable $\mathbf{A} \models T$.

Problem 20.
Let T be a complete theory of countable signature that has infinite models. Prove that T is ω-categorical if and only if $\operatorname{Aut}\mathbf{A}$ partitions A^n into only finitely many orbits for every natural number n, for some countable $\mathbf{A} \models T$.

Problem 21.
Let T be an elementary theory in a countable signature and suppose that T is κ-categorical for some infinite cardinal κ. Let $\mathcal{K} = \{A \mid A \models T \text{ and } A \text{ is infinite}\}$. Prove that $\mathcal{K}$ is is an elementary class and that $\operatorname{Th}\mathcal{K}$ is complete.

Problem 22.
Consider a signature provided with one 2-place relation symbol $\leq$ and a countable list $c_0, c_1, c_2, \ldots$ of constant symbols. Let T be the theory asserting that $\leq$ is a dense linear order without endpoints and $c_i < c_{i+1}$ for each $i \in \omega$. Prove that T is a complete theory and find out (with proofs), up to isomorphism, how many countable models T has.

LECTURE 8

Categoricity in Uncountable Cardinalities

8.1 Morley's Categoricity Theorem

In 1954 Jerzy Łoś published the conjecture that an elementary theory of countable signature that is categorical in some uncountable cardinal should be categorical in all uncountable cardinals. So the elementary theory of algebraically closed fields of a given characteristic would offer a paradigm. This conjecture was confirmed by Michael Morley in his 1962 PhD dissertation written under the direction of Saunders MacLane at the University of Chicago. I also note that Morley was a visiting graduate student in Berkeley and worked closely with Robert Vaught. Morley's original proof used in an essential way the ideas from the previous chapter, but also developed model-theoretic extensions of older ideas that had emerged in the study of algebraically closed fields, beginning with the 1910 treatise of Ernst Steinitz. The proof given here has an involved heritage. After Morley's original proof was in hand, William Marsh found a rather simple proof of a special case. John Baldwin and Alistair Lachlan put the ideas of Marsh together with some ideas of H. Jerome Keisler, to give a new proof of Morley's Theorem. C. C. Chang and Keisler worked out an exposition along the lines of the Baldwin-Lachlan proof in their magnificent 1973 monograph Model Theory. Finally, Peter Hinman, in his 2005 text Fundamentals of Mathematical Logic reworked this proof. The proof given here takes advantage of all these earlier expositions. This line of reasoning has the advantages of being direct and relying principally on methods closely connected to those developed in these lecture notes. It has the disadvantages of avoiding the methods with more explicit heritage in the theory of algebraically closed fields. In particular, Morley's original proof provided a powerful analysis of the complexity of formulas that became the starting point of many deep investigations over the ensuing decades. Morley's Theorem, it seems to me, is a theorem were it makes sense to know more than one proof.

Morley's Categoricity Theorem, [1962]. *Let T be an elementary theory of countable signature. If T is categorical in some uncountable cardinality, then T is categorical in every uncountable cardinal.*

We'll prove the following two theorems, which when combined, will give us Morley's Theorem.

Downward Morley Theorem. *If T is categorical in some uncountable cardinal, then T is $\aleph_1$-categorical.*

Upward Morley Theorem. *If T is $\aleph_1$-categorical, then T is κ-categorical for all uncountable cardinals κ.*

To prove these, we need some new concepts.

T is κ-**stable** means if $\mathbf{A} \models T$ and $C \subseteq A$ with $|C| = \kappa$, then the number of 1-types is itself equal to κ. That is, $|S_1(\mathrm{Th}\langle \mathbf{A}, c\rangle_{c\in C})| = \kappa$.

$\mathbf{A}$ is **modest** means if $C \subseteq A$ and C is countable, then at most countably many types of $\langle \mathbf{A}, c\rangle_{c\in C}$ are realized in $\langle \mathbf{A}, c\rangle_{c\in C}$.

Stability says you have a small number of types. Modesty says only countably many of those types are realized.

B is **modest over A** provided $\mathbf{A} \preccurlyeq \mathbf{B}$ and for all countable $C \subseteq A$, every type of $\langle \mathbf{B}, c\rangle_{c\in C}$ which is realized in $\langle \mathbf{B}, c\rangle_{c\in C}$ is already realized in $\langle \mathbf{A}, c\rangle_{c\in C}$. We also say that **B** is a **modest extension** of **A**.

First Modesty Theorem. *Let T be a theory in a countable signature which has infinite models and let κ be an infinite cardinal. Then T has a modest model of cardinality κ.*

The Categoricity Implies Stability Corollary. *Let T be a theory of countable signature and κ be an uncountable cardinal. If T is κ-categorical, then T is ω-stable.*

Proof of the Corollary. Suppose T is not ω-stable. Then there is some model $\mathbf{A} \models T$ and some $C \subseteq A$ with $|C| = \aleph_0$ and so that $|S_1(\mathrm{Th}\langle \mathbf{A}, c\rangle_{c\in C})| > \aleph_0$. By the Downward (or Upward) Löwenheim-Skolem-Tarski Theorem, we can assume that $|A| = \aleph_1$. Add $\aleph_1$ constant symbols. Pick $\aleph_1$ many complete 1-types over $\langle \mathbf{A}, c\rangle_{c\in C}$. Let Δ be the set of sentences saying that the constants realize the 1-types. Then $\Delta \cup \mathrm{Th}(\langle \mathbf{A}, c\rangle_{c\in C})$ has a model. It must have a model $\langle \mathbf{B}^+, c\rangle_{c\in C}$ of cardinality κ. Now too many types are realized in this model for **B** to be modest. So T cannot be κ-categorical. □

Second Modesty Theorem. *Let T be an ω-stable theory of countable signature. Let* **A** *be an uncountable model of T and let $\kappa \geq |A|$. Then* **A** *has a modest extension of cardinality κ.*

The Characterization Theorem for Categoricity of ω-Stable Theories. *Let T be an ω-stable complete theory of countable signature and let κ be an infinite cardinal. Then T is κ-categorical if and only if T has a model of cardinality κ and all models of T of cardinality κ are saturated.*

Proof of the Downward Morley Theorem. Say κ is uncountable and T is κ-categorical. So T is ω-stable. We will show that every model $\mathbf{A} \models T$ of cardinality $\aleph_1$ is saturated. Pick $C \subseteq A$ with C countable. We need to show that every 1-type of $\langle \mathbf{A}, c\rangle_{c\in C}$ is realized in $\rangle \mathbf{A}, c\rangle_{c\in C}$. By the Second Modesty Theorem, there is a modest extension $\langle \mathbf{B}, c\rangle_{c\in C}$ of $\langle \mathbf{A}, c\rangle_{c\in C}$ of cardinality κ. So every 1-type of $\langle \mathbf{B}, c\rangle_{c\in C}$ is realized in $\langle \mathbf{B}, c\rangle_{c\in C}$ because **B** is saturated. So every 1-type of $\langle \mathbf{A}, c\rangle_{c\in C}$ is realized in $\langle \mathbf{B}, c\rangle_{c\in C}$ because **B** is saturated. So it is realized in $\langle \mathbf{A}, c\rangle_{c\in C}$ because $\langle \mathbf{B}, c\rangle_{c\in C}$ is a modest extension of $\langle \mathbf{A}, c\rangle_{c\in C}$. □

We still need to prove the Upward Morley Theorem. First we need a definition and a theorem. Let $\theta(x)$ be a formula in one free variable, x, and **A** is a structure. Define

$$\theta^{\mathbf{A}}(x) = \{a \mid a \in A \text{ and } \mathbf{A} \models \theta(x)\}.$$

Keisler's Two Cardinal Theorem. *Let* **A** *be an structure of countable signature and $\theta(x)$ be a formula of the signature. Suppose that*

$$|A| > |\theta^{\mathbf{A}}(x)| \geq \aleph_0.$$

Then there are structures **B** *and* **C** *so that*

(a) $\mathbf{B} \preccurlyeq \mathbf{A}$ *and* **B** *is countable;*

(b) $\mathbf{B} \preccurlyeq \mathbf{C}$ *and* $|C| = \aleph_1$*;*

(c) $\theta^{\mathbf{B}}(x) = \theta^{\mathbf{C}}(x)$

Proof of the Upward Morley Theorem. Without loss of generality, T is complete. We'll prove that if κ is an uncountable cardinal, and T has a model $\mathbf{A}$ of cardinality κ which is not saturated, then T has a model $\mathbf{C}$ of cardinality $\aleph_1$ which is not saturated.

$\mathbf{A}$ fails to be saturated means that we can pick $D \subseteq A$ with $|D| < |A|$ and a complete 1-type $\Gamma(x)$ over $\langle \mathbf{A}, d \rangle_{d \in D}$ that is not realized in $\langle \mathbf{A}, d \rangle_{d \in D}$. We'll use the Keisler's Two Cardinal Theorem.

We'll do this by expanding the language so we can talk about types and talk about the parameters that occur in the formulas.

Observe: $|\Gamma(x)| < |A|$.

We'll pull out a subset of A to index the formulas in $\Gamma(x)$. Let $U^{\mathbf{A}^*}$ be a subset of A with cardinality the same as $\Gamma(x)$. That is, $|U^{\mathbf{A}^*}| = |\Gamma(x)|$ and $U^{\mathbf{A}^*} \subseteq A$. So we have a one-one correspond $a \mapsto \gamma_a(x)$ between $U^{\mathbf{A}^*}$ and $\Gamma(x)$. So

$$\Gamma(x) = \left\{ \gamma_a(x) : a \in U^{\mathbf{A}^*} \right\}.$$

Add to the signature the following relation symbols:

- U - a one-place relation symbol;
- R - a two-place relation symbol;
- S - a two-place relation symbol.

Expand $\mathbf{A}$ to $\mathbf{A}^* = \langle \mathbf{A}, U^{\mathbf{A}^*}, R^{\mathbf{A}^*}, S^{\mathbf{A}^*} \rangle$. We've already described $U^{\mathbf{A}^*}$. Now we'll describe $R^{\mathbf{A}^*}$ and $S^{\mathbf{A}^*}$:

$$S^{\mathbf{A}^*} = \left\{ (a, b) \mid a \in U^{\mathbf{A}^*} \text{ and } \mathbf{A} \models \gamma_a(b) \right\} \text{ and}$$
$$R^{\mathbf{A}^*} = \left\{ (a, d) \mid a \in U^{\mathbf{A}^*}, d \in D, \text{ and the parameter } d \text{ occurs in } \gamma_a(x) \right\}$$

By Keisler's Two Cardinal Theorem, we can get $\mathbf{B}^*$ and $\mathbf{C}^*$ so that

0. $\mathbf{B}^* \preccurlyeq \mathbf{A}^*$ and B is countable;
1. $\mathbf{B}^* \preccurlyeq \mathbf{C}^*$ and $|C| = \aleph_1$;
2. $U^{\mathbf{B}^*} = U^{\mathbf{C}^*}$.

Let $\Gamma_B(x) = \left\{ \gamma_a(x) \mid a \in U^{\mathbf{B}^*} \right\}$. Remember that $U^{\mathbf{B}^*} = U^{\mathbf{A}^*} \cap B$ and $R^{\mathbf{B}^*} = R^{\mathbf{A}^*} \cap (B \times B)$ and $S^{\mathbf{B}^*} = S^{\mathbf{A}^*} \cap (B \times B)$, all of these since $\mathbf{B}^*$ is a substructure of $\mathbf{A}^*$. Let $D' = D \cap B$.

$\gamma_a(x)$ has some fixed finite number of parameters. In our expanded signature we can say $\gamma_a(x)$ has exactly the right number of parameters:

$$\exists y_0 \dots y_{n-1} \left[\neg y_0 \approx y_1 \wedge \cdots \wedge \neg y_{n-2} \approx y_{n-1} \wedge R(a, y_0) \wedge \cdots \wedge R(a, y_{n-1}) \right]$$

For any formula $\theta(x)$, if $\mathbf{A}^* \models \theta(a)$ and $a \in B$, then $\mathbf{B}^* \models \theta(a)$ since $\mathbf{B}^* \preccurlyeq \mathbf{A}^*$. So all the parameters of $\gamma_a(x)$ belong to D'. So $|D'| < \aleph_1$.

We need $\Gamma_B(x)$ to be compatible with $\langle \mathbf{C}, d\rangle_{d\in D'}$ and we need that $\Gamma_B(x)$ is not realized in $\langle \mathbf{C}, d\rangle_{d\in D'}$. Some sentences:

$$\forall x\, [S(a,x) \leftrightarrow \gamma_a(x)] \tag{$\star$}$$

$$\forall u_0, \ldots, u_{n-1}\, [U(u_0) \wedge \cdots \wedge U(u_{n-1})] \to \exists x\, [S(u_0,x) \wedge \cdots \wedge S(u_{n-1},x)] \tag{$\star\star$}$$

$$\neg\exists x \forall u\, [U(u) \to S(u,x)] \tag{$\star\star\star$}$$

In $\mathbf{A}^*$ we have $(\star)$ for each $a \in U^{A^*}$, we have $(\star\star)$ for each n, and we have $(\star\star\star)$. We also have $(\star)$ for each $a \in U^{\mathbf{B}^*}$.

We know that $(\star)$ holds in $\mathbf{B}^*$ for all $a \in U^{\mathbf{B}^*}$. We have that $(\star)$ holds in $\mathbf{C}^*$ for all $a \in U^{\mathbf{B}^*}$. We have that $(\star\star)$ holds in $\mathbf{C}^*$ for all n. We have that $(\star\star\star)$ holds in $\mathbf{C}^*$.

By $(\star\star)$, in $\mathbf{C}^*$ we have by $(\star)$

$$\langle \mathbf{C}, d\rangle^*_{d\in D'} \models \exists x \left[\gamma_{a_0}(x) \wedge \cdots \wedge \gamma_{a_{n-1}}\right] \text{ for any finite sequence } a_0, \ldots, a_{n-1} \in U^{\mathbf{B}^*}$$

so $\Gamma_B(x)$ is compatible with $\langle \mathbf{C}, d\rangle_{d\in D'}$. $(\star\star\star)$ in addition tells us that $\Gamma_B(x)$ is not realized in $\langle \mathbf{C}, d\rangle_{d\in D'}$. So $\mathbf{C}$ is not saturated. □

In this way, Morley's Categoricity Theorem is established—except we need to prove the Characterization Theorem for Categoricity of ω-Stable Theories, the First Modesty Theorem, the Second Modesty Theorem, and Keisler's Two Cardinal Theorem.

8.2 Keisler's Two Cardinal Theorem

Keisler's Two Cardinal Theorem. *Let* $\mathbf{A}$ *be a structure of countable signature and* $\theta(x)$ *be a formula of the signature. Suppose that*

$$|A| > |\theta^{\mathbf{A}}(x)| \geq \aleph_0.$$

Then there are structures $\mathbf{B}$ *and* $\mathbf{C}$ *so that*

(a) $\mathbf{B} \preccurlyeq \mathbf{A}$ *and* $\mathbf{B}$ *is countable;*

(b) $\mathbf{B} \preccurlyeq \mathbf{C}$ *and* $|C| = \aleph_1$*;*

(c) $\theta^{\mathbf{B}}(x) = \theta^{\mathbf{C}}(x)$

Proof. In view of the Downward Löwenheim-Skolem-Tarski Theorem, we can suppose, without loss of generality, that $|\theta^A(x)| = \lambda$ and $|A| = \lambda^+$, the smallest cardinal larger than λ.

We expand the signature by a new binary relation symbol $\leq$. We expand $\mathbf{A}$ to $\langle \mathbf{A}, \leq^*\rangle$ where $\leq^*$ is a well-ordering of A of order type λ^+ such that $\theta^{\mathbf{A}}(x)$ is an initial segment. Let $\psi(x)$ be a formula of the expanded signature . We will use $\exists^{\mathrm{cf}} x\, \psi(x)$ to abbreviate

$$\forall z \exists x\, [z \leq x \wedge \psi(x)]$$

This says there are arbitrarily large x's so that ψ holds. Let $\varphi(x,y)$ be a formula of the expanded signature. We assert the following:

$$\langle \mathbf{A}, \leq^*\rangle \models \exists^{\mathrm{cf}} y\, \exists x\, \left[\theta(x) \wedge \varphi(x,y)\right] \to \exists x\, \left[\theta(x) \wedge \exists^{\mathrm{cf}} y\, \varphi(x,y)\right] \tag{$\circledast$}$$

To see this, For $b \in \theta^{\mathbf{A}}(x)$, we can let $Y_b = \{a \mid a \in A \text{ and } \langle \mathbf{A}, \leq^* \rangle \models \varphi(b, a)\}$. Then let

$$Y = \bigcup_{b \in \theta^{\mathbf{A}}(x)} Y_b.$$

So Y is cofinal in A, so $|Y| = \lambda^+ = |\bigcup Y_b|$.

If $|Y_b| \leq \lambda$ for all $b \in \theta^{\mathbf{A}}(x)$, then $\lambda^+ \leq |\theta^{\mathbf{A}}(x)| \cdot \lambda = \lambda \cdot \lambda = \lambda$. This is impossible.

Hence one of the Y_b has to be larger than λ, that is, $\lambda < |Y_b|$ for some b. So $\lambda < |Y_b| \leq \lambda^+$. But there's only one cardinal in the interval $(\lambda, \lambda^+]$, so $|Y_b| = \lambda^+$ for some $b \in \theta^{\mathbf{A}}(x)$. So Y_b is cofinal for some $b \in \theta^{\mathbf{A}}(x)$. But that's precisely what ($\circledast$) says.

Let $\langle \mathbf{B}, \leq^\circ \rangle \preccurlyeq \langle \mathbf{A}, \leq^* \rangle$ so that $|B|$ is countable. We can do that by the Downward Löweheim-Skolem-Tarski Theorem.

> Claim: Because $\langle \mathbf{A}, \leq^* \rangle \equiv \langle \mathbf{B}, \leq^\circ \rangle$, then $\langle \mathbf{B}, \leq^\circ \rangle$ has a proper elementary extension $\langle \mathbf{B}', \leq' \rangle$ which is countable and so that $\theta^{\mathbf{B}}(x) = \theta^{\mathbf{B}'}(x)$.

We'll see how the rest of the proof works given this claim, and then we'll prove the claim.

Construct the following, using the claim recursively through $\aleph_1$ steps:

$$\begin{array}{ccc}
\langle \mathbf{B}_0, \leq_0 \rangle & \text{is} & \langle \mathbf{B}, \leq^\circ \rangle \\
\curlywedge & & \curlywedge \\
\langle \mathbf{B}_1, \leq_1 \rangle & \text{is} & \langle \mathbf{B}'_0, \leq'_0 \rangle \\
\curlywedge & & \curlywedge \\
\langle \mathbf{B}_2, \leq_2 \rangle & \text{is} & \langle \mathbf{B}'_1, \leq'_1 \rangle \\
\curlywedge & & \curlywedge \\
 & \vdots & \\
\langle \mathbf{B}_\omega, \leq_\omega \rangle & \text{is} & \bigcup_{i \in \omega} \langle \mathbf{B}_i, \leq_i \rangle \\
\curlywedge & & \curlywedge \\
\langle \mathbf{B}_{\omega+1}, \leq_{\omega+1} \rangle & \text{is} & \langle \mathbf{B}'_\omega, \leq'_\omega \rangle \\
 & \vdots &
\end{array}$$

By invoking Tarski's Elementary Chain Theorem at the limit stages, we have that this makes an elementary chain of length ω_1. Let $\langle \mathbf{C}, \leq^* \rangle$ by the limit of this elementary chain. Then $\langle \mathbf{B}, \leq^\circ \rangle \preccurlyeq \langle \mathbf{C}, \leq^* \rangle$ by a last application of Tarski's Elementary Chain Theorem.

Now $b \in \theta^{\mathbf{C}}(x)$ if and only if $b \in \theta^{\mathbf{B}_\alpha}(x)$ and $b \in B_\alpha$ for some ordinal $\alpha < \omega_1$. This happens if and only if $b \in \theta^B(x)$, as can be established by transfinite induction on α. This is what we needed for Keisler's Two Cardinal Theorem. But we still have the Claim to establish.

To prove the Claim, we start by adding constant symbols to name the elements of B and one more constant c. Let $T^+ = \mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B} \cup \{b < c \mid b \in B\}$. By the Compactness Theorem, T^+ has infinite models.

Let

$$\Phi := \{\theta(x)\} \cup \{\neg x \approx b \mid b \in \theta^B(x)\}.$$

We want a countable model of T^+ which omits Φ.

Subclaim: Let $\eta(x)$ be a formula of newly expanded signature. Then

$$T^+ \models \eta(c) \text{ if and only if } \langle \mathbf{B}, \leq^\circ \rangle \models \neg \exists^{\mathrm{cf}}\, y\, [\neg\, \eta(y)]$$

Proof of the Subclaim: ($\Leftarrow$) Well,

$$\begin{aligned} \langle \mathbf{B}, \leq^\circ \rangle \not\models \exists^{\mathrm{cf}}\, y\, [\neg\, \eta(y)] &\iff \langle \mathbf{B}, \leq^\circ \rangle \models \exists\, z\, \forall\, y\, [z \leq y \to \eta(y)] \\ &\iff \mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B} \models \forall\, y \;\; (b^* \leq y \to \eta(y)) \text{ for some } b^* \in B \end{aligned}$$

We know $T^+ \models b^* < c$. Then $T^+ \models \forall\; y \;\; (b^* < y \to \eta(y))$, so $T^* \models b^* < c \to \eta(c)$, so $T^+ \models \eta(c)$.

($\Rightarrow$) Suppose $T^+ \models \eta(c)$. By the Compactness Theorem,

$$\mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B} \cup \{b_0 < c, \ldots, b_{m-1} < c\} \models \eta(c) \text{ for some } b_0, \ldots, b_{m-1} \in B,$$

which is the same as

$$\mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B} \models (b_0 < c \wedge b_1 < c \wedge \cdots \wedge b_{m-1} < c) \to \eta(c).$$

Because the constant symbol c does not occur in $\mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B}$, we have

$$\mathrm{Th}\langle \mathbf{B}, \leq^\circ, b \rangle_{b \in B} \models \forall y \;\; [(b_0 < y \wedge \cdots \wedge b_{m-1} < y) \to \eta(y)]\,.$$

Now just pick b^* to be bigger than each b_i for $i = 0, \ldots, m-1$; that is, $b_0, \ldots, b_{m-1} \leq^\circ b^*$. Since the b_i are linearly ordered, we know b^* dominates the biggest one. So we can conclude $\mathrm{Th}(B, \leq, b)_{b \in B} \models \forall y\, [b^* \leq y \to \eta(y)]$. This finishes the proof of the Subclaim.

We're going to use the Subclaim to find a countable model of T^+ which omits Φ. According to the Omitting Types Theorem, it is enough to show that Φ is an unsupported type. For a contradiction, suppose $\psi(x, c)$ supports Φ for T^+. Then

$$\begin{aligned} &T^+ \models \forall x \;\; [\psi(x, c) \to \neg x \approx b] \;\text{ for all } b \in \theta^{\mathbf{B}}(x), \\ &T^+ \models \forall x \;\; [\psi(x, c) \to \theta(x)]\,, \text{ and} \\ &T^+ \cup \{\exists x\; \psi(x, c)\} \;\text{ has a model.} \end{aligned}$$

But the following three assertions are logically equivalent:

$$\begin{aligned} &T^+ \models \forall x \;\; [\psi(x, c) \to \neg x \approx b] \\ &T^+ \models \forall x \;\; [x \approx b \to \neg \psi(x, c)] \\ &T^+ \models \neg \psi(b, c) \end{aligned}$$

By the Subclaim,

$$\langle \mathbf{B}, \leq^\circ \rangle \models \neg \exists^{\mathrm{cf}} y \; \psi(b, y) \text{ for all } b \in \theta^B(x).$$

But this is the same as

$$\langle \mathbf{B}, \leq^\circ \rangle \models \forall x \;\; \left[\theta(x) \to \neg \exists^{\mathrm{cf}} y \; \psi(x, y)\right]$$

Then, as $\langle \mathbf{B}, \leq^\circ \rangle \equiv \langle \mathbf{A}, \leq^* \rangle$, we see

$$\langle \mathbf{A}, \leq^* \rangle \models \forall x \ \left[\theta(x) \to \neg \exists^{\mathrm{cf}} y \ \psi(x,y)\right]$$

or, equivalently,

$$\langle \mathbf{A}, \leq^* \rangle \models \neg \exists x \ \left[\theta(x) \wedge \exists^{\mathrm{cf}} y \ \psi(x,y)\right]$$

Remember what (⊛) said:

$$\langle \mathbf{A}, \leq^* \rangle \models \exists^{\mathrm{cf}} y \, \exists x \ [\theta(x) \wedge \varphi(x,y)] \to \exists x \ \left[\theta(x) \wedge \exists^{\mathrm{cf}} y \, \varphi(x,y)\right] \tag{$\circledast$}$$

Applying the contrapositive of (⊛), we see that

$$\langle \mathbf{A}, \leq^* \rangle \models \neg \exists^{\mathrm{cf}} y \, \exists x \ [\theta(x) \wedge \psi(x,y)] \, .$$

As $\langle \mathbf{B}, \leq^\circ \rangle \equiv \langle \mathbf{A}, \leq^\circ \rangle$,

$$\langle \mathbf{B}, \leq^\circ \rangle \models \neg \exists^{\mathrm{cf}} y \, \exists x \ [\theta(x) \wedge \psi(x,y)] \, .$$

By the Subclaim,

$$T^+ \models \neg \exists x \ [\theta(x) \wedge \psi(x,c)]$$

which is the same as

$$T^+ \models \forall x \ [\psi(x,c) \to \neg \theta(x)] \, .$$

But our assumption that $\psi(x,c)$ supports Φ gives us

$$T^+ \models \forall x \ [\psi(x,c) \to \theta(x)] \, .$$

Our assumption also yields that $T^+ \cup \{\exists x \psi(x,c)\}$ has a model $\mathbf{D}$. So there must be $d \in D$ such that d realizes both $\theta(x)$ and $\neg\theta(x)$. This is a contradiction. So the proof of Keisler's Two Cardinal Theorem is finished. □

Keisler's Two Cardinal Theorem is a relative of the Löwenheim-Skolem-Tarski Theorem, but with two cardinal parameters rather than one. The first theorem of this kind was established by Robert Vaught and was put to use by Morley in his proof. Another two cardinal theorem was found by C. C. Chang. These two-cardinal theorems all have among their hypotheses a structure $\mathbf{A}$ of some infinite cardinality κ with a subset $\theta^{\mathbf{A}}(x)$ of infinite cardinality $\lambda < \kappa$, and among their conclusions a structure $\mathbf{B} \equiv \mathbf{A}$ where $|B| = \mu$ and $|\theta^{\mathbf{B}}(x)| = \nu$, where $\mu > \nu \geq \aleph_0$. At issue in these theorems is the relationship among the four cardinals. In the most general setting, results of this kind require properties of cardinals that cannot be proved in ZFC (provided, of course, that ZFC is consistent). On the other hand, restricting $\mathbf{A}$ in some way (e.g. that $\mathrm{Th}\,\mathbf{A}$ is ω-stable) can be fruitful. Over the last few decades there has been a cottage industry in two-cardinal theorems.

8.3 Order Indiscernibles and the First Modesty Theorem

We still have to prove the two modesty theorems. We also still have to convince ourselves that if we have an ω-stable theory then it has models of every size.

Let $\mathbf{A}$ be a structure. Suppose $D \subseteq A$ and D is strictly linearly ordered by $<$, where this relation may be quite independent of the basic operations and relations of $\mathbf{A}$. We say D is a set of **order indiscernibles** for $\mathbf{A}$ provided for all natural numbers n, whenever $\bar{a}$ and $\bar{b}$ are strictly $<$-increasing n-tuples of elements of D, then

$$\mathbf{A} \models \varphi(\bar{x})[\bar{a}] \text{ if and only if } \mathbf{A} \models \varphi(\bar{x})[\bar{b}]$$

for all formulas $\varphi(x_0, \ldots, x_{n-1})$ of the signature.

Ehrenfeucht-Mostowski Theorem, [1956]. *Let T be a theory with infinite models and let $\langle D, < \rangle$ be any linearly ordered set. Then T has a model $\mathbf{A}$ in which D is a set of order indiscernibles.*

Proof. We have to make a model $\mathbf{A}$ of T. Let L be the signature of T. Expand the signature by new constants to name elements of D. Let

$$\begin{aligned} \Delta = \{\varphi(\bar{c}) \leftrightarrow \varphi(\bar{d}) \mid \bar{c} \text{ and } \bar{d} \text{ name increasing tuples and } \varphi(\bar{x}) \text{ is a formula of signature } L\} \\ \cup \{\neg c_r \approx c_s \mid r, s \in L \text{ and } r \neq s\} \end{aligned}$$

What we need is a model of $T \cup \Delta$. To invoke the Compactness Theorem, let $\Delta' \subseteq \Delta$ be finite. We show $T \cup \Delta'$ has a model. Let $\varphi_0, \ldots, \varphi_{k-1}$ and $c_0, \ldots, c_{\ell-1}$ and $x_0, \ldots, x_{n-1}$ be formulas, constants, and variables of Δ'. We construe each φ_j as a formula $\varphi(\bar{x})$ with free variables from the n-tuple $\bar{x}\langle x_0, \ldots, x_{n-1}\rangle$. We also index the constant symbols to reflect the order of the elements of D named by the constant symbols.

Let $\mathbf{A}$ be an infinite model of T. Let $\sqsubset$ be an arbitrary strict linear ordering of A. We use $\binom{A}{n}$ to denote the collection of subsets of A of cardinality n. We represent a subset of A of cardinality n has an n-tuple $\bar{a}$ by ordering the subset with $\sqsubset$. On $\binom{A}{n}$ define an equivalence relation $\sim$:

$$\bar{a} \sim \bar{b} \text{ means } \mathbf{A} \models \varphi_j(\bar{a}) \leftrightarrow \varphi_j(\bar{b}) \text{ for all } j < k.$$

We leave in the hands of the graduate students the duty to check that this is an equivalence relation. How many equivalence classes are there? There are no more than 2^k equivalence classes. At this point we want Ramsey's Theorem.

> **Ramsey's Theorem**
> Let n be a natural number and A be an infinite set. However the collection of all n-element subsets of A is partitioned into finitely many blocks, there is an infinite $H \subseteq A$ so that all the n-element subsets of H belong to the same block.

We supply a proof of Ramsey's Theorem in an appendix.

By Ramsey's Theorem, there is an infinite $H \subseteq A$ so that all the n-element subsets of H are included in a single equivalence class. Since H is infinite, pick

$$a_0 \sqsubset a_1 \sqsubset \cdots \sqsubset a_{\ell-1}$$

from H. Then $\langle \mathbf{A}, a_0, a_1, \ldots, a_{\ell-1}\rangle$ is a model of $T \cup \Delta'$. So by the Compactness Theorem $T \cup \Delta$ has a model in which the elements named by the new constant symbols form a set of order indiscernibles isomorphic to $\langle D, <\rangle$. This is enough to establish the theorem. □

Skolemization

Consider $\exists x\ \varphi(x, \bar{y}) \to \varphi(F(\bar{y}), \bar{y})$. If we had the use of operation symbols like F, we could eliminate the quantifier $\exists$. We call F a **Skolem function** for φ. Each Skolem function is associated with a **Skolem Axiom:**

$$\forall \bar{y}\,[\exists x \varphi\,(x, \bar{y}) \to \varphi\,(F_\varphi(\bar{y}), \bar{y})]$$

Let L be a signature and let T be a set of L-sentences. Make L_1 by adding all Skolem functions for all $\exists$-formulas of L. Let Σ_1 be the set of Skolem axioms that go along with these new symbols. But this isn't quite good enough because expanding our signature with all these Skolem function symbols leads to formulas that we didn't have before.

Repeating this process recursively through countably many stages will fix this.

Let L^* be the resulting signature and let $\Sigma = \bigcup_{i\in\omega} \Sigma_{i+1}$. Let $T^* = T \cup \Sigma$.

Then the following are evident:

- T^* is a Skolem theory: it has Skolem axioms for all appropriate formulas.
- L and L^* have the same number of formulas. In particular, if L is countable, so is L^*.
- Every model $\mathbf{A} \models T$ can be expanded to a model $\mathbf{A}^* \models T^*$.
- For every formula $\varphi(\bar{x})$, there is a quantifier-free formula φ^* such that

$$T^* \models \forall \bar{x}\ \ (\varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x}))$$

- If $\mathbf{B} \models T^*$ and $\mathbf{C}$ is a substructure of $\mathbf{B}$, then $\mathbf{C}$ is an elementary substructure of $\mathbf{B}$.

The idea of Skolemizing a theory traces back the Thoralf Skolem in 1920. He introduced this technique to give a simple correct proof of the the Downward Löwenheim-Skolem Theorem. At that time the notion of elementary substructure was not available, but had it been Skolem could have easily reached a stronger conclusion.

The First Modesty Theorem

Recall that $\mathbf{A}$ is **modest** means $\langle \mathbf{A}, c\rangle_{c\in C}$ realizes at most countably many complete 1-types whenever $C \subseteq A$ is countable.

The First Modesty Theorem. *Let T be a theory of countable signature that has infinite models. Then T has modest models of every infinite cardinality.*

Proof. Let κ be an infinite cardinal. We will produce a modest model of T of cardinality κ. Let Y be a set of cardinality κ and let $<$ be a well-ordering of Y of order type κ.

Let T^* be a Skolemization of T. By the Ehrenfeucht-Mostowski Theorem, we can get a model $\mathbf{B}^*$ of T^* with Y a set of order indiscernibles in B^*. Let $\mathbf{A}^*$ be the substructure of $\mathbf{B}^*$ generated by Y. Then

- $|A^*| = |Y| = \kappa$;
- $\mathbf{A}^* \preccurlyeq \mathbf{B}^*$.

We claim that $\mathbf{A}^*$ is modest. (This is not quite our goal, but almost.)

Let $C \subseteq A$ with C countable. For each $c \in C$ pick a term $t(\bar{x})$ and some increasing tuple $\bar{d}$ of Y so that $c = t^{A^*}(\bar{d})$. Let Z be the countable subset of Y consisting of all the entries in the chosen $\bar{d}$'s. Let n be a natural number. Define $\doteqdot_n$ on collection of all increasing n-tuples of Y by

$$\bar{a} \doteqdot_n \bar{b}$$
$$\text{means}$$

> Whenever $W \subseteq Z$ is finite, then the strictly increasing tuple made from W and the entries of $\bar{a}$ is the same length as the strictly increasing tuple made from W and the entries of $\bar{b}$ and a_i and b_i occupy corresponding positions for all $i < n$.

The relation $\doteqdot_n$ is an equivalence relation on the n-element subsets of Y. Let us estimate the number of equivalence classes. Suppose $a \in Y \setminus Z$ and there is $z \in Z$ with $a < z$. In this case, let d_a be the least such element of Z. In case $a \in Y \setminus Z$ so that a is an upper bound on Z, we give d_a the default value ∞. Now observe that for any increasing n-tuples $\bar{a}$ and $\bar{b}$ from Y we have

$$\bar{a} \doteqdot_n \bar{b}$$
$$\text{if and only if}$$
$$d_{a_i} = d_{b_i} \text{ for all } i < n \text{ such that neither } a_i \in Z \text{ nor } b_i \in Z$$
$$\text{and if } a_i \in Z, \text{ then } a_i = b_i \text{ and if } b_i \in Z, \text{ then } a_i = b_i.$$

Since $Z \cup \{\infty\}$ is countable, we see that there are only countably many $\doteqdot_n$-equivalence classes.

We need to count the complete 1-types with parameters from C that are realized in $\langle \mathbf{A}^*, c\rangle_{c\in C}$. Let $\varphi(x, \bar{c})$ be a formula. Let $\bar{t}$ and $\bar{\bar{d}}$ be the sequences of terms and tuple so that $c_j = t_j^{\mathbf{A}^*}(\bar{d}_j)$, according to the selections made above.

Let s be any term and let $\bar{a}$ and $\bar{b}$ be increasing n-tuples of elements of Y with $\bar{a} \doteqdot_n \bar{b}$.

$$\begin{aligned}
\langle \mathbf{A}^*, c\rangle_{c\in C} \models \varphi\left(x, \bar{c}\right)[s^{\mathbf{A}^*}(\bar{a})] &\Leftrightarrow \mathbf{A}^* \models \varphi(s, \bar{t})[\bar{a}, \bar{\bar{d}}] \\
&\Leftrightarrow \mathbf{A}^* \models \varphi(s, \bar{t})[\bar{b}, \bar{\bar{d}}] \text{ by indiscernibility} \\
&\Leftrightarrow \langle \mathbf{A}^*, c\rangle_{c\in C} \models \varphi\left(x, \bar{c}\right)[s^{\mathbf{A}^*}(\bar{b})].
\end{aligned}$$

In the step above that uses indiscernibility, we must assemble the tuple of tuples indicated by $\bar{a}, \bar{\bar{d}}$ into one strictly increasing tuple. We must also do the same with $\bar{b}, \bar{\bar{d}}$. It is the definition of $\doteqdot_n$ that ensures that the positions of the a_i and b_i in these extended increasing tuples match up.

So we have

$$\langle \mathbf{A}^*, c\rangle_{c\in C} \models \varphi\left(x, \bar{c}\right)[s^{\mathbf{A}^*}(\bar{a})] \Leftrightarrow \langle \mathbf{A}^*, c\rangle_{c\in C} \models \varphi\left(x, \bar{c}\right)[s^{\mathbf{A}^*}(\bar{b})].$$

What this means is that as long as $\bar{a} \doteqdot_n \bar{b}$, we see that $s^{\mathbf{A}^*}(\bar{a})$ and $s^{\mathbf{A}^*}(\bar{b})$ realize the same complete 1-type with parameters from C. Since there are only countably many $\doteqdot_n$-equivalence classes there are only countably many complete 1-types realized in $\langle \mathbf{A}^*, c\rangle_{c\in C}$ by elements that can be represented using terms $s^{\mathbf{A}^*}(x_0,\dots,x_{n-1})$. By letting n run through the natural numbers, we see that every element of A is represented. But since the union of countably many countable sets is countable, we see that only countably many complete 1-types are realized in $\langle \mathbf{A}^*, c\rangle_{c\in C}$. So $\mathbf{A}^*$ is modest.

It is evident that $\mathbf{A} \models T$ and that $|A| = \kappa$. To see that $\mathbf{A}$ is modest let $C \subseteq A$ with C countable. Let $\mathcal{S}^*$ be the collection of complete 1-types realized in $\langle \mathbf{A}^*, c\rangle_{c\in C}$. Let $\mathcal{S}$ be the collection of complete 1-types realized in $\langle \mathbf{A}, c\rangle_{c\in C}$. We have seen above that $\mathcal{S}^*$ is countable.

Define $F : \mathcal{S}^* \to \mathcal{S}$ by

$$F(\Gamma^*) = \{\varphi(x) : \varphi(x) \text{ is an } L\text{-formula and } \varphi(x) \in \Gamma^*\}$$

It is easy to see $F(\Gamma^*) \in \mathcal{S}$. Also easy to see that F is onto $\mathcal{S}$. So $\mathcal{S}$ is countable and $\mathbf{A}$ is modest. $\square$

8.4 Saturated Models of ω-Stable Theories

Let κ be an infinite cardinal. Recall that a structure $\mathbf{A}$ is **κ-saturated** provided whenever $D \subseteq A$ with $|D| < \kappa$ and $\Gamma(x)$ is a 1-type compatible with $\mathrm{Th}\langle \mathbf{A}, d\rangle_{d\in D}$, then $\Gamma(x)$ is realized in $\langle \mathbf{A}, d\rangle_{d\in D}$. We say that $\mathbf{A}$ is **saturated** if $\mathbf{A}$ is $|A|$-saturated. Recall also that a theory T is **κ-stable** means that for every $\mathbf{A} \models T$ and every $D \subseteq A$ with $|D| \leq \kappa$, we have that $\langle \mathbf{A}, d\rangle_{d\in D}$ realizes exactly κ complete 1-types.

Our aim here is to prove the following

The Characterization Theorem for Categoricity of ω-Stable Theories. *Let T be a complete ω-stable theory in a countable signature and let κ be any infinite cardinal. T is κ categorical if and only if T has a model of cardinality κ and all models of T of cardinality κ are saturated.*

The proof of this theorem that will be given here depends on two other theorems. We deal with these first.

The Stability Theorem. *Let T be a theory of countable signature that has infinite models. If T is ω-stable, then T is κ-stable for every infinite cardinal κ.*

Proof. Suppose not. The pick an infinite cardinal κ, a model $\mathbf{A}$ of T, and a subset $D \subseteq A$ with $|D| \leq \kappa$ so that there are more than κ complete 1-types realized in $\langle \mathbf{A}, d\rangle_{d\in D}$. Call a formula $\theta(x)$ **large** if it is a member of more than κ complete 1-types. The formula $x \approx x$ is an example of a large formula. Suppose $\theta(x)$ is a large formula. Let $\mathcal{Q}$ be the collection of all complete 1-types $\Gamma(x)$ such that there is formula $\psi(x)$ so that $\theta(x) \wedge \psi(x) \in \Gamma(x)$ and $\theta(x) \wedge \psi(x)$ is not large. Now to estimate the size of $\mathcal{Q}$ observe that there are only κ candidates for $\psi(x)$ (since the original signature is countable and there are only $\kappa \geq |D|$ additional constant symbols). Now since $\theta(x) \wedge \psi(x)$ is not large, for each choice of $\psi(x)$ there are at most κ choices for $\Gamma(x)$ that qualify for membership in $\mathcal{Q}$. This means that $|\mathcal{Q}| \leq \kappa^2 = \kappa$. So let $\Gamma(x)$ and $\Psi(x)$ be two distinct complete 1-types to which $\theta(x)$ belongs but which do not lie in $\mathcal{Q}$. Pick $\varphi(x) \in \Gamma(x)$ so that $\varphi(x) \notin \Psi(x)$. Since $\Psi(x)$ is complete, we have that $\neg\varphi(x) \in \Psi(x)$. Since neither $\Gamma(x)$ nor $\Psi(x)$ belong to $\mathcal{Q}$ we deduce that both $\theta(x) \wedge \varphi(x)$ and $\theta(x) \wedge \neg\varphi(x)$ are large. Put $\theta_0(x) = \theta(x) \wedge \varphi(x)$ and $\theta_1(x) = \theta(x) \wedge \neg\varphi(x)$.

So for each large formula $\theta(x)$ we have obtained two large formulas $\theta_0(x)$ and $\theta_1(x)$ such that

1. $\langle \mathbf{A}, d\rangle_{d\in D} \models \forall x[\theta_0(x) \to \theta(x)]$,
2. $\langle \mathbf{A}, d\rangle_{d\in D} \models \forall x[\theta_1(x) \to \theta(x)]$, and
3. $\langle \mathbf{A}, d\rangle_{d\in D} \models \forall x\neg(\theta_0(x) \wedge \theta_1(x))$.

This allows us do build a fully binary tree of large formulas, leading, as in a number of previous arguments, to a violation of the ω-stability of T. □

Theorem on Stability and Saturation. *Let T be an ω-stable theory of countable signature that has infinite models. Let κ and λ be cardinals such that $\omega \le \lambda < \kappa$. Then T has a model of cardinality κ which is λ^+-saturated.*

Proof. Let $\mathbf{A} \models T$ with $|A| = \kappa$. Invoke the Compactness Theorem to get a model $\langle \mathbf{D}, a\rangle_{a\in A} \models \mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A}$ in which every complete 1-type on $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A}$ is realized. By the Stability Theorem we know that T is κ-stable. So the number of complete 1-types realized in $\langle \mathbf{D}, a\rangle_{a\in A}$ is exactly κ. By the Downward Löwenheim-Skolem-Tarski Theorem, there is $\langle \mathbf{A}', a\rangle_{a\in A}$ with cardinality κ and $\langle \mathbf{A}', a\rangle_{a\in A} \preccurlyeq \langle \mathbf{D}, a\rangle_{a\in A}$. Notice all the complete 1-types of $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A}$ are realized in $\langle \mathbf{A}', a\rangle_{a\in A}$. Observe that by Tarski's Criterion for Elementary Substructures, we have $\langle \mathbf{A}, a\rangle_{a\in A} \preccurlyeq \langle \mathbf{A}', a\rangle_{a\in A}$. What we have accomplished is

> Every model $\mathbf{A}$ of T of cardinality κ has an elementary extension $\mathbf{A}'$, also of cardinality κ, such that every complete 1-type of $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A}$ is realized in $\langle \mathbf{A}', a\rangle_{a\in A}$.

We iterate this process λ^+ times to make an elementary chain

$$\mathbf{A} = \mathbf{A}_0 \preccurlyeq \mathbf{A}_1 \preccurlyeq \cdots \preccurlyeq \mathbf{A}_\alpha \preccurlyeq \dots \text{ where } \alpha \text{ is an ordinal and } \alpha \in \lambda^+.$$

Let $\mathbf{B} = \bigcup_{\alpha\in\lambda^+} \mathbf{A}_\alpha$. Then we know the following:

- $\mathbf{A}_\alpha \preccurlyeq \mathbf{B}$ for all $\alpha \in \lambda^+$, by Tarski's Elementary Chain Theorem and
- $|B| = \lambda^+ \cdot \kappa = \kappa$.

So it remains to show that $\mathbf{B}$ is λ^+ saturated.

To this end, let $D \subseteq B = \bigcup_{\alpha\in\lambda^+} A_\alpha$ with $|D| < \lambda^+$. For each $d \in D$ pick $\delta_d \in \lambda^+$ so that $d \in A_{\delta_d}$. In this way no more than λ ordinals δ_d have been selected. Each such ordinal is a set of cardinality at most λ. So $|\bigcup_{d\in D} \delta_d| \le \lambda \cdot \lambda = \lambda < \lambda^+$. So pick $\beta \in \lambda^+$ so that $\delta_d < \beta$ for all $d \in D$. This implies that $D \subseteq A_\beta$. Now let $\Gamma(x)$ be any 1-type of $\mathrm{Th}\langle \mathbf{B}, d\rangle_{d\in D}$. Then $\Gamma(x)$ is also a 1-type of $\mathrm{Th}\langle \mathbf{A}_\beta, d\rangle_{d\in D}$, since $\mathbf{A}_\beta \preccurlyeq \mathbf{B}$. By our construction, $\Gamma(x)$ is realized in $\langle \mathbf{A}_{\beta+1}, d\rangle_{d\in D}$ and hence also in $\langle \mathbf{B}, d\rangle_{d\in D}$. This means $\mathbf{B}$ is λ^+ saturated, as desired. □

Proof of the Characterization of Categoricity for ω-Stable Theories. First notice that if $\kappa = \omega$, then we can appeal the the ω-Categoricity Theorem. So from this point on, we take κ to be uncountable.

The implication from right to left results from the familiar back-and-forth argument we already saw for countable saturated structures. The eager graduate student should take up the task of figuring out how to lift the back-and-forth method into the transfinite. After all, what is the break between semester for anyway.

To see the implication in the other direction suppose that T is κ categorical. By the corollary to the Fist Modesty Theory we know that T is ω-stable. We also know that T has a model of the uncountable cardinality κ. By the Theorem on Stability and Saturation for each infinite $\lambda < \kappa$, the theory T has a model of cardinality κ which is λ^+-saturated. Since all models of T of cardinality κ are isomorphic with each other, we see that the (essentially unique) model $\mathbf{A}$ of T of cardinality κ must be λ^+-saturated for each infinite $\lambda < \kappa$. To see that $\mathbf{A}$ is κ-saturated, let $D \subseteq A$ with $|D| < \kappa$. Put $\lambda = |D|$. Let $\Gamma(x)$ be a complete 1-type for $\mathrm{Th}\langle \mathbf{A}, d\rangle_{d\in D}$. Since $\mathbf{A}$ is λ^+-saturated and $|D| < \lambda^+$, then $\Gamma(x)$ is realized in $\langle \mathbf{A}, d\rangle_{d\in D}$. Hence A is saturated, as desired. □

8.5 The Second Modesty Theorem

Recall that an L-structure $\mathbf{B}$ is a **modest extension** of the structure $\mathbf{A}$ provided

- $\mathbf{A} \preccurlyeq \mathbf{B}$;
- Every 1-type realized in $\langle \mathbf{B}, c\rangle_{c\in C}$ is realized in $\langle \mathbf{A}, a\rangle_{c\in C}$ whenever C is a countable subset of A.

A theory T is **stable in power** α provided for every $\mathbf{A} \models T$ and every set $D \subseteq A$ with $|D| \leq \alpha$, there are exactly α complete 1-types realized in $\langle \mathbf{A}, d\rangle_{d\in D}$.

The Second Modesty Theorem. *Let T be an ω-stable theory of countable signature, let $\mathbf{A}$ be an uncountable model of T, and let κ be a cardinal no smaller than $|A|$. Then $\mathbf{A}$ has a modest extension of cardinality κ.*

Proof. Let $\lambda = |A|$. The idea of the proof is to first see how to get a proper modest extension of $\mathbf{A}$ which still has cardinality λ. We then iterate this construction κ times, forming an elementary chain. At the successor stages we employ the proper extension method and at limit stages we just take the union, invoking Tarski's Elementary Chain Theorem. In the end we form the union one last time to complete the construction.

Let L_A be the signature of the elementary diagram of $\mathbf{A}$. That is L_A is obtained by adding new constants to name the elements of A.

Lemma 8.5.1. *There is an L_A-formula $\theta(x)$ such that*

- *The subset of A defined by $\theta(x)$ is uncountable.*
- *For each L_A-formula $\varphi(x)$ at least one of the sets*

$$\{b \mid \langle \mathbf{A}, a\rangle_{a\in A} \models \theta(b) \wedge \varphi(b)\} \text{ and } \{b \mid \langle \mathbf{A}, a\rangle_{a\in A} \models \theta(b) \wedge \neg\varphi(b)\}$$

  is only countable.

Proof. Suppose not. Then for every $\theta(x)$ such that $\theta^{\mathbf{A}}(x)$ is uncountable we can find $\varphi(x)$ so that both $\theta(x) \wedge \varphi(x)$ and $\theta(x) \wedge \neg\varphi(x)$ define uncountable subsets of A. Put $\theta_0(x)$ equal to $\theta(x) \wedge \varphi(x)$ and put $\theta_1(x)$ equal to $\theta(x) \wedge \neg\varphi(x)$. Then

- Both $\theta_0(x)$ and $\theta_1(x)$ define uncountable subsets of A.

- $\langle \mathbf{A}, a \rangle_{a \in A} \models \forall x[\theta_0(x) \to \theta(x)]$ and $\langle \mathbf{A}, a \rangle_{a \in A} \models \forall x[\theta_1(x) \to \theta(x)]$.
- $\langle \mathbf{A}, a \rangle_{a \in A} \models \forall x \neg[\theta_0(x) \land \theta_1(x)]$.

We can repeat this construction to obtain $\theta_{00}(x)$ and $\theta_{01}(x)$ and likewise $\theta_{10}(x)$ and $\theta_{11}(x)$. In this way we get a binary tree whose first few levels are displayed below.

$$\begin{tikzcd}
& & \theta_{00}(x) \arrow[dl] \\
& \theta_0(x) \arrow[ddl] & \\
& & \theta_{01}(x) \arrow[ul] \\
\theta(x) & & \\
& & \theta_{10}(x) \arrow[dl] \\
& \theta_1(x) \arrow[uul] & \\
& & \theta_{11}(x) \arrow[ul]
\end{tikzcd}$$

This tree has 2^ω branches, each associated with some simply infinite $0,1$-sequence. On the other hand there are only finitely many formulas on any level and only countably many levels. So only countably many formulas appear in this tree. Let C be the set of constants occurring in these formulas. C is countable. The construction ensures that any finite initial segment of any branch is realized in $\langle \mathbf{A}, a \rangle_{a \in A}$. Thus each branch is included in a complete 1-type of $\mathrm{Th}\langle \mathbf{A}, c \rangle_{c \in C}$ by the Compactness Theorem. But different branches are incompatible. So there are uncountably many complete 1-types of $\mathrm{Th}\langle \mathbf{A}, c \rangle_{c \in C}$. By the Compactness Theorem there is a structure in which they are all realized. This violates the ω-stability of T, finishing the proof of the Lemma. $\square$

Now expand the signature by a new constant symbol c. Let

$$\Delta = \{\varphi(c) \mid \theta(x) \land \varphi(x) \text{ defines an uncountable set in } \langle \mathbf{A}, a \rangle_{a \in A}\}.$$

Observe that Δ is itself a complete theory and that $\mathrm{Th}\langle \mathbf{A}, a \rangle_{a \in A} \subseteq \Delta$. Now let $\mathbf{E}$ be a model of Δ and let $\mathbf{E}'$ be the reduct of $\mathbf{E}$ to the original signature. Let F be the set of elements of E named by the constants added to the original signature. So $\mathbf{E}$ looks like $\langle \mathbf{E}', f, c' \rangle_{f \in F}$. According to the Atomic Model Theorem for ω-Stable Theories (proved just below), since T is ω-stable and $\mathbf{E}' \models T$, then $\mathrm{Th}\langle \mathbf{E}', f \rangle_{f \in F}$ has an atomic model $\langle \mathbf{B}, a, c \rangle_{a \in A}$. Because $\mathrm{Th}\langle \mathbf{A}, a \rangle_{a \in A} \subseteq \Delta$, we have that $\mathbf{A} \preccurlyeq \mathbf{B}$. Moreover, $\mathbf{B}$ is a proper extension of $\mathbf{A}$ since $\Delta \models \neg c \approx a$ for all $a \in A$.

Now suppose $\Gamma(x)$ is a countable set of formulas of L_A which is realized in $\langle \mathbf{B}, a \rangle_{a \in A}$. To show that $\mathbf{B}$ is a modest extension of $\mathbf{A}$ we only need to show that $\Gamma(x)$ is realized in $\langle \mathbf{A}, a \rangle_{a \in A}$. Pick $b \in B$ that realizes $\Gamma(x)$. Because $\langle \mathbf{B}, a, c \rangle_{a \in A}$ is atomic there is a complete formula $\varphi(c, x)$ such that $\langle \mathbf{B}, a, c \rangle_{a \in A} \models \varphi(c, b)$. Since Δ is a complete theory, we have

$$\Delta \models \exists x \varphi(c, x) \text{ and } \Delta \models \forall x[\varphi(c, x) \to \psi(x)] \tag{$\star$}$$

for all $\psi(x) \in \Gamma(x)$. Since $\Gamma(x)$ is countable, using the Compactness Theorem we can extract a countable $\Delta_0 \subseteq \Delta$ so that ($\star$) holds with Δ_0 in place of Δ. Now for each $\delta(x)$ so that $\delta(c) \in \Delta_0$, we have that $\delta(x)$ is satisfied by all but countably many elements of A. Since Δ_0 is countable there must be an element $c^* \in A$ such that $\langle \mathbf{A}, a, c^* \rangle_{a \in A} \models \Delta_0$. But then there is $d \in A$ so that $\langle \mathbf{A}, a, c^* \rangle_{a \in A} \models \varphi(c^*, d)$. By the second part of ($\star$), we see that d realizes $\Gamma(x)$.

So $\mathbf{B}$ is a proper modest extension of $\mathbf{A}$. By using the Downward Löwenheim-Skolem-Tarski Thoerem if necessary we can have $|B| = |A|$ as well.

This completes the proof of the Second Modesty Theorem—except we need to prove the Atomic Model Theorem for ω-Stable Theories. □

The Atomic Model Theorem for ω-Stable Theories. *Let T be an ω-stable theory of countable signature, let $\mathbf{A} \models T$, and let $A' \subseteq A$. Then* $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$ *has an atomic model.*

Proof. First we argue that every formula is completeable with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$, whenever $\mathbf{A}$ and A' are chosen as described.

Suppose not. Pick n as small as possible so that some formula $\psi(x_0, \dots, x_{n-1})$ in not completeable. Were $n > 1$ then $\exists x_0[\psi(x_0, x_1, \dots, x_{n-1})]$ could be completed by, say $\varphi(x_1, \dots, x_{n-1})$. Add new constants $c_1, \dots, c_{n-1}$, let $\bar{c}$ denote $\langle c_1, \dots, c_{n-1}\rangle$ and put $T' = \mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'} \cup \{\varphi(\bar{c})\}$. Because $\varphi(x_1, \dots, x_{n-1})$ is a complete formula with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$, we see that T' is a complete theory in the expanded language. We contend that $\psi(x_0, \bar{c})$ is not completeable with respect to T'. For assume $\theta(x_0, \bar{c})$ completed $\psi(x_0, \bar{c})$ with respect to T'. Then we would get

$$\begin{aligned}
\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'} \cup \{\varphi(\bar{c})\} &\models \forall x_0[\theta(x_0, c_1, \dots, c_{n-1}) \to \psi(x_0, \bar{c})]\\
\mathrm{Th}\rangle \mathbf{A}, a\rangle_{a\in A'} &\models \forall x_0[\varphi(\bar{c}) \wedge \theta(x_0, \bar{c}) \to \psi(x_0, \bar{c})]\\
\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'} &\models \forall x_0, \dots, x_{n-1}[\varphi(x_1, \dots, x_{n-1}) \wedge \theta(x_0, \dots, x_{n-1}) \to \psi(x_0, \dots, x_{n-1})]
\end{aligned}$$

But then $\varphi(x_1, \dots, x_{n-1}) \wedge \theta(x_0, \dots, x_{n-1})$ would complete $\psi(x_0, \dots, x_{n-1})$. So the contention that $\psi(x_0, \bar{c})$ is not completeable with respect to T' holds. Let $\mathbf{B}'$ be any model of T' and let Y be the set of elements named by the constants in $A' \cup \{c_1, \dots, c_{n-1}\}$. Let $\mathbf{B}$ be the reduct of $\mathbf{B}'$ to the original signature. Then $\mathbf{B}' = \langle \mathbf{B}, y\rangle_{y\in Y}$. Now since T' is complete, we see that $\psi(x_0, c_1, \dots, c_{n-1})$ is not completeable with respect to $\mathrm{Th}\langle \mathbf{B}, y\rangle_{y\in Y}$. So since n was as small as possible, we see that $n = 1$.

So now suppose that $\psi(x)$ is compatible with $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$ but not completeable with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$. So $\psi(x)$ is not complete. Pick $\varphi(x)$ so that both $\langle \mathbf{A}, a\rangle_{a\in A'} \models \psi(x) \to \varphi(x)$ and $\langle \mathbf{A}, a\rangle_{a\in A'} \models \psi(x) \to \neg\varphi(x)$ fail. Put $\psi_0 = \psi(x) \wedge \varphi(x)$ and $\psi_1 = \psi(x) \wedge \neg\varphi(x)$. Then $\langle \mathbf{A}, a\rangle_{a\in A'} \models \psi_0 \to \psi(x)$ and $\langle \mathbf{A}, a\rangle_{a\in A'} \models \psi_1 \to \psi(x)$ and $\langle \mathbf{A}, a\rangle_{a\in A'} \models \neg(\psi_0 \wedge \psi_1)$. Also both ψ_0 and ψ_1 are compatible with $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$. So once more we can grow a full binary tree and obtain a violation of ω-stability.

So at this point we know that every formula is completeable with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$.

We will extract an elementary substructure of $\langle \mathbf{A}, a\rangle_{a\in A'}$ which is atomic. Let α be the number of formulas in the signature of $\langle \mathbf{A}, a\rangle_{a\in A'}$. We let $\alpha \cdot \omega$ be the ordinal that looks like countably many copies of α arranged one after the other in a sequence of order type ω. We are going to construct a sequence $a_0, a_1, \dots, a_\beta, \dots$ where β ranges through $\alpha \cdot \omega$ that has the following properties:

(1) a_β realizes a complete formula with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\gamma\rangle_{\gamma<\beta, a\in A'}$, for each $\beta < \alpha \cdot \omega$.

(2) For all $n < \omega$ and every formula $\varphi(x)$ that is complete with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\gamma\rangle_{\gamma<\alpha\cdot n, a\in A'}$, there is $\delta < \alpha \cdot (n+1)$ such that a_δ realizes $\varphi(x)$.

We build this sequence recursively. That is, having built an initial segment of our sequence we describe how to get the next bit of it.

So suppose $m < \omega$ and we have built all the a_β's where $\beta < \alpha \cdot m$ so that (1) holds for all $\beta < \alpha \cdot m$ and (2) holds for all $n < m$. Now let $\varphi_\delta(x)$ where $\delta < \alpha$ enumerates all the formulas complete with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\gamma\rangle_{\gamma<\alpha\cdot m, a\in A'}$. For $a_{\alpha\cdot m}$ pick an element of A which satisfies $\varphi_0(x)$. Now observe that $\varphi_1(x)$ is completeable with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\gamma\rangle_{\gamma<\alpha\cdot m+1, a\in A'}$. Let $\varphi_1'(x)$ complete $\varphi_1(x)$. For $a_{\alpha\cdot m+1}$ pick an element of A that satisfies $\varphi_1'(x)$ and hence also $\varphi_1(x)$. We continue in this way to obtain $a_{\alpha\cdot m+\delta}$ for all $\delta < \alpha$ so that (1) holds for all $\beta < \alpha \cdot (m+1)$ and (2) holds for $n = m$. In this way the recursive construction proceeds to completion.

Now let $\langle \mathbf{B}, a\rangle_{a\in A'}$ be the substructure of $\langle \mathbf{A}, a\rangle_{a\in A'}$ with domain $B = \{a_\beta \mid \beta < \alpha\cdot\omega\}$. We note that this set is closed with respect to any basic operation of $\langle \mathbf{A}, a\rangle_{a\in A'}$. For example, suppose $\star$ is a two-place operation and $a_\beta, a_\gamma \in B$. Then the formula $a_\beta \star a_\gamma = x$ is compatible with $\mathrm{Th}\langle \mathbf{A}, a, a_\delta\rangle_{\delta<\eta, a\in A'}$ for some sufficiently large η. Hence the formula is completeable by some $\varphi(x)$. So one of the members of B must satisfy it. This means $a_\beta \star a_\gamma \in B$.

Claim. $\langle \mathbf{B}, a\rangle_{a\in A'} \preccurlyeq \langle \mathbf{A}, a\rangle_{a\in A'}$.

To see this claim, let $\psi(x, a_{\beta_0}, \dots, a_{\beta_{n-1}})$ be a formula which is satisfied in $\langle \mathbf{A}, a\rangle_{a\in A'}$. Using Tarski's Criterion for elementary substructures, it is enough to show it is satisfied in $\langle \mathbf{B}, a\rangle_{a\in A'}$. Evidently this formula is completeable by $\varphi(x)$ with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\gamma\rangle_{\gamma<\eta, a\in A'}$ for some sufficiently large η. By (2) this will be satisfied by some member of B. So will $\psi(x, a_{\beta_0}, \dots, a_{\beta_{n-1}})$. Hence $\langle \mathbf{B}, a\rangle_{a\in A'} \preccurlyeq \langle \mathbf{A}, a\rangle_{a\in A}$, as claimed.

Finally,

Claim. $\langle \mathbf{B}, a\rangle_{a\in A'}$ is atomic.

That is we claim that any n-tuple of elements of B satisfies a formula which is complete with respect to $\mathrm{Th}\langle \mathbf{B}, a\rangle_{a\in A'} (= \mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'})$.

To accomplish this we will argue by transfinite induction on $\beta < \alpha\cdot\omega$ that

> The n-tuple $a_{\delta_0}, \dots, a_{\delta_{n-1}}$ satisfies a complete formula with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$, whenever each $\delta_i < \beta$ and n is a natural number.

Suppose that the statement above holds for all $\beta < \gamma$. If γ is a limit ordinal, then it is easy to see that the statement hold when $\beta = \gamma$. So consider the case when $\gamma = \eta + 1$. What we have to do is to show that any finite tuple of a_δ's, with each $\delta \le \eta$ and one actually η itself, satisfies a complete formula with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$. So let $a_{\delta_0}, \dots, a_{\delta_{n-1}}, a_{\delta_\eta}$ be such a tuple. Now according to (1) we know that a_η satisfies a complete formula $\varphi(x)$ with respect to $\mathrm{Th}\langle \mathbf{A}, a, a_\beta\rangle_{\beta<\eta, a\in A'}$. Let $a_{\lambda_0}, \dots, a_{\lambda_{m-1}}$ be the elements of B that appear as constants in $\varphi(x)$. So we have the formula $\varphi(x, u_0, \dots, u_{m-1})$ of the signature of $\langle \mathbf{A}, a\rangle_{a\in A'}$ so that $\varphi(x) = \varphi(x, a_{\lambda_0}, \dots, a_{\lambda_{m-1}})$.

Now, according to our induction hypothesis, the $n+m$ tuple $a_{\delta_0}, \dots, a_{\delta_{n-1}}, a_{\lambda_0}, \dots, a_{\lambda_{m-1}}$ satisfies a complete formula $\theta(x_0, \dots, x_{n-1}, u_0, \dots, u_{m-1})$ with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$.

For any formula $\psi(x)$ of $\langle \mathbf{A}, a, a_\beta\rangle_{\beta<\eta, a\in A'}$ one of the formulas

$$\forall x[\varphi(x) \to \psi(x)] \text{ and } \forall x[\varphi(x) \to \psi(x)]$$

holds in the model. This entails that for any formula $\psi(x, x_0, \dots, x_{n-1}, u_0, \dots, u_{m-1})$ of $L_{A'}$ either

$$\langle \mathbf{A}, a\rangle_{a\in A'} \models \theta(x_0, \dots, x_{n-1}, u_0, \dots, u_{m-1}) \to \forall x[\varphi(x, u_0, \dots, u_{m-1}) \to \psi]$$

or else

$$\langle \mathbf{A}, a\rangle_{a\in A'} \models \theta(x_0, \ldots, x_{n-1}, u_0, \ldots, u_{m-1}) \to \forall x[\varphi(x, u_0, \ldots, u_{m-1}) \to \neg\psi]$$

Using a bit of logical rearranging we find that either

$$\langle \mathbf{A}, a\rangle_{a\in A'} \models \forall x, \bar{x}, \bar{u}[\theta(\bar{x}, \bar{u}) \wedge \varphi(x, \bar{u}) \to \psi(x, \bar{x}, \bar{u})]$$

or else

$$\langle \mathbf{A}, a\rangle_{a\in A'} \models \forall x, \bar{x}, \bar{u}[\theta(\bar{x}, \bar{u}) \wedge \varphi(x, \bar{u}) \to \neg\psi(x, \bar{x}, \bar{u})]$$

This means that the formula $\theta(\bar{x}, \bar{u}) \wedge \varphi(x, \bar{u})$ is complete with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$. It follows that the formula $\exists \bar{u}\theta(\bar{x}, \bar{u}) \wedge \varphi(x, \bar{u})$ is complete with respect to $\mathrm{Th}\langle \mathbf{A}, a\rangle_{a\in A'}$ as well. But this formula is satisfied by the tuple $a_{\delta_0}, \ldots, a_{\delta_{n-1}}, a_\eta$, as desired. Hence, $\langle \mathbf{B}, a\rangle_{a\in A'}$ is atomic. □

Morley's Categoricity Theorem became a turning point in the development of model theory. On the one hand, it's proof is an extended application of many of the concepts and tools developed in model theory up to 1960. On the other hand, the new notions involved in its various proofs led to the flowering of model theory. It also held out the possibility that the ideas and methods of model theory could be used to open a way forward in parts of mathematics that have close connections to algebraically closed fields. Indeed, just such an enterprise has gotten well under way since the 1990's.

While many mathematicians have made spectacular contributions in the flowering of model theory in the decades following Morley's work, there is one who has been the driving force and who has been a real fountain of ideas, beginning with his extension of Morley's Theorem in uncountable signatures. Just as Alfred Tarski was the first mathematician named in these notes, it is entirely suitable that this man who has led the way forward since 1970 should be the last one named: Saharon Shelah.

APPENDIX A

Ramsey's Theorem

In 1928 Frank P. Ramsey, then 25 years old, found an algorithm for determining whether a universal sentence without operation symbols has a model. His reasoning was based on a combinatorial result that Ramsey realized was of independent interest. This result, Ramsey's Theorem, has indeed become a central result in combinatorics. Ramsey died of liver failure before his paper was published in 1930.

Ramsey's Theorem has many proofs has well as many generalizations—there is the whole mathematical field of partition relations, for example. The proof given here traces its heritage back to Stephen Simpson in 1970, then a graduate student at MIT working under the supervision of Gerald Sacks.

Ramsey's Theorem, Infinite Form. *For any positive natural numbers n and k and any infinite set X, if the collection of all n-element subsets of X is partitioned into k blocks, then there is an infinite $Y \subseteq X$ so that every n-element subset of Y belongs to a single block of the partition.*

Proof. Every infinite set has a countably infinite subset, so it is clear that Ramsey's Theorem would follow once we establish it is the special case when $X = \omega$. We demonstrate this special case.

The proof is by induction on n.

Base Step: $n = 1$

Suppose we have partitioned the singleton subsets of ω into k blocks. Because the union of k finite sets is finite but the collection of singletons in countably infinite, on of the blocks is infinite. Forming the union of the singletons in this infinite block produces the desired set Y.

Inductive Step: $n = m + 1$

Our inductive hypothesis is that Ramsey's Theorem holds with m in place of n.

Let us try to express the hypotheses of Ramsey's Theorem with elementary sentences. Knowing the connection between partitions and functions, we provide ourselves with an n-place operation $Q^{\mathbf{A}} : \omega^n \to \omega$ so that the values produced by $Q^{\mathbf{A}}$ are $0, 1, \ldots, k-1$ and the blocks of the partition are the sets $\{\langle a_0, \ldots, a_{n-1}\rangle \mid a_0, \ldots, a_{n-1} \in \omega \text{ and } Q^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = j\}$, for $j = 0, \ldots, k-1$. There is a small trouble with this idea: the n-tuple $\langle a_0, \ldots, a_{n-1}\rangle$ is *not* the set $\{a_0, \ldots, a_{n-1}\}$, and even if it were, this set might have fewer than n elements. To handle this difficulty, we employ the usual ordering of the natural numbers. A set of size n can the associated with the strictly increasing n-tuple made by arranging its elements in strictly ascending order. Since we want $Q^{\mathbf{A}}$ to be defined for all n-tuples, we use k as a default value by insisting that $Q^{\mathbf{A}}(\bar{a}) = k$ whenever $\bar{a}$ is an n-tuple that is not strictly increasing.

Using this device, we can take any partition of the n-element subsets of ω and construct a structure $\mathbf{A} = \langle \omega, Q^{\mathbf{A}}, <, \ell\rangle_{\ell \in \omega}$ from which the partition can be recovered. The signature of this structure provides one n-place operation symbol Q, one 2-place relation symbol $\sqsubset$ to name $<$, and a countably infinite list $c_0, c_1, c_2, \ldots$ of constant symbols to name the elements of ω.

We can express, with elementary sentences, all of the following

- $\sqsubset$ is a linear ordering.
- The constant symbols are ordered in the expected way.
- The predecessors of the constants are as expected: e.g.

$$\forall x\, [x \sqsubset c_r \to (x \approx c_0 \vee \cdots \vee x \approx c_{r-1})]$$

- The values produced by Q are as expected. We have to express three things: all k of the desired values are achieved (so no block of the partition is empty), all n-tuples that are not strictly increasing are assigned to default value, and that no other values are assigned.

The details of how to express these things with elementary sentences are left in the sure hands of the graduate students.

Since $\mathbf{A}$ is infinite, it has a proper elementary extension $\mathbf{B}$, as a by now routine use of the Compactness Theorem shows. Pick $b \in B$ so that $b \notin A$. Because the constant symbols name the elements of ω and we have insisted that predecessors of named elements are also named, we see that in $\mathbf{B}$ the element b must lie above all the elements of ω.

We are going to construct an infinite set $W \subseteq \omega$ so that

$$Q^{\mathbf{B}}(w_0, \ldots, w_{m-1}, w_m) = Q^{\mathbf{B}}(w_0, \ldots, w_{m-1}, b), \tag{$\circledcirc$}$$

for any $w_0 < \cdots < w_m \in W$. We construct W is stages, adding one element to get to the next stage each time. To begin, let $W_0 = \{0, \ldots, m-1\}$. Suppose W_r has been constructed so that ($\circledcirc$) holds for elements drawn from W_r.

For any increasing m-tuple $\bar{w}$ from W_r, let $\bar{c}_{\bar{w}}$ denote the corresponding m-tuple of constant symbol and let $d_{\bar{w}}$ be the constant symbol denoting $Q^{\mathbf{B}}(\bar{w}, b)$. Then ($\circledcirc$) can be rendered as

$$\mathbf{B} \models \Big(Q\bar{c}_{\bar{w}}x \approx d_{\bar{w}}\Big)[w_m]. \tag{$\odot$}$$

Let $\varphi(x)$ be the conjunction of the following formulas:

$$\begin{array}{rl} c_m \sqsubset x & \text{for all } m \in W_r \\ Q\bar{c}_{\bar{w}}x \approx d_{\bar{w}} & \text{for all increasing } m\text{-tuples } \bar{w} \text{ of elements of } W_r \end{array}$$

Evidently $\mathbf{B} \models \varphi(x)[b]$. So $\mathbf{B} \models \exists x \varphi(x)$. But then $\mathbf{A} \models \exists x \varphi(x)$. Let $w \in \omega$ satisfy $\varphi(x)$ in $\mathbf{A}$. Put $W_{r+1} = W_r \cup \{w\}$. So ($\circledcirc$) holds for W_{r+1}.

Now just let $W = \bigcup_{r \in \omega} W_r$. Because $Q^{\mathbf{B}}(w_0, \ldots, w_{m-1}, w_m) = Q^{\mathbf{A}}(w_0, \ldots, w_{m-1}, w_m)$, we see from ($\circledcirc$), that on increasing n-tuples from W, the value of $Q^{\mathbf{A}}$ does not depend on the last entry in the n-tuple. Recalling that $n = m + 1$ we see that $Q^{\mathbf{A}}$ induces a partition on the increasing m-tuple of elements of W into finitely many (no more than k) blocks. According to our inductive hypothesis there is an infinite subset $Y \subseteq W \subseteq X$ so that all the increasing m-tuples of elements of Y belong to a single block of the partition. Because $Q^{\mathbf{A}}(y_0, \ldots, y_{m-1}, y_m)$ does not depend on y_m as long as $y_0 < y_1 < \cdots < y_m$ and $y_j \in Y$ for $j < m + 1 = n$, we conclude that all the increasing n-tuples of elements of Y belong to a single block of our original partition. This finishes the inductive step and the proof. □

Here is one way to construe the infinite version of Ramsey's Theorem in the case $n = 2$. Think of the infinite set X as a set of vertices and the two-elements subsets of X as potential edges. Any given graph on the vertex set X amounts to a partition of the two-element subsets of X into two blocks: those two-element sets that are actually edges of the given graph belong to one block and those two-element sets that are not edges of the graph belong to the other block. Then Ramsey's Theorem says that either the graph an infinite complete induced subgraph (an infinite clique) or the graph has an infinite independent set of vertices.

Ramsey needed a finite version of his theorem to obtain the algorithm he sought. Here is the finite version.

Ramsey's Theorem, Finite Form. *For any positive natural numbers n, k and ℓ there is a positive natural number r such that for any set X of cardinality at least r, if the collection of all n-element subsets of X is partitioned into k blocks, then there is a $Y \subseteq X$, with $|Y| \geq \ell$, so that every n-element subset of Y belongs to a single block of the partition.*

Proof. Our idea is to use the Compactness Theorem as a link between the finite and the infinite.

Fix the positive natural numbers n, k, and ℓ. Were the Finite Form of Ramsey's Theorem to fail for this choice of n, k, and ℓ, it would mean that for arbitrarily large finite sets X the n-element subsets of X can be partitioned into k blocks in such a way that if $Y \subseteq X$ and all the n-element subsets of Y belong to the same block of the partition, then $|Y| < \ell$. We can express this by a set Γ of elementary sentences of suitable signature.

As in the proof of the Infinite Form, we employ a signature supplied with an n-place operation symbol Q, a binary relation symbol $\sqsubset$. We also need $k + 1$ constant symbols $c_0, \ldots, c_k$. The set Γ should contain sentences asserting each of the following:

(a) That $\sqsubset$ is a linear order;

(b) That Q sends each nonincreasing n-tuple to the default value denoted by c_k;

(c) That Q sends each increasing n-tuple to one of the values denoted by $c_0, \ldots, c_{k-1}$;

(d) That among any given ℓ distinct elements one can find two increasing n-tuples that Q sends to different values.

The sentences required by (d) did not arise in the proof of the Infinite Form. To see how they might be devised, consider the case when $n = 3$ and $\ell = 5$. The sentence we want is

$$\forall x_0, \ldots, x_4 \, [(x_0 \sqsubset x_1 \wedge \cdots \wedge x_3 \sqsubset x_4) \rightarrow \theta]$$

where θ is the disjunction of all formulas of the form $\neg Q y_0 y_1 y_2 \approx Q z_0 z_1 z_2$ where y_0, y_1, y_2 are selected, in order from $x_0, \ldots, x_4$ and the same applies to z_0, z_1, z_2.

According to the Infinite Form of Ramsey's Theorem, Γ cannot have an infinite model. By a corollary of the Compactness Theorem, Γ cannot have arbitrarily large finite models. This means that there is a natural number r so that any model of (a)–(c) of cardinality at least r, the sentence (d) must fail. This is the Finite Form of Ramsey's Theorem. $\square$

Bibliography

Beth, Evert W.

(1953) “On Padoa’s method in the theory of definition”. In: *Nederl. Akad. Wetensch. Proc. Ser. A.* ***56*** *= Indagationes Math.* 15, pp. 330–339.

Craig, William

(1957a) “Linear reasoning. A new form of the Herbrand-Gentzen theorem”. In: *J. Symb. Logic* 22, pp. 250–268. ISSN: 0022-4812.

(1957b) “Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory”. In: *J. Symb. Logic* 22, pp. 269–285. ISSN: 0022-4812.

Ehrenfeucht, A. and A. Mostowski

(1956) “Models of axiomatic theories admitting automorphisms”. In: *Fund. Math.* 43, pp. 50–68. ISSN: 0016-2736.

Grzegorczyk, A., A. Mostowski, and C. Ryll-Nardzewski

(1958) “The classical and the ω-complete arithmetic”. In: *J. Symb. Logic* 23, pp. 188–206. ISSN: 0022-4812.

(1961) “Definability of sets in models of axiomatic theories”. In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 9, pp. 163–167. ISSN: 0001-4117.

Henkin, Leon

(1947) “The Completeness of Formal Systems”. Ph.D. dissertation. Princeton University.

(1949a) “Fragments of the propositional calculus”. In: *J. Symbolic Logic* 14, pp. 42–48. ISSN: 0022-4812.

(1949b) “The completeness of the first-order functional calculus”. In: *J. Symbolic Logic* 14, pp. 159–166. ISSN: 0022-4812.

(1954) “A generalization of the concept of ω-consistency”. In: *J. Symbolic Logic* 19, pp. 183–196. ISSN: 0022-4812.

Łoś, Jerzy

(1954) “On the categoricity in power of elementary deductive systems and some related problems”. In: *Colloquium Math.* 3, pp. 58–62.

Łoś, Jerzy

(1955) “Quelques remarques, théorèmes et problèmes sur les classes définissables d’algèbres”. In: *Mathematical interpretation of formal systems*. Amsterdam: North-Holland Publishing Co., pp. 98–113.

Löwenheim, Leopold

(1915) “Über Möglichkeiten im Relativkalkül”. In: *Math. Ann.* 76.4, pp. 447–470. ISSN: 0025-5831. DOI: `10.1007/BF01458217`. URL: `http://dx.doi.org/10.1007/BF01458217`.

Morley, Michael

(1962) “Categoricity in Power”. Ph.D. dissertation. University of Chicago.

(1965) “Categoricity in power”. In: *Trans. Amer. Math. Soc.* 114, pp. 514–538. ISSN: 0002-9947.

Morley, Michael and Robert L. Vaught

(1962) “Homogeneous universal models”. In: *Math. Scand.* 11, pp. 37–57. ISSN: 0025-5521.

Orey, Steven

(1956) “On ω-consistency and related properties”. In: *J. Symb. Logic* 21, pp. 246–252. ISSN: 0022-4812.

Ramsey, Frank Plumpton

(1960) *The foundations of mathematics and other logical essays*. Edited by R. B. Braithwaite with a preface by G. E. Moore. International Library of Psychology, Philosophy and Scientific Method. Paterson, N.J.: Littlefield Adams & Co., pp. xviii+292.

Robinson, Abraham

(1951) *On the metamathematics of algebra*. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Publishing Co., pp. ix+195.

(1956) “A result on consistency and its application to the theory of definition”. In: *Nederl. Akad. Wetensch. Proc. Ser. A.* ***59*** *= Indag. Math.* 18, pp. 47–58.

Ryll-Nardzewski, Czseslaw

(1959) “On the categoricity in power $\leq \aleph_0$”. In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 7, 545–548. (unbound insert). ISSN: 0001-4117.

Svenonius, Lars

(1959a) “A theorem on permutations in models”. In: *Theoria (Lund)* 25, pp. 173–178. ISSN: 0040-5825.

(1959b) “$\aleph_0$-categoricity in first-order predicate calculus”. In: *Theoria (Lund)* 25, pp. 82–94. ISSN: 0040-5825.

Tarski, Alfred

(1948) *A Decision Method for Elementary Algebra and Geometry.* RAND Corporation, Santa Monica, Calif., pp. iii+60.

(1951) *A decision method for elementary algebra and geometry.* 2nd ed. Berkeley and Los Angeles, Calif.: University of California Press, pp. iii+63.

(1952) "Some notions and methods on the borderline of algebra and metamathematics". In: *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1.* Providence, R. I.: Amer. Math. Soc., pp. 705–720.

(1954a) "Contributions to the theory of models. I". In: *Nederl. Akad. Wetensch. Proc. Ser. A.* 57, 572–581 = Indagationes Math. 16, 572–581 (1954).

(1954b) "Contributions to the theory of models. II". In: *Nederl. Akad. Wetensch. Proc. Ser. A.* 57, 582–588 = Indagationes Math. 16, 582–588 (1954).

(1955) "Contributions to the theory of models. III". In: *Nederl. Akad. Wetensch. Proc. Ser. A.* 58, 56–64 = Indagationes Math. 17, 56–64 (1955).

(1956) *Logic, semantics, metamathematics. Papers from 1923 to 1938.* Translated by J. H. Woodger. Oxford at the Clarendon Press, pp. xiv+471.

Tarski, Alfred and Robert L. Vaught

(1958) "Arithmetical extensions of relational systems". In: *Compositio Math* 13, pp. 81–102. ISSN: 0010-437X.

Vaught, Robert L.

(1954) "Topics in the Theory of Arithmetical Classes and Boolean Algebras". Ph.D. dissertation. University of California, Berkeley, 96 pp. ISBN: 0198119.

(1961) "Denumerable models of complete theories". In: *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959).* Oxford: Pergamon, pp. 303–321.

(1965) "A Löwenheim-Skolem theorem for cardinals far apart". In: *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley).* Amsterdam: North-Holland, pp. 390–401.

Index