Fundamentals of Model Theory

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Introduction

Model Theory is the part of mathematics which shows how to apply logic to the study of structures in pure mathematics. On the one hand it is the ultimate abstraction; on the other, it has immediate applications to every-day mathematics. The fundamental tenet of Model Theory is that mathematical truth, like all truth, is relative. A statement may be true or false, depending on how and where it is interpreted. This isn't necessarily due to mathematics itself, but is a consequence of the language that we use to express mathematical ideas.

What at first seems like a deficiency in our language, can actually be shaped into a powerful tool for understanding mathematics. This book provides an introduction to Model Theory which can be used as a text for a reading course or a summer project at the senior undergraduate or graduate level. It is also a primer which will give someone a self contained overview of the subject, before diving into one of the more encyclopedic standard graduate texts.

Any reader who is familiar with the cardinality of a set and the algebraic closure of a field can proceed without worry. Many readers will have some acquaintance with elementary logic, but this is not absolutely required, since all necessary concepts from logic are reviewed in Chapter 0. Chapter 1 gives the motivating examples; it is short and we recommend that you peruse it first, before studying the more technical aspects of Chapter 0. Chapters 2 and 3 are selections of some of the most important techniques in Model Theory. The remaining chapters investigate the relationship between Model Theory and the algebra of the real and complex numbers. Thirty exercises develop familiarity with the definitions and consolidate understanding of the main proof techniques.

Throughout the book we present applications which cannot easily be found elsewhere in such detail. Some are chosen for their value in other areas of mathematics: Ramsey's Theorem, the Tarski-Seidenberg Theorem. Some are chosen for their immediate appeal to every mathematician: existence of infinitesimals for calculus, graph colouring on the plane. And some, like Hilbert's Seventeenth Problem, are chosen because of how amazing it is that logic can play an important role in the solution of a problem from high school algebra. In each case, the derivation is shorter than any which tries to avoid logic. More importantly, the methods of Model Theory display clearly the structure of the main ideas of the proofs, showing how theorems of logic combine with theorems from other areas of mathematics to produce stunning results.

The theorems here are all are more than thirty years old and due in great part to the cofounders of the subject, Abraham Robinson and Alfred Tarski. However, we have not attempted to give a history. When we attach a name to a theorem, it is simply because that is what mathematical logicians popularly call it.

The bibliography contains a number of texts that were helpful in the preparation of this manuscript. They could serve as avenues of further study and in addition, they contain many other references and historical notes. The more recent titles were added to show the reader where the subject is moving today. All are worth a look.

This book began life as notes for William Weiss's graduate course at the University of Toronto. The notes were revised and expanded by Cherie D'Mello and

Contents

CHAPTER 0

Models, Truth and Satisfaction

We will use the following symbols:

- logical symbols:
  - the connectives $\wedge$,$\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$ called "and", "or", "not", "implies" and "iff" respectively
  - the quantifiers $\forall$, $\exists$ called "for all" and "there exists"
  - an infinite collection of variables indexed by the natural numbers $\mathbb{N}$ v_0 ,v_1 , v_2 , ...
  - the two parentheses), (
  - the symbol = which is the usual "equal sign"
- constant symbols : often denoted by the letter c with subscripts
- function symbols : often denoted by the letter F with subscripts; each function symbol is an m-placed function symbol for some natural number $m \geq 1$
- relation symbols : often denoted by the letter R with subscripts; each relational symbol is an n-placed relation symbol for some natural number $n \geq 1$.

We now define terms and formulas.

DEFINITION 1. A *term* is defined as follows:

(1) a variable is a term
(2) a constant symbol is a term
(3) if F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms, then $F(t_1 \ldots t_m)$ is a term.
(4) a string of symbols is a term if and only if it can be shown to be a term by a finite number of applications of (1), (2) and (3).

REMARK. This is a recursive definition.

DEFINITION 2. A *formula* is defined as follows :

(1) if t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula.
(2) if R is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms, then $(R(t_1 \ldots t_n))$ is a formula.
(3) if φ is a formula, then $(\neg\varphi)$ is a formula
(4) if φ and ψ are formulas then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$
(5) if v_i is a variable and φ is a formula, then $(\exists v_i)\varphi$ and $(\forall v_i)\varphi$ are formulas
(6) a string of symbols is a formula if and only if it can be shown to be a formula by a finite number of applications of (1), (2), (3), (4) and (5).

REMARK. This is another recursive definition. $\neg\varphi$ is called the negation of φ; $\varphi \wedge \psi$ is called the conjunction of φ and ψ; and $\varphi \vee \psi$ is called the disjunction of φ and ψ.

DEFINITION 3. A *subformula* of a formula φ is defined as follows:

(1) φ is a subformula of φ
(2) if $(\neg\psi)$ is a subformula of φ then so is ψ
(3) if any one of $(\theta \wedge \psi)$, $(\theta \vee \psi)$, $(\theta \rightarrow \psi)$ or $(\theta \leftrightarrow \psi)$ is a subformula of φ , then so are both θ and ψ
(4) if either $(\exists v_i)\psi$ or $(\forall v_i)\psi$ is a subformula of φ for some natural number i , then ψ is also a subformula of φ
(5) A string of symbols is a subformula of φ, if and only if it can be shown to be such by a finite number of applications of (1), (2), (3) and (4).

DEFINITION 4. A variable v_i is said to occur *bound* in a formula φ iff for some subformula ψ of φ either $(\exists v_i)\psi$ or $(\forall v_i)\psi$ is a subformula of φ. In this case each occurence of v_i in $(\exists v_i)\psi$ or $(\forall v_i)\psi$ is said to be a *bound occurence* of v_i. Other occurences of v_i which do not occur bound in φ are said to be *free*.

EXAMPLE 1.

$$F(v_3) \text{ is a term.}$$
$$(((\exists v_3)(v_0 = v_3) \vee (\exists v_2)(v_1 = v_2)) \wedge (\forall v_0)(v_0 = v_0))$$

is a formula. In this formula the variables v_2 and v_3 occurs bound, the variable v_1 occurs free, but the variable v_0 occurs both bound and free.

EXERCISE 1. Using the previous definitions as a guide, define the substitution of a term t for a variable v_i in a formula φ. In particular, demonstrate how to substitute the term for the variable v_0 in the formula of the example above.

DEFINITION 5. A *language* $\mathcal{L}$ is a set consisting of all the logical symbols with perhaps some constant, function and/or relational symbols included. It is understood that the formulas of $\mathcal{L}$ are made up from this set in the manner prescribed above. Note that all the formulas of $\mathcal{L}$ are uniquely described by listing only the constant, function and relation symbols of $\mathcal{L}$.

We use $t(v_0, \ldots, v_k)$ to denote a term t all of whose variables occur among $v_0, \ldots, v_k$.

We use $\varphi(v_0, \ldots, v_k)$ to denote a formula φ all of whose free variables occur among $v_0, \ldots, v_k$.

EXAMPLE 2. These would be formulas of any language :

- For any variable v_i: $(v_i = v_i)$
- for any term $t(v_0, \ldots, v_k)$ and other terms t_1 and t_2:

$$((t_1 = t_2) \rightarrow (t(v_0, \ldots, v_{i-1}, t_1, v_{i+1}, \ldots, v_k) = t(v_0, \ldots, v_{i-1}, t_2, v_{i+1}, \ldots, v_k)))$$

- for any formula $\varphi(v_0, \ldots, v_k)$ and terms t_1 and t_2:

$$((t_1 = t_2) \rightarrow (\varphi(v_0, \ldots, v_{i-1}, t_1, v_{i+1}, \ldots, v_k) \leftrightarrow \varphi(v_0, \ldots, v_{i-1}, t_2, v_{i+1}, \ldots, v_k)))$$

Note the simple way we denote the substitution of t_1 for v_i.

DEFINITION 6. A *model* (or structure) $\mathfrak{A}$ for a language $\mathcal{L}$ is an ordered pair $\langle \mathbf{A}, \mathcal{I} \rangle$ where $\mathbf{A}$ is a nonempty set and $\mathcal{I}$ is an *interpretation function* with domain the set of all constant, function and relation symbols of $\mathcal{L}$ such that:

(1) if c is a constant symbol, then $\mathcal{I}(c) \in \mathbf{A}$; $\mathcal{I}(c)$ is called a constant
(2) if F is an m-placed function symbol, then $\mathcal{I}(F)$ is an m-placed function on $\mathbf{A}$

(3) if R is an n-placed relation symbol, then $\mathcal{I}(R)$ is an n-placed relation on $\mathbf{A}$.

$\mathbf{A}$ is called the *universe* of the model $\mathfrak{A}$. We generally denote models with Gothic letters and their universes with the corresponding Latin letters in boldface. One set may be involved as a universe with many different interpretation functions of the language $\mathcal{L}$. The model is *both* the universe *and* the interpretation function.

Remark. The importance of Model Theory lies in the observation that mathematical objects can be cast as models for a language. For instance, the real numbers with the usual ordering $<$ and the usual arithmetic operations, addition $+$ and multiplication $\cdot$ along with the special numbers $\mathbf{0}$ and $\mathbf{1}$ can be described as a model. Let $\mathcal{L}$ contain one two-placed (i.e. binary) relation symbol R_0, two two-placed function symbols F_1 and F_2 and two constant symbols c_0 and c_1. We build a model by letting the universe $\mathbf{A}$ be the set of real numbers. The interpretation function $\mathcal{I}$ will map R_0 to $<$, i.e. R_0 will be interpreted as $<$. Similarly, $\mathcal{I}(F_1)$ will be $+$, $\mathcal{I}(F_2)$ will be $\cdot$, $\mathcal{I}(c_0)$ will be $\mathbf{0}$ and $\mathcal{I}(c_1)$ will be $\mathbf{1}$. So $\langle \mathbf{A}, \mathcal{I}\rangle$ is an example of a model for the language described by $\{R_0, F_1, F_2, c_0, c_1\}$.

We now wish to show how to use formulas to express mathematical statements about elements of a model. We first need to see how to interpret a term in a model.

Definition 7. The value $t[x_0, \ldots, x_q]$ of a term $t(v_0, \ldots, v_q)$ at $x_0, \ldots, x_q$ in the universe $\mathbf{A}$ of the model $\mathfrak{A}$ is defined as follows:

(1) if t is v_i then $t[x_0, \cdots, x_q]$ is x_i,
(2) if t is the constant symbol c, then $t[x_0, \ldots, x_q]$ is $\mathcal{I}(c)$, the interpretation of c in $\mathbf{A}$,
(3) if t is $F(t_1 \ldots t_m)$ where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms, then $t[x_0, \ldots, x_q]$ is $G(t_1[x_0, \ldots, x_q], \ldots, t_m[x_0, \ldots, x_q])$ where G is the m-placed function $\mathcal{I}(F)$, the interpretation of F in $\mathbf{A}$.

Definition 8. Suppose $\mathfrak{A}$ is a model for a language $\mathcal{L}$. The sequence $x_0, \ldots, x_q$ of elements of $\mathbf{A}$ *satisfies* the formula $\varphi(v_0, \ldots, v_q)$ all of whose free and bound variables are among $v_0, \ldots, v_q$, in the model $\mathfrak{A}$, written $\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ provided we have:

(1) if $\varphi(v_0, \ldots, v_q)$ is the formula $(t_1 = t_2)$, then

$\mathfrak{A} \models (t_1 = t_2)[x_0, \ldots, x_q]$ means that $t_1[x_0, \ldots, x_q]$ equals $t_2[x_0, \ldots, x_q]$,

(2) if $\varphi(v_0, \ldots, v_q)$ is the formula $(R(t_1 \ldots t_n))$ where R is an n-placed relation symbol, then

$\mathfrak{A} \models (R(t_1 \ldots t_n))[x_0, \ldots, x_q]$ means $S(t_1[x_0, \ldots, x_q], \ldots, t_n[x_0, \ldots, x_q])$

where S is the n-placed relation $\mathcal{I}(R)$, the interpretation of R in $\mathbf{A}$,

(3) if φ is $(\neg\theta)$, then

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_q] \text{ means not } \mathfrak{A} \models \theta[x_0, \ldots, x_q],$$

(4) if φ is $(\theta \wedge \psi)$, then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means both $\mathfrak{A} \models \theta[x_0, \ldots, x_q]$ and $\mathfrak{A} \models \psi[x_0, \ldots x_q]$,

(5) if φ is $(\theta \vee \psi)$ then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means either $\mathfrak{A} \models \theta[x_0, \ldots, x_q]$ or $\mathfrak{A} \models \psi[x_0, \ldots, x_q]$,

(6) if φ is $(\theta \to \psi)$ then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means that $\mathfrak{A} \models \theta[x_0, \ldots, x_q]$ implies $\mathfrak{A} \models \psi[x_0, \ldots, x_q]$,

(7) if φ is $(\theta \leftrightarrow \psi)$ then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means that $\mathfrak{A} \models \theta[x_0, \ldots, x_q]$ iff $\mathfrak{A} \models \psi[x_0, \ldots, x_q]$,

(8) if φ is $\forall v_i \theta$, then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means for every $x \in \mathbf{A}, \mathfrak{A} \models \theta[x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_q]$,

(9) if φ is $\exists v_i \theta$, then

$\mathfrak{A} \models \varphi[x_0, \ldots, x_q]$ means for some $x \in \mathbf{A}, \mathfrak{A} \models \theta[x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_q]$.

Exercise 2. Each of the formulas of Example 2 is satisfied in any model $\mathfrak{A}$ for any language $\mathcal{L}$ by any (long enough) sequence $x_0, x_1, \ldots, x_q$ of $\mathbf{A}$. This is where you test your solution to Exercise 1.

We now prove two lemmas which show that the preceding concepts are well-defined. In the first one, we see that the value of a term only depends upon the values of the variables which actually occur in the term. In this lemma the equal sign = is used, not as a logical symbol in the formal sense, but in its usual sense to denote equality of mathematical objects — in this case, the values of terms, which are elements of the universe of a model.

Lemma 1. *Let $\mathfrak{A}$ be a model for $\mathcal{L}$ and let $t(v_0, \ldots, v_p)$ be a term of $\mathcal{L}$. Let $x_0, \ldots, x_q$ and $y_0, \ldots, y_r$ be sequences from $\mathbf{A}$ such that $p \leq q$ and $p \leq r$, and let $x_i = y_i$ whenever v_i actually occurs in $t(v_0, \ldots, v_p)$. Then*

$$t[x_0, \ldots, x_q] = t[y_0, \ldots, y_r]$$

.

Proof. We use induction on the complexity of the term t.

(1) If t is v_i then $x_i = y_i$ and so we have

$$t[x_0, \ldots, x_q] = x_i = y_i = t[y_0, \ldots, y_r] \text{ since } p \leq q \text{ and } p \leq r.$$

(2) If t is the constant symbol c, then

$$t[x_0, \ldots, x_q] = \mathcal{I}(c) = t[y_0, \ldots, y_r]$$

where $\mathcal{I}(c)$ is the interpretation of c in $\mathbf{A}$.

(3) If t is $F(t_1 \ldots t_m)$ where F is an m-placed function symbol, $t_1, \ldots, t_m$ are terms and $\mathcal{I}(F) = G$, then
$t[x_0, \ldots, x_q] = G(t_1[x_0, \ldots, x_q], \ldots, t_m[x_0, \ldots, x_q])$ and
$t[y_0, \ldots, y_r] = G(t_1[y_0, \ldots, y_r], \ldots, t_m[y_0, \ldots, y_r])$.
By the induction hypothesis we have that $t_i[x_0, \ldots, x_q] = t_i[y_0, \ldots, y_r]$ for $1 \leq i \leq m$ since $t_1, \ldots, t_m$ have all their variables among $\{v_0, \ldots, v_p\}$. So we have $t[x_0, \ldots, x_q] = t[y_0, \ldots, y_r]$.

□

In the next lemma the equal sign = is used in both senses — as a formal logical symbol in the formal language $\mathcal{L}$ and also to denote the usual equality of mathematical objects. This is common practice where the context allows the reader to distinguish the two usages of the same symbol. The lemma confirms that satisfaction of a formula depends only upon the values of its free variables.

LEMMA 2. *Let $\mathfrak{A}$ be a model for $\mathcal{L}$ and φ a formula of $\mathcal{L}$, all of whose free and bound variables occur among $v_0, \dots, v_p$. Let $x_0, \dots, x_q$ and $y_0, \dots, y_r$ $(q, r \geq p)$ be two sequences such that x_i and y_i are equal for all i such that v_i occurs free in φ. Then*

$$\mathfrak{A} \models \varphi[x_0, \dots, x_q] \textit{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]$$

PROOF. Let $\mathfrak{A}$ and $\mathcal{L}$ be as above. We prove the lemma by induction on the complexity of φ.

(1) If $\varphi(v_0, \dots, v_p)$ is the formula $(t_1 = t_2)$, then we use Lemma 1 to get:

$$\begin{aligned} \mathfrak{A} \models (t_1 = t_2)[x_0, \dots, x_q] &\text{ iff } t_1[x_0, \dots, x_q] = t_2[x_0, \dots, x_q] \\ &\text{ iff } t_1[y_0, \dots, y_r] = t_2[y_0, \dots, y_r] \\ &\text{ iff } \mathfrak{A} \models (t_1 = t_2)[y_0, \dots, y_r]. \end{aligned}$$

(2) If $\varphi(v_0, \dots, v_p)$ is the formula $(R(t_1 \dots t_n))$ where R is an n-placed relation symbol with interpretation S, then again by Lemma 1, we get:

$$\begin{aligned} \mathfrak{A} \models (R(t_1 \dots t_n))[x_0, \dots, x_q] &\text{ iff } S(t_1[x_0, \dots, x_q], \dots, t_n[x_0, \dots, x_q]) \\ &\text{ iff } S(t_1[y_0, \dots, y_r], \dots, t_n[y_0, \dots, y_r]) \\ &\text{ iff } \mathfrak{A} \models R(t_1 \dots t_n)[y_0, \dots, y_r]. \end{aligned}$$

(3) If φ is $(\neg\theta)$, the inductive hypothesis gives that the lemma is true for θ. So,

$$\begin{aligned} \mathfrak{A} \models \varphi[x_0, \dots, x_q] &\text{ iff not } \mathfrak{A} \models \theta[x_0, \dots, x_q] \\ &\text{ iff not } \mathfrak{A} \models \theta[y_0, \dots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]. \end{aligned}$$

(4) If φ is $(\theta \wedge \psi)$, then using the inductive hypothesis on θ and ψ we get

$$\begin{aligned} \mathfrak{A} \models \varphi[x_0, \dots, x_q] &\text{ iff both } \mathfrak{A} \models \theta[x_0, \dots, x_q] \text{ and } \mathfrak{A} \models \psi[x_0, \dots x_q] \\ &\text{ iff both } \mathfrak{A} \models \theta[y_0, \dots, y_r] \text{ and } \mathfrak{A} \models \psi[y_0, \dots y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]. \end{aligned}$$

(5) If φ is $(\theta \vee \psi)$ then

$$\begin{aligned} \mathfrak{A} \models \varphi[x_0, \dots, x_q] &\text{ iff either } \mathfrak{A} \models \theta[x_0, \dots, x_q] \text{ or } \mathfrak{A} \models \psi[x_0, \dots, x_q] \\ &\text{ iff either } \mathfrak{A} \models \theta[y_0, \dots, y_r] \text{ or } \mathfrak{A} \models \psi[y_0, \dots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]. \end{aligned}$$

(6) If φ is $(\theta \rightarrow \psi)$ then

$$\begin{aligned} \mathfrak{A} \models \varphi[x_0, \dots, x_q] &\text{ iff } \mathfrak{A} \models \theta[x_0, \dots, x_q] \text{ implies } \mathfrak{A} \models \psi[x_0, \dots, x_q] \\ &\text{ iff } \mathfrak{A} \models \theta[y_0, \dots, y_r] \text{ implies } \mathfrak{A} \models \psi[y_0, \dots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]. \end{aligned}$$

(7) If φ is $(\theta \leftrightarrow \psi)$ then

$$\begin{aligned} \mathfrak{A} \models \varphi[x_0, \dots, x_q] &\text{ iff we have } \mathfrak{A} \models \theta[x_0, \dots, x_q] \text{ iff } \mathfrak{A} \models \psi[x_0, \dots, x_q] \\ &\text{ iff we have } \mathfrak{A} \models \theta[y_0, \dots, y_r] \text{ iff } \mathfrak{A} \models \psi[y_0, \dots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \dots, y_r]. \end{aligned}$$

(8) If φ is $(\forall v_i)\theta$, then

$$\begin{aligned}\mathfrak{A} \models \varphi[x_0, \ldots, x_q] &\text{ iff for every } z \in \mathbf{A}, \mathfrak{A} \models \theta[x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_q] \\ &\text{ iff for every } z \in \mathbf{A}, \mathfrak{A} \models \theta[y_0, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \ldots, y_r].\end{aligned}$$

The inductive hypothesis uses the sequences $x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_q$ and $y_0, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_r$ with the formula θ.

(9) If φ is $(\exists v_i)\theta$, then

$$\begin{aligned}\mathfrak{A} \models \varphi[x_0, \ldots, x_q] &\text{ iff for some } z \in \mathbf{A}, \mathfrak{A} \models \theta[x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_q] \\ &\text{ iff for some } z \in \mathbf{A}, \mathfrak{A} \models \theta[y_0, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_r] \\ &\text{ iff } \mathfrak{A} \models \varphi[y_0, \ldots, y_r].\end{aligned}$$

The inductive hypothesis uses the sequences $x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_q$ and $y_0, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_r$ with the formula θ.

□

DEFINITION 9. A *sentence* is a formula with no free variables.

If φ is a sentence, we can write $\mathfrak{A} \models \varphi$ without any mention of a sequence from $\mathbf{A}$ since by the previous lemma, it doesn't matter which sequence from $\mathbf{A}$ we use. In this case we say:

- $\mathfrak{A}$ *satisfies* φ
- or $\mathfrak{A}$ is a *model* of φ
- or φ *holds* in $\mathfrak{A}$
- or φ is *true* in $\mathfrak{A}$

If φ is a sentence of $\mathcal{L}$, we write $\models \varphi$ to mean that $\mathfrak{A} \models \varphi$ for every model $\mathfrak{A}$ for $\mathcal{L}$. Intuitively then, $\models \varphi$ means that φ is true under any relevant interpretation (model for $\mathcal{L}$). Alternatively, no relevant example (model for $\mathcal{L}$) is a counterexample to φ — so φ is true.

LEMMA 3. *Let $\varphi(v_0, \ldots, v_q)$ be a formula of the language $\mathcal{L}$. There is another formula $\varphi'(v_0, \ldots, v_q)$ of $\mathcal{L}$ such that*

(1) *φ' has exactly the same free and bound occurences of variables as φ.*
(2) *φ' can possibly contain $\neg$, $\wedge$ and $\exists$ but no other connective or quantifier.*
(3) $\models (\forall v_0) \ldots (\forall v_q)(\varphi \leftrightarrow \varphi')$

EXERCISE 3. Prove the above lemma by induction on the complexity of φ. As a reward, note that this lemma can be used to shorten future proofs by induction on complexity of formulas.

DEFINITION 10. A formula φ is said to be in *prenex normal form* whenever

(1) there are no quantifiers occurring in φ, or
(2) φ is $(\exists v_i)\psi$ where ψ is in prenex normal form and v_i does not occur bound in ψ, or
(3) φ is $(\forall v_i)\psi$ where ψ is in prenex normal form and v_i does not occur bound in ψ.

REMARK. If φ is in prenex normal form, then no variable occurring in φ occurs both free and bound and no bound variable occurring in φ is bound by more than one quantifier. In the written order, all of the quantifiers precede all of the connectives.

LEMMA 4. *Let $\varphi(v_0, \ldots, v_p)$ be any formula of a language $\mathcal{L}$. There is a formula φ^* of $\mathcal{L}$ which has the following properties:*

(1) *φ^* is in prenex normal form*
(2) *φ and φ^* have the same free occurences of variables, and*
(3) $\models (\forall v_0) \ldots (\forall v_p)(\varphi \leftrightarrow \varphi^*)$

EXERCISE 4. Prove this lemma by induction on the complexity of φ.

There is a notion of rank on prenex formulas — the number of alternations of quantifiers. The usual formulas of elementary mathematics have prenex rank 0, i.e. no alternations of quantifiers. For example:

$$(\forall x)(\forall y)(2xy \leq x^2 + y^2).$$

However, the $\epsilon - \delta$ definition of a limit of a function has prenex rank 2 and is much more difficult for students to comprehend at first sight:

$$(\forall \epsilon)(\exists \delta)(\forall x)((0 < \epsilon \wedge 0 < |x - a| < \delta) \rightarrow |F(x) - L| < \epsilon).$$

A formula of prenex rank 4 would make any mathematician look twice.

CHAPTER 1

Notation and Examples

Although the formal notation for formulas is precise, it can become cumbersome and difficult to read. Confident that the reader would be able, if necessary, to put formulas into their formal form, we will relax our formal behaviour. In particular, we will write formulas any way we want using appropriate symbols for variables, constant symbols, function and relation symbols. We will omit parentheses or add them for clarity. We will use binary function and relation symbols between the arguments rather than in front as is the usual case for "plus", "times" and "less than".

Whenever a language $\mathcal{L}$ has only finitely many relation, function and constant symbols we often write, for example:

$$\mathcal{L} = \{<, R_0, +, F_1, c_0, c_1\}$$

omitting explicit mention of the logical symbols (including the infinitely many variables) which are always in $\mathcal{L}$. Correspondingly we may denote a model $\mathfrak{A}$ for $\mathcal{L}$ as:

$$\mathfrak{A} = \langle \mathbf{A}, \mathbf{<}, \mathbf{S_0}, \mathbf{+}, \mathbf{G_1}, \mathbf{a_0}, \mathbf{a_1} \rangle$$

where the interpretations of the symbols in the language $\mathcal{L}$ are given by $\mathcal{I}(<) = \mathbf{<}$, $\mathcal{I}(R_0) = \mathbf{S_0}$, $\mathcal{I}(+) = \mathbf{+}$, $\mathcal{I}(F_1) = \mathbf{G_1}$, $\mathcal{I}(c_0) = \mathbf{a_0}$ and $\mathcal{I}(c_1) = \mathbf{a_1}$.

Example 3. $\mathfrak{R} = \langle \mathbb{R}, \mathbf{<}, \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1} \rangle$ and $\mathfrak{Q} = \langle \mathbb{Q}, \mathbf{<}, \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1} \rangle$, where $\mathbb{R}$ is the reals, $\mathbb{Q}$ the rationals , are models for the language $\mathcal{L} = \{<, +, \cdot, 0, 1\}$. Here $<$ is a binary relation symbol, $+$ and $\cdot$ are binary function symbols, 0 and 1 are constant symbols whereas $\mathbf{<}$, $\mathbf{+}$, $\cdot$, $\mathbf{0}$, $\mathbf{1}$ are the well known relations, arithmetic functions and constants.

Similarly, $\mathfrak{C} = \langle \mathbb{C}, \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1} \rangle$, where $\mathbb{C}$ is the complex numbers, is a model for the language $\mathcal{L} = \{+, \cdot, 0, 1\}$. Note the exceptions to the boldface convention for these popular sets.

Example 4. Here $\mathcal{L} = \{<, +, \cdot, 0, 1\}$, where $<$ is a binary relation symbol, $+$ and $\cdot$ are binary function symbols and 0 and 1 are constant symbols. The following formulas are sentences.

(1) $(\forall x)\neg(x < x)$
(2) $(\forall x)(\forall y)\neg(x < y \wedge y < x)$
(3) $(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \rightarrow x < z)$
(4) $(\forall x)(\forall y)(x < y \vee y < x \vee x = y)$
(5) $(\forall x)(\forall y)(x < y \rightarrow (\exists z)(x < z \wedge z < y))$
(6) $(\forall x)(\exists y)(x < y)$
(7) $(\forall x)(\exists y)(y < x)$
(8) $(\forall x)(\forall y)(\forall z)(x + (y + z) = (x + y) + z)$
(9) $(\forall x)(x + 0 = x)$

(10) $(\forall x)(\exists y)(x + y = 0)$
(11) $(\forall x)(\forall y)(x + y = y + x)$
(12) $(\forall x)(\forall y)(\forall z)(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$
(13) $(\forall x)(x \cdot 1 = x)$
(14) $(\forall x)(x = 0 \vee (\exists y)(y \cdot x = 1)$
(15) $(\forall x)(\forall y)(x \cdot y = y \cdot x)$
(16) $(\forall x)(\forall y)(\forall z)(x \cdot (y + z) = (x \cdot y) + (y \cdot z))$
(17) $0 \neq 1$
(18) $(\forall x)(\forall y)(\forall z)(x < y \rightarrow x + z < y + z)$
(19) $(\forall x)(\forall y)(\forall z)(x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z)$
(20) for each $n \geq 1$ we have the formula

$$(\forall x_0)(\forall x_1) \cdots (\forall x_n)(\exists y)(x_n \cdot y^n + x_{n-1} \cdot y^{n-1} + \cdots + x_1 \cdot y + x_0 = 0 \vee x_n = 0)$$

where, as usual, y^k abbreviates $\overbrace{y \cdot y \cdot \cdots \cdot y}^{k}$

The latter formulas express that each polynomial of degree n has a root. The following formulas express the intermediate value property for polynomials of degree n: if the polynomial changes sign from w to z, then it is zero at some y between w and z.

(21) for each $n \geq 1$ we have

$$\begin{aligned}(\forall x_0) \ldots (\forall x_n)(\forall w)(\forall z)[(x_n \cdot w^n + x_{n-1} \cdot w^{n-1} + \cdots + x_1 \cdot w + x_0) \cdot \\ (x_n \cdot z^n + x_{n-1} \cdot z^{n-1} + \cdots + x_1 \cdot z + x_0) < 0 \\ \rightarrow (\exists y)(((w < y \wedge y < z) \vee (z < y \wedge y < w)) \\ \wedge (x_n \cdot y^n + x_{n-1} \cdot y^{n-1} + \cdots + x_1 \cdot y + x_0 = 0))]\end{aligned}$$

The most fundamental concept is that of a sentence σ being true when interpreted in a model $\mathfrak{A}$. We write this as $\mathfrak{A} \models \sigma$, and we extend this concept in the following definitions.

Definition 11. If Σ is a set of sentences, $\mathfrak{A}$ is said to be a *model* of Σ, written $\mathfrak{A} \models \Sigma$, whenever $\mathfrak{A} \models \sigma$ for each $\sigma \in \Sigma$. Σ is said to be *satisfiable* iff there is some $\mathfrak{A}$ such that $\mathfrak{A} \models \Sigma$.

Definition 12. A *theory* $\mathcal{T}$ is a set of sentences. If $\mathcal{T}$ is a theory and σ is a sentence, we write $\mathcal{T} \models \sigma$ whenever we have that for all $\mathfrak{A}$ if $\mathfrak{A} \models \mathcal{T}$ then $\mathfrak{A} \models \sigma$. We say that σ is a *consequence* of $\mathcal{T}$. A theory is said to be *closed* whenever it contains all of its consequences.

Definition 13. If $\mathfrak{A}$ is a model for the language $\mathcal{L}$, the *theory of* $\mathfrak{A}$, denoted by $\mathrm{Th}\mathfrak{A}$, is defined to be the set of all sentences of $\mathcal{L}$ which are true in $\mathfrak{A}$,

$$\{\sigma \text{ of } \mathcal{L} : \mathfrak{A} \models \sigma\}.$$

This is one way that a theory can arise. Another way is through axioms.

Definition 14. $\Sigma \subseteq \mathcal{T}$ is said to be a set of *axioms* for $\mathcal{T}$ whenever $\Sigma \models \sigma$ for every σ in $\mathcal{T}$; in this case we write $\Sigma \models \mathcal{T}$.

Remark. We will generally assume our theories are closed and we will often describe theories by specifying a set of axioms Σ. The theory will then be all consequences σ of Σ.

EXAMPLE 5. We will consider the following theories and their axioms:

(1) The theory of Linear Orderings (LOR) which has as axioms sentences 1-4 from Example 4.
(2) The theory of Dense Linear Orders (DLO) which has as axioms all the axioms of LOR, and sentence 5, 6 and 7 of Example 4.
(3) The theory of Fields (FEI) which has as axioms sentences 8-17 from Example 4.
(4) The theory of Ordered Fields (ORF) which has as axioms all the axioms of FEI, LOR and sentences 18 and 19 from Example 4.
(5) The theory of Algebraically Closed Fields (ACF) which has as axioms all the axioms of FEI and all sentences from 20 of Example 4, i.e. infinitely many sentences, one for each $n \geq 1$.
(6) The theory of Real Closed Ordered Fields (RCF) which has as axioms all the axioms of ORF, and all sentences from 21 of Example 4, i.e. infinitely many sentences, one for each $n \geq 1$.

EXERCISE 5. Show that :

(1) $\mathfrak{Q} \models$ DLO
(2) $\mathfrak{R} \models$ RCF using the Intermediate Value theorem
(3) $\mathfrak{C} \models$ ACF using the Fundamental Theorem of Algebra

where $\mathfrak{Q}$, $\mathfrak{R}$ and $\mathfrak{C}$ are as in Example 3.

REMARK. The theory of Real Closed Ordered Fields is sometimes axiomatized differently. All the axioms of ORF are retained, but the sentences from 21 of Example 4, which amount to an Intermediate Value Property, are replaced by the sentences from 20 for odd n and the sentence

$$(\forall x)(0 < x \rightarrow (\exists y)y^2 = x)$$

which states that every positive element has a square root. A significant amount of algebra would then be used to verify the Intermediate Value Property from these axioms.

CHAPTER 2

Compactness and Elementary Submodels

THEOREM 1. *The Compactness Theorem (Malcev)*
A set of sentences is satisfiable iff every finite subset is satisfiable.

PROOF. There are several proofs. We only point out here that it is an easy consequence of the following, a theorem which appears in all elementary logic texts:

PROPOSITION. *The Completeness Theorem (Gödel, Malcev)*
A set of sentences is consistent iff it is satisfiable.

Although we do not here formally define "consistent", it does mean what you think it does. In particular, a set of sentences is consistent if and only if each finite subset is consistent.

□

REMARK. The Compactness Theorem is the only one for which we do not give a complete proof. If the reader has not previously seen the Completeness Theorem, there are other proofs of the Compactness Theorem which may be more easily absorbed: set theoretic (using ultraproducts), topological (using compact spaces, hence the name) or Boolean algebraic. However these topics are too far afield to enter into the proofs here. We will use the Compactness Theorem as a starting point — in fact, all that follows can be seen as its corollaries.

EXERCISE 6. Suppose $\mathcal{T}$ is a theory for the language $\mathcal{L}$ and σ is a sentence of $\mathcal{L}$ such that $\mathcal{T} \models \sigma$. Prove that there is some finite $\mathcal{T}' \subseteq \mathcal{T}$ such that $\mathcal{T}' \models \sigma$. Recall that $\mathcal{T} \models \sigma$ iff $\mathcal{T} \cup \{\neg\sigma\}$ is not satisfiable.

DEFINITION 15. If $\mathcal{L}$, and $\mathcal{L}'$ are two languages such that $\mathcal{L} \subseteq \mathcal{L}'$ we say that $\mathcal{L}'$ is an *expansion* of $\mathcal{L}$ and $\mathcal{L}$ is a *reduction* of $\mathcal{L}'$. Of course when we say that $\mathcal{L} \subseteq \mathcal{L}'$ we also mean that the constant, function and relation symbols of $\mathcal{L}$ remain (respectively) constant, function and relation symbols of $\mathcal{L}'$.

DEFINITION 16. Given a model $\mathfrak{A}$ for the language $\mathcal{L}$, we can expand it to a model $\mathfrak{A}'$ of $\mathcal{L}'$ by giving appropriate interpretations to the symbols in $\mathcal{L}' \setminus \mathcal{L}$. We say that $\mathfrak{A}'$ is an *expansion* of $\mathfrak{A}$ to $\mathcal{L}'$ and that $\mathfrak{A}$ is a *reduct* of $\mathfrak{A}'$ to $\mathcal{L}$. We also use the notation $\mathfrak{A}'|\mathcal{L}$ for the reduct of $\mathfrak{A}'$ to $\mathcal{L}$.

THEOREM 2. *If a theory $\mathcal{T}$ has arbitrarily large finite models, then it has an infinite model.*

PROOF. Consider new constant symbols c_i for $i \in \mathbb{N}$, the usual natural numbers, and expand from $\mathcal{L}$, the language of $\mathcal{T}$, to $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in \mathbb{N}\}$.

Let

$$\Sigma = \mathcal{T} \cup \{\neg c_i = c_j : i \neq j, i, j \in \mathbb{N}\}.$$

We first show that every finite subset of Σ has a model by interpreting the finitely many relevant constant symbols as different elements in an expansion of some finite model of $\mathcal{T}$. Then we use compactness to get a model $\mathfrak{A}'$ of Σ.

The model that we require is for the language $\mathcal{L}$, so we take $\mathfrak{A}$ to be the reduct of $\mathfrak{A}'$ to $\mathcal{L}$.

□

Definition 17. Two models $\mathfrak{A}$ and $\mathfrak{A}'$ for $\mathcal{L}$ are said to be *isomorphic* whenever there is a bijection $f : \mathbf{A} \to \mathbf{A}'$ such that

(1) for each n-placed relation symbol R of $\mathcal{L}$ and corresponding interpretations S of $\mathfrak{A}$ and S' of $\mathfrak{A}'$ we have
$$S(x_1, \dots, x_n) \text{ iff } S'(f(x_1), \dots, f(x_n)) \text{ for all } x_1, \dots, x_n \text{ in } \mathbf{A}$$
(2) for each n-placed function symbol F of $\mathcal{L}$ and corresponding interpretations G of $\mathfrak{A}$ and G' of $\mathfrak{A}'$ we have
$$f(G(x_1, \dots, x_n)) = G'(f(x_1), \dots, f(x_n)) \text{ for all } x_1, \dots, x_n \text{ in } \mathbf{A}$$
(3) for each constant symbol c of $\mathcal{L}$ and corresponding constant elements a of $\mathfrak{A}$ and a' of $\mathfrak{A}'$ we have $f(a) = a'$.

We write $\mathfrak{A} \cong \mathfrak{A}'$. This is an equivalence relation.

Example 6. Number theory is $\mathrm{Th}\langle \mathbb{N}, \mathbf{+}, \mathbf{\cdot}, \mathbf{<}, \mathbf{0}, \mathbf{1} \rangle$, the set of all sentences of $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ which are true in $\langle \mathbb{N}, \mathbf{+}, \mathbf{\cdot}, \mathbf{<}, \mathbf{0}, \mathbf{1} \rangle$, the standard model which we all learned in school. Any model not isomorphic to the standard model of number theory is said to be a non-standard model of number theory.

Theorem 3. *(T. Skolem)*
There exist non-standard models of number theory.

Proof. Add a new constant symbol c to $\mathcal{L}$. Consider
$$\mathrm{Th}\langle \mathbb{N}, +, \cdot, <, 0, 1 \rangle \cup \{\overbrace{1 + 1 + \cdots + 1}^{n} < c : n \in \mathbb{N}\}$$
and use the Compactness Theorem. The interpretation of the constant symbol c will not be a natural number. □

Definition 18. Two models $\mathfrak{A}$ and $\mathfrak{A}'$ for $\mathcal{L}$ are said to be *elementarily equivalent* whenever we have that for each sentence σ of $\mathcal{L}$
$$\mathfrak{A} \models \sigma \text{ iff } \mathfrak{A}' \models \sigma$$
We write $\mathfrak{A} \equiv \mathfrak{A}'$. This is another equivalence relation.

Exercise 7. Suppose $f : \mathfrak{A} \to \mathfrak{A}'$ is an isomorphism and φ is a formula such that $\mathfrak{A} \models \varphi[a_0, \dots, a_k]$ for some $a_0, \dots, a_k$ from $\mathbf{A}$; prove $\mathfrak{A}' \models \varphi[f(a_0), \dots, f(a_k)]$. Use this to show that $\mathfrak{A} \cong \mathfrak{A}'$ implies $\mathfrak{A} \equiv \mathfrak{A}'$.

Definition 19. A model $\mathfrak{A}'$ is called a *submodel* of $\mathfrak{A}$ iff $\phi \neq \mathbf{A}' \subseteq \mathbf{A}$ and

(1) each n-placed relation S' of $\mathfrak{A}'$ is the restriction to $\mathbf{A}'$ of the corresponding relation S of $\mathfrak{A}$, i. e. $S' = S \cap (\mathbf{A}')^n$
(2) each m-placed function G' of $\mathfrak{A}'$ is the restriction to $\mathbf{A}'$ of the corresponding function G of $\mathfrak{A}$, i. e. $G' = G \restriction (\mathbf{A}')^m$
(3) each constant of $\mathfrak{A}'$ is the corresponding constant of $\mathfrak{A}$.
We write $\mathfrak{A}' \subseteq \mathfrak{A}$.

Definition 20. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models for $\mathcal{L}$. We say $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$ and $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$ and we write $\mathfrak{A} \prec \mathfrak{B}$ whenever

(1) $\mathbf{A} \subseteq \mathbf{B}$ and
(2) for all formulas $\varphi(v_0, \ldots, v_k)$ of $\mathcal{L}$ and all $a_0, \ldots, a_k \in \mathbf{A}$

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_k] \text{ iff } \mathfrak{B} \models \varphi[a_0, \ldots, a_k].$$

Exercise 8. Prove that if $\mathfrak{A} \prec \mathfrak{B}$ then $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$.

Example 7. Let $\mathbb{N}$ be the usual natural numbers with $<$ as the usual ordering. Let $\mathfrak{B} = \langle \mathbb{N}, < \rangle$ and $\mathfrak{A} = \langle \mathbb{N} \setminus \{0\}, < \rangle$ be models for the language with one binary relation symbol $<$. Then $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$; in fact $\mathfrak{A} \cong \mathfrak{B}$. But we do not have $\mathfrak{A} \prec \mathfrak{B}$; 1 satisfies the formula describing the least element of the ordering in $\mathfrak{A}$ but in $\mathfrak{B}$. So we see that being an elementary submodel is a very strong condition indeed. Nevertheless, later in the chapter we will obtain many examples of elementary submodels.

Exercise 9. Show that

- if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$ then $\mathfrak{A} \subseteq \mathfrak{C}$ and
- if $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \prec \mathfrak{C}$ then $\mathfrak{A} \prec \mathfrak{C}$.

Definition 21. A *chain of models* for a language $\mathcal{L}$ is an increasing sequence of models

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_n \subseteq \cdots \quad n \in \mathbb{N}.$$

The *union* of the chain is defined to be the model $\mathfrak{A} = \cup\{\mathfrak{A}_n : n \in \mathbb{N}\}$ where the universe of $\mathfrak{A}$ is $\mathbf{A} = \cup\{\mathbf{A}_n : n \in \mathbb{N}\}$ and:

(1) each relation S on $\mathfrak{A}$ is the union of the corresponding relations S_n of $\mathfrak{A}_n$; $S = \cup\{S_n : n \in \mathbb{N}\}$, i.e. the relation extending each S_n
(2) each function G on $\mathfrak{A}$ is the union of the corresponding functions G_n of $\mathfrak{A}_n$; $G = \cup\{G_n : n \in \mathbb{N}\}$, i.e. the function extending each G_n
(3) all the models $\mathfrak{A}_n$ and $\mathfrak{A}$ have the same constant elements.

Note that each $\mathfrak{A}_n \subseteq \mathfrak{A}$.

Example 8. For each $n \in \mathbb{N}$, let

$$\mathbf{A}_n = \{-n, -n+1, -n+2, \ldots, 0, 1, 2, 3, \ldots\} \subseteq \mathbb{Z}.$$

Let $\mathfrak{A}_n = \langle \mathbf{A}_n, \leq \rangle$. Each $\mathfrak{A}_n \equiv \mathfrak{A}_0$, but we don't have $\mathfrak{A}_0 \equiv \cup\{\mathfrak{A}_n : n \in \mathbb{N}\}$.

Remark. To be sure, what is defined here is a chain of models indexed by the natural numbers $\mathbb{N}$. More generally, a chain of models could be indexed by any ordinal. However we will not need the concept of an ordinal at this point.

Definition 22. An *elementary chain* is a chain of models $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ such that for each $m < n$ we have $\mathfrak{A}_m \prec \mathfrak{A}_n$.

Theorem 4. *(Tarski's Elementary Chain Theorem)*
Let $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ be an elementary chain. For all $n \in \mathbb{N}$ we have

$$\mathfrak{A}_n \prec \cup\{\mathfrak{A}_n : n \in \mathbb{N}\}.$$

Proof. Denote the union of the chain by $\mathfrak{A}$. We have $\mathfrak{A}_k \subseteq \mathfrak{A}$ for each $k \in \mathbb{N}$.

Claim. *If t is a term of the language $\mathcal{L}$ and $a_0, \ldots, a_p$ are in $\mathbf{A}_k$, then the value of the term $t[a_0, \ldots, a_p]$ in $\mathfrak{A}$ is equal to the value in $\mathfrak{A}_k$.*

Proof of Claim. We prove this by induction on the complexity of the term.

(1) If t is the variable v_i then both values are just a_i.
(2) If t is the constant symbol c then the values are equal because c has the same interpretation in $\mathfrak{A}$ and in $\mathfrak{A}_k$.
(3) If t is $F(t_1 \dots t_m)$ where F is a function symbol and $t_1, \dots, t_m$ are terms such that each value $t_i[a_0, \dots, a_p]$ is the same in both $\mathfrak{A}$ and $\mathfrak{A}_k$, then the value
$$F(t_1 \dots t_m)[a_0, \dots, a_p]$$
in $\mathfrak{A}$ is
$$G(t_1[a_0, \dots, a_p], \dots, t_m[a_0, \dots, a_p])$$
where G is the interpretation of F in $\mathfrak{A}$ and the value of
$$F(t_1 \dots t_m)[a_0, \dots, a_p]$$
in $\mathfrak{A}_k$ is
$$G_k(t_1[a_0, \dots, a_p], \dots, t_m[a_0, \dots, a_p])$$
where G_k is the interpretation of F in $\mathfrak{A}_k$. But G_k is the restriction of G to $\mathbf{A}_k$ so these values are equal.

In order to show that each $\mathfrak{A}_k \prec \mathfrak{A}$ it will suffice to prove the following statement for each formula $\varphi(v_0, \dots, v_p)$ of $\mathcal{L}$.

" For all $k \in \mathbb{N}$ and all $a_0, \dots, a_p$ in $\mathbf{A}_k$:
$$\mathfrak{A} \models \varphi[a_0, \dots, a_p] \text{ iff } \mathfrak{A}_k \models \varphi[a_0, \dots, a_p]."$$

Claim. *The statement is true whenever φ is $t_1 = t_2$ where t_1 and t_2 are terms.*

Proof of Claim. Fix $k \in \mathbb{N}$ and $a_0, \dots, a_p$ in $\mathbf{A}_k$.
$$\begin{aligned} \mathfrak{A} \models (t_1 = t_2)[a_0, \dots, a_p] &\text{ iff } t_1[a_0, \dots, a_p] = t_2[a_0, \dots, a_p] \text{ in } \mathfrak{A} \\ &\text{ iff } t_1[a_0, \dots, a_p] = t_2[a_0, \dots, a_p] \text{ in } \mathfrak{A}_k \\ &\text{ iff } \mathfrak{A}_k \models (t_1 = t_2)[a_0, \dots, a_p]. \end{aligned}$$

Claim. *The statement is true whenever φ is $R(t_1 \dots t_n)$ where R is a relation symbol and $t_1, \dots, t_n$ are terms.*

Proof of Claim. Fix $k \in \mathbb{N}$ and $a_0, \dots, a_p$ in $\mathbf{A}_k$. Let S be the interpretation of R in $\mathfrak{A}$ and S_k be the interpretation in $\mathfrak{A}_k$; S_k is the restriction of S to $\mathbf{A}_k$.
$$\begin{aligned} \mathfrak{A} \models R(t_1 \dots t_n)[a_0, \dots, a_p] &\text{ iff } S(t_1[a_0, \dots, a_p], \dots, t_n[a_0, \dots, a_p]) \\ &\text{ iff } S_k(t_1[a_0, \dots, a_p], \dots, t_n[a_0, \dots, a_p]) \\ &\text{ iff } \mathfrak{A}_k \models R(t_1 \dots t_n)[a_0, \dots, a_p] \end{aligned}$$

Claim. *If the statement is true when φ is θ, then the statement is true when φ is $\neg\theta$.*

Proof of Claim. Fix $k \in \mathbb{N}$ and $a_0, \ldots, a_p$ in $\mathbf{A}_k$.

$$\begin{aligned}\mathfrak{A} \models (\neg\theta)[a_0, \ldots, a_p] &\text{ iff not } \mathfrak{A} \models \theta[a_0, \ldots, a_p]\\ &\text{ iff not } \mathfrak{A}_k \models \theta[a_0, \ldots, a_p]\\ &\text{ iff } \mathfrak{A}_k \models (\neg\theta)[a_0, \ldots, a_p].\end{aligned}$$

Claim. *If the statement is true when φ is θ_1 and when φ is θ_2 then the statement is true when φ is $\theta_1 \wedge \theta_2$.*

Proof of Claim. Fix $k \in \mathbb{N}$ and $a_0, \ldots, a_p$ in $\mathbf{A}_k$.

$$\begin{aligned}\mathfrak{A} \models (\theta_1 \wedge \theta_2)[a_0, \ldots, a_p] &\text{ iff } \mathfrak{A} \models \theta_1[a_0, \ldots, a_p] \text{ and } \mathfrak{A} \models \theta_2[a_0, \ldots, a_p]\\ &\text{ iff } \mathfrak{A}_k \models \theta_1[a_0, \ldots, a_p] \text{ and } \mathfrak{A}_k \models \theta_2[a_0, \ldots, a_p]\\ &\text{ iff } \mathfrak{A}_k \models (\theta_1 \wedge \theta_2)[a_0, \ldots, a_p].\end{aligned}$$

Claim. *If the statement is true when φ is θ then the statement is true when φ is $\exists v_i\theta$.*

Proof of Claim. Fix $k \in \mathbb{N}$ and $a_0, \ldots, a_p$ in $\mathbf{A}_k$. Note that $\mathbf{A} = \cup\{\mathbf{A}_j : j \in \mathbb{N}\}$.

$$\mathfrak{A} \models \exists v_i\theta[a_0, \ldots, a_p] \text{ iff } \mathfrak{A} \models \exists v_i\theta[a_0, \ldots, a_q]$$

$$\text{where } q \text{ is the maximum of } i \text{ and } p \text{ (by Lemma 2),}$$

$$\text{iff } \mathfrak{A} \models \theta[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q] \text{ for some } a \in \mathbf{A},$$
$$\text{iff } \mathfrak{A} \models \theta[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q]$$

$$\text{for some } a \in \mathbf{A}_l \text{ for some } l \geq k$$

$$\text{iff } \mathfrak{A}_l \models \theta[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q]$$

$$\text{since the statement is true for } \theta,$$

$$\text{iff } \mathfrak{A}_l \models \exists v_i\theta[a_0, \ldots, a_q]$$
$$\text{iff } \mathfrak{A}_k \models \exists v_i\theta[a_0, \ldots, a_q] \text{ since } \mathfrak{A}_k \prec \mathfrak{A}_l$$
$$\text{iff } \mathfrak{A}_k \models \exists v_i\theta[a_0, \ldots, a_p] \text{ (by Lemma 2).}$$

By induction on the complexity of φ, we have proven the statement for all formulas φ which do not contain the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ or the quantifier $\forall$. To verify the statement for all φ we use Lemma 3. Let φ be any formula of $\mathcal{L}$. By Lemma 3 there is a formula ψ which does not use $\vee$, $\rightarrow$, $\leftrightarrow$ nor $\forall$ such that

$$\models (\forall v_0) \ldots (\forall v_p)(\varphi \leftrightarrow \psi).$$

Now fix $k \in \mathbb{N}$ and $a_0, \ldots, a_p$ in $\mathbf{A}_k$. We have

$$\mathfrak{A} \models (\varphi \leftrightarrow \psi)[a_0, \ldots, a_p] \text{ and } \mathfrak{A}_k \models (\varphi \leftrightarrow \psi)[a_0, \ldots, a_p].$$

$$\begin{aligned}\mathfrak{A} \models \varphi[a_0, \ldots, a_p] &\text{ iff } \mathfrak{A} \models \psi[a_0, \ldots, a_p]\\ &\text{ iff } \mathfrak{A}_k \models \psi[a_0, \ldots, a_p]\\ &\text{ iff } \mathfrak{A}_k \models \varphi[a_0, \ldots, a_p]\end{aligned}$$

which completes the proof of the theorem.

□

LEMMA 5. *(The Tarski-Vaught Condition)*
Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. The following are equivalent:

(1) $\mathfrak{A} \prec \mathfrak{B}$
(2) *for any formula $\psi(v_0, \ldots, v_q)$ and any $i \leq q$ and any $a_0, \ldots, a_q$ from $\mathbf{A}$: if there is some $b \in \mathbf{B}$ such that*

$$\mathfrak{B} \models \psi[a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_q]$$

then we have some $a \in \mathbf{A}$ such that

$$\mathfrak{B} \models \psi[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q].$$

PROOF. Only the implication (2) $\Rightarrow$ (1) requires a lot of proof. We will prove that for each formula $\varphi(v_0, \ldots, v_p)$ and all $a_0, \ldots, a_p$ from $\mathbf{A}$ we will have:

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_p] \text{ iff } \mathfrak{B} \models \varphi[a_0, \ldots, a_p]$$

by induction on the complexity of φ using only the negation symbol $\neg$, the connective $\wedge$ and the quantifier $\exists$ (recall Lemma 3).

(1) The cases of formulas of the form $t_1 = t_2$ and $R(t_1 \ldots t_n)$ come immediately from the fact that $\mathfrak{A} \subseteq \mathfrak{B}$.
(2) For negation: suppose φ is $\neg\psi$ and we have it for ψ, then

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_p] \text{ iff not } \mathfrak{A} \models \psi[a_0, \ldots, a_p]$$

$$\text{iff not } \mathfrak{B} \models \psi[a_0, \ldots, a_p] \text{ iff } \mathfrak{B} \models \varphi[a_0, \ldots, a_p].$$

(3) The $\wedge$ case proceeds similarly.
(4) For the $\exists$ case we consider φ as $\exists v_i \psi$. If $\mathfrak{A} \models \exists v_i \psi[a_0, \ldots, a_p]$, then the inductive hypothesis for ψ and the fact that $\mathbf{A} \subseteq \mathbf{B}$ ensure that $\mathfrak{B} \models \exists v_i \psi[a_0, \ldots, a_p]$. It remains to show that if $\mathfrak{B} \models \varphi[a_0, \ldots, a_p]$ then $\mathfrak{A} \models \varphi[a_0, \ldots, a_p]$.

Assume $\mathfrak{B} \models \exists v_i \psi[a_0, \ldots, a_p]$. By Lemma 2, $\mathfrak{B} \models \exists v_i \psi[a_0, \ldots, a_q]$ where q is the maximum of i and p. By the definition of satisfaction, there is some $b \in \mathbf{B}$ such that

$$\mathfrak{B} \models \psi[a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_q].$$

By (2), there is some $a \in \mathbf{A}$ such that

$$\mathfrak{B} \models \psi[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q].$$

By the inductive hypothesis on ψ, for that same $a \in \mathbf{A}$,

$$\mathfrak{A} \models \psi[a_0, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_q].$$

By the definition of satisfaction,

$$\mathfrak{A} \models \exists v_i \psi[a_0, \ldots, a_q].$$

Finally, by Lemma 2, $\mathfrak{A} \models \phi[a_0, \ldots, a_p]$.

□

Recall that $|\mathbf{B}|$ is used to represent the cardinality, or size, of the set $\mathbf{B}$. Note that since any language $\mathcal{L}$ contains infinitely many variables, $|\mathcal{L}|$ is always infinite, but may be countable or uncountable depending on the number of other symbols. We often denote an arbitrary infinite cardinal by the lower case Greek letter κ.

THEOREM 5. *(Downward Löwenheim-Skolem Theorem)*
Let $\mathfrak{B}$ be a model for $\mathcal{L}$ and let κ be any cardinal such that $|\mathcal{L}| \le \kappa < |\mathbf{B}|$. Then $\mathfrak{B}$ has an elementary submodel $\mathfrak{A}$ of cardinality κ.

Furthermore if $X \subseteq \mathbf{B}$ and $|X| \le \kappa$, then we can also have $X \subseteq \mathbf{A}$.

PROOF. Without loss of generosity assume $|X| = \kappa$. We recursively define sets X_n for $n \in \mathbb{N}$ such that $X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$ and such that for each formula $\varphi(v_0, \ldots, v_p)$ of $\mathcal{L}$ and each $i \le p$ and each $a_0, \ldots, a_p$ from X_n such that

$$\mathfrak{B} \models \exists v_i \varphi[a_0, \ldots, a_p]$$

we have $x \in X_{n+1}$ such that

$$\mathfrak{B} \models \varphi[a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_p].$$

Since $|\mathcal{L}| \le \kappa$ and each formula of $\mathcal{L}$ is a finite string of symbols from $\mathcal{L}$, there are at most κ many formulas of $\mathcal{L}$. So there are at most κ elements of $\mathbf{B}$ that need to be added to each X_n and so, without loss of generosity each $|X_n| = \kappa$. Let $\mathbf{A} = \cup\{X_n : n \in \mathbb{N}\}$; then $|\mathbf{A}| = \kappa$. Since $\mathbf{A}$ is closed under functions from $\mathfrak{B}$ and contains all constants from $\mathfrak{B}$, $\mathbf{A}$ gives rise to a submodel $\mathfrak{A} \subseteq \mathfrak{B}$.

The Tarski-Vaught Condition is used to show that $\mathfrak{A} \prec \mathfrak{B}$.

□

THEOREM 6. *(The Upward Löwenheim-Skolem Theorem)*
Let $\mathfrak{A}$ be an infinite model for $\mathcal{L}$ and κ be any cardinal such that $|\mathcal{L}| \le \kappa$ and $|\mathbf{A}| < \kappa$. Then $\mathfrak{A}$ has an elementary extension of cardinality κ.

PROOF. For each $a \in \mathbf{A}$, let c_a be a new constant symbol; let

$$\mathcal{L}' = \mathcal{L} \cup \{c_a : a \in \mathbf{A}\}.$$

Note that sentences of $\mathcal{L}'$ are just formulas of $\mathcal{L}$ with all free variables replaced by constant symbols.

In addition, add κ many new constant symbols to $\mathcal{L}'$ to make $\mathcal{L}''$. Define Σ to be the following set of sentences of $\mathcal{L}''$:

$$\begin{aligned}
&\{\neg d = d' : d \text{ and } d' \text{ are distinct new constant symbols of } \mathcal{L}'' \setminus \mathcal{L}'\} \cup \\
&\{\sigma : \sigma \text{ is a sentence of } \mathcal{L}' \text{ obtained from the formula } \varphi(v_0, \ldots, v_p) \text{ of } \mathcal{L} \\
&\quad \text{by replacing each free variable } v_i \text{ by the constant symbol } c_{a_i} \\
&\quad \text{and } \mathfrak{A} \models \varphi[a_0, \ldots, a_p]\}
\end{aligned}$$

By interpreting c_a as a and the new constant symbols as distinct elements of $\mathbf{A}$ we can transform $\mathfrak{A}$ into a model of any finite subset of Σ. Using the Compactness Theorem, we obtain a model $\mathfrak{D}''$ for $\mathcal{L}''$ such that $\mathfrak{D}'' \models \Sigma$.

Note that $\mathfrak{D}''$ has size at least κ and that for any sentence σ of $\mathcal{L}'$, $\mathfrak{D}'' \models \sigma$ iff $\sigma \in \Sigma$.

Obtain a model $\mathfrak{C}''$ for $\mathcal{L}''$ from $\mathfrak{D}''$ by simply switching elements of the universe of $\mathfrak{D}''$ with $\mathbf{A}$ to ensure that for each $a \in \mathbf{A}$ the interpretation of c_a in $\mathfrak{C}''$ is a.

Hence the universe of $\mathfrak{C}''$ contains $\mathbf{A}$ and $\mathfrak{C}'' \models \Sigma$. Let $\mathfrak{C}$ be the reduct of $\mathfrak{C}''$ to $\mathcal{L}$. The following argument will show that $\mathfrak{A} \prec \mathfrak{C}$.

Let φ be any formula of $\mathcal{L}$ and $a_0, \ldots, a_p$ any elements from $\mathbf{A}$. Let σ be the sentence of $\mathcal{L}'$ formed by replacing free occurences of v_i with c_{a_i}. We have

$$\begin{aligned}\mathfrak{A} \models \varphi[a_0, \ldots, a_p] &\text{ iff } \sigma \in \Sigma \\ &\text{ iff } \mathfrak{D}'' \models \sigma \\ &\text{ iff } \mathfrak{C}'' \models \sigma \\ &\text{ iff } \mathfrak{C} \models \varphi[a_0, \ldots, a_p].\end{aligned}$$

However, $\mathfrak{C}$ may have size strictly larger than κ. In this case we obtain our final $\mathfrak{B}$ by using the previous theorem to get $\mathfrak{B} \prec \mathfrak{C}$ with $\mathbf{A} \subseteq \mathbf{B}$. It is now straightforward to conclude that $\mathfrak{A} \prec \mathfrak{B}$.

□

Definition 23. A theory $\mathcal{T}$ for a language $\mathcal{L}$ is said to be *complete* whenever for each sentence σ of $\mathcal{L}$ either $\mathcal{T} \models \sigma$ or $\mathcal{T} \models \neg\sigma$.

Lemma 6. *A theory $\mathcal{T}$ for $\mathcal{L}$ is complete iff any two models of $\mathcal{T}$ are elementarily equivalent.*

Proof. ($\Rightarrow$) easy. ($\Leftarrow$) easy. □

Definition 24. A theory $\mathcal{T}$ is said to be *categorical* in cardinality κ whenever any two models of $\mathcal{T}$ of cardinality κ are isomorphic. We also say that $\mathcal{T}$ is κ-*categorical.*

The most interesting cardinalities in the context of categorical theories are $\aleph_0$, the cardinality of countably infinite sets, and $\aleph_1$, the first uncountable cardinal.

Exercise 10. Show that DLO is $\aleph_0$-categorical. There are two well-known proofs. One uses a back-and-forth construction of an isomorphism. The other constructs, by recursion, an isomorphism from the set of dyadic rational numbers between 0 and 1:

$$\{\frac{n}{2^m} : m \text{ is a positive integer and } n \text{ is an integer } 0 < n < 2^m\},$$

onto a countable dense linear order without endpoints.

Now use the following theorem to show that DLO is complete.

Theorem 7. *(The Łoś-Vaught Test)*
Suppose that a theory $\mathcal{T}$ has only infinite models for a language $\mathcal{L}$ and that $\mathcal{T}$ is κ-categorical for some cardinal $\kappa \geq |\mathcal{L}|$. Then $\mathcal{T}$ is complete.

Proof. We will show that any two models of $\mathcal{T}$ are elementarily equivalent. Let $\mathfrak{A}$ of cardinality λ_1, and $\mathfrak{B}$ of cardinality λ_2, be two models of $\mathcal{T}$.

If $\lambda_1 < \kappa$ use the Upward Löwenheim-Skolem Theorem to get $\mathfrak{A}'$ such that $|\mathbf{A}'| = \kappa$ and $\mathfrak{A} \prec \mathfrak{A}'$.

If $\lambda_1 > \kappa$ use the Downward Löwenheim-Skolem Theorem to get $\mathfrak{A}'$ such that $|\mathbf{A}'| = \kappa$ and $\mathfrak{A}' \prec \mathfrak{A}$.

Either way, we can get $\mathfrak{A}'$ such that $|\mathbf{A}'| = \kappa$ and $\mathfrak{A}' \equiv \mathfrak{A}$. Similarly, we can get $\mathfrak{B}'$ such that $|\mathbf{B}'| = \kappa$ and $\mathfrak{B}' \equiv \mathfrak{B}$. Since $\mathcal{T}$ is κ -categorical, $\mathfrak{A}' \cong \mathfrak{B}'$. Hence $\mathfrak{A} \equiv \mathfrak{B}$. □

Recall that the characteristic of a field is the prime number p such that

$$\overbrace{1+1+\cdots+1}^{p}=0$$

provided that such a p exists, and, if no such p exists the field has characteristic 0. All of our best-loved fields: $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have characteristic 0. On the other hand, fields of characteristic p include the finite field of size p (the prime Galois field).

THEOREM 8. *The theory of algebraically closed fields of characteristic 0 is complete.*

PROOF. We use the Łoś-Vaught Test and the following Lemma. □

LEMMA 7. *Any two algebraically closed fields of characteristic 0 and cardinality $\aleph_1$ are isomorphic.*

PROOF. Let $\mathfrak{A}$ be such a field containing the rationals $\mathfrak{Q} = \langle \mathbb{Q}, \mathbf{+}, \mathbf{\cdot}, \mathbf{0}, \mathbf{1} \rangle$ as a prime subfield. In a manner completely analogous to finding a basis for a vector space, we can find a transcendence basis for $\mathfrak{A}$, that is, an indexed subset $\{a_\alpha : \alpha \in I\} \subseteq \mathbf{A}$ such that $\mathfrak{A}$ is the algebraic closure of the subfield $\mathfrak{A}'$ generated by $\{a_\alpha : \alpha \in I\}$ but no a_β is in the algebraic closure of the subfield generated by the rest: $\{a_\alpha : \alpha \in I \text{ and } \alpha \neq \beta\}$.

Since the subfield generated by a countable subset would be countable and the algebraic closure of a countable subfield would also be countable, we must have that the transcendence base is uncountable. Since $|\mathbf{A}| = \aleph_1$, the *least* uncountable cardinal, we must have in fact that $|I| = \aleph_1$.

Now let $\mathfrak{B}$ be any other algebraically closed field of characteristic 0 and size $\aleph_1$. As above, obtain a transcendence basis $\{b_\beta : \beta \in J\}$ with $|J| = \aleph_1$ and its generated subfield $\mathfrak{B}'$. Since $|I| = |J|$, there is a bijection $g : I \to J$ which we can use to build an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Since $\mathfrak{B}$ has characteristic 0, a standard theorem of algebra gives that the rationals are isomorphically embedded into $\mathfrak{B}$. Let this embedding be:

$$f : \mathfrak{Q} \hookrightarrow \mathfrak{B}.$$

We extend f as follows: for each $\alpha \in I$, let $f(a_\alpha) = b_{g(\alpha)}$, which maps the transcendence basis of $\mathfrak{A}$ into the transcendence basis of $\mathfrak{B}$.

We now extend f to map $\mathfrak{A}'$ onto $\mathfrak{B}'$ as follows: Each element of $\mathfrak{A}'$ is given by

$$\frac{p(a_{\alpha_1}, \ldots, a_{\alpha_m})}{q(a_{\alpha_1}, \ldots, a_{\alpha_m})},$$

where p and q are polynomials with rational coefficients and the a's come, of course, from the transcendence basis.

Let f map such an element to

$$\frac{\bar{p}(b_{g(\alpha_1)}, \ldots, b_{g(\alpha_m)})}{\bar{q}(b_{g(\alpha_1)}, \ldots, b_{g(\alpha_m)})}$$

where $\bar{p}$ and $\bar{q}$ are polynomials whose coefficients are the images under f of the rational coefficients of p and q.

The final extension of f to all of $\mathfrak{A}$ and $\mathfrak{B}$ comes from the uniqueness of algebraic closures.

□

REMARK. Lemma 7 is also true when 0 is replaced by any fixed characteristic and $\aleph_1$ by any uncountable cardinal.

THEOREM 9. *Let $\mathcal{H}$ be a set of sentences in the language of field theory which are true in algebraically closed fields of arbitrarily high characteristic. Then $\mathcal{H}$ holds in some algebraically closed field of characteristic 0.*

PROOF. A field is a model in the language $\{+, \cdot, 0, 1\}$ of the axioms of field theory. Let ACF be the set of axioms for the theory of algebraically closed fields; see Example 5. For each $n \geq 2$, let τ_n denote the sentence

$$\neg(\overbrace{1 + 1 + \cdots + 1}^{n}) = 0$$

Let $\Sigma = \text{ACF} \cup \mathcal{H} \cup \{\tau_n : n \geq 2\}$

Let Σ' be any finite subset of Σ and let m be the largest natural number such that $\tau_m \in \Sigma'$ or let $m = 1$ by default.

Let $\mathfrak{A}$ be an algebraically closed field of characteristic $p > m$ such that $\mathfrak{A} \models \mathcal{H}$; then in fact $\mathfrak{A} \models \Sigma'$.

So by compactness there is $\mathfrak{B}$ such that $\mathfrak{B} \models \Sigma$. $\mathfrak{B}$ is the required field.

$\square$

COROLLARY 1. *Let $\mathbb{C}$ denote, as usual, the complex numbers. Every one-to-one polynomial map $f : \mathbb{C}^m \to \mathbb{C}^m$ is onto.*

PROOF. A *polynomial map* is a function of the form

$$f(x_1, \ldots, x_m) = \langle p_1(x_1, \ldots, x_m), \ldots, p_m(x_1, \ldots, x_m)\rangle$$

where each p_i is a polynomial in the variables $x_1, \ldots, x_m$.

We call max { degree of $p_i : i \leq m\}$ the *degree* of f.

Let $\mathcal{L}$ be the language of field theory and let $\theta_{m,n}$ be the sentence of $\mathcal{L}$ which expresses that "each polynomial map of m variables of degree < n which is one-to-one is also onto".

We wish to show that there are algebraically closed fields of arbitrarily high characteristic which satisfy $\mathcal{H} = \{\theta_{m,n} : m, n \in \mathbb{N}\}$. We will then apply Theorem 9, Theorem 8, Lemma 6 and Exercise 5 and be finished.

Let p be any prime and let F_p be the prime Galois field of size p. The algebraic closure $\tilde{F_p}$ is the countable union of a chain of finite fields

$$F_p = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq A_{k+1} \subseteq \cdots$$

obtained by recursively adding roots of polynomials.

We finish the proof by showing that each $\langle \tilde{F_p}, \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1}\rangle$ satisfies $\mathcal{H}$.

Given any polynomial map $f : (\tilde{F_p}^m) \to (\tilde{F_p}^m)$ which is one-to-one, we show that f is also onto. Given any elements $b_1, \ldots, b_m \in \tilde{F_p}$, there is some A_k containing $b_1, \ldots, b_m$ as well as all the coefficients of f.

Since f is one-to-one, $f \restriction A_k^m : A_k^m \to A_k^m$ is a one-to-one polynomial map.

Hence, since A_k^m is finite, $f \restriction A_k^m$ is onto and so there are $a_1, \ldots, a_m \in A_k$ such that $f(a_1, \ldots, a_m) = \langle b_1, \ldots, b_m\rangle$. Therefore f is onto.

Thus, for each prime number p and each $m, n \in \mathbb{N}$, $\theta_{m,n}$ holds in a field of characteristic p, i.e. $\langle \tilde{F_p}, \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1}\rangle$ satisfies $\mathcal{H}$.

$\square$

It is a significant problem to replace "one-to-one" with "locally one-to-one".

CHAPTER 3

Diagrams and Embeddings

Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{I} \rangle$ be a model for a language $\mathcal{L}$ and $X \subseteq \mathbf{A}$. Expand $\mathcal{L}$ to $\mathcal{L}_X = \mathcal{L} \cup \{c_a : a \in X\}$ by adding new constant symbols to $\mathcal{L}$. We can expand $\mathfrak{A}$ to a model $\mathfrak{A}_X = \langle \mathbf{A}, \mathcal{I}' \rangle$ for $\mathcal{L}_X$ by choosing $\mathcal{I}'$ extending $\mathcal{I}$ such that $\mathcal{I}'(c_a) = a$ for each $a \in X$. We sometimes write this as $\langle \mathfrak{A}, x \rangle_{x \in X}$. We often deal with the case $X = \mathbf{A}$, to obtain $\mathfrak{A}_{\mathbf{A}}$.

Exercise 11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}$ with $X \subseteq \mathbf{A} \subseteq \mathbf{B}$. Prove:

(i) if $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A}_X \subseteq \mathfrak{B}_X$.
(ii) if $\mathfrak{A} \prec \mathfrak{B}$ then $\mathfrak{A}_X \prec \mathfrak{B}_X$.

Hint: $\mathfrak{A} \models \varphi[a_1, \dots, a_p]$ iff $\mathfrak{A}_{\mathbf{A}} \models \varphi^*$ where φ^* is the sentence of $\mathcal{L}_{\mathbf{A}}$ formed by replacing each free occurence of v_i with c_{a_i}.

Definition 25. Let $\mathfrak{A}$ be a model for $\mathcal{L}$.

(1) The *elementary diagram* of $\mathfrak{A}$ is $\mathrm{Th}\mathfrak{A}_{\mathbf{A}}$, the set of all sentences of $\mathcal{L}_{\mathbf{A}}$ which hold in $\mathfrak{A}_{\mathbf{A}}$.
(2) The *diagram* of $\mathfrak{A}$, denoted by $\triangle_{\mathfrak{A}}$, is the set of all those sentences in $\mathrm{Th}\mathfrak{A}_{\mathbf{A}}$ without quantifiers.

Remark. There is a notion of *atomic formula*, which is a formula of the form $t_1 = t_2$ or $R(t_1 \dots t_n)$ where $t_1, \dots, t_n$ are terms. Sometimes $\triangle_{\mathfrak{A}}$ is defined to be the set of all atomic formulas and negations of atomic formulas which occur in $\mathrm{Th}\mathfrak{A}_{\mathbf{A}}$. However this is not substantially different from Definition 25, since the reader can quickly show that for any model $\mathfrak{B}$, $\mathfrak{B} \models \triangle_{\mathfrak{A}}$ in one sense iff $\mathfrak{B} \models \triangle_{\mathfrak{A}}$ in the other sense.

Definition 26. $\mathfrak{A}$ is said to be *isomorphically embedded* into $\mathfrak{B}$ whenever

(1) there is a model $\mathfrak{C}$ such that $\mathfrak{A} \cong \mathfrak{C}$ and $\mathfrak{C} \subseteq \mathfrak{B}$

or

(2) there is a model $\mathfrak{D}$ such that $\mathfrak{A} \subseteq \mathfrak{D}$ and $\mathfrak{D} \cong \mathfrak{B}$.

Exercise 12. Prove that, in fact, (1) and (2) are equivalent conditions.

Definition 27. $\mathfrak{A}$ is said to be *elementarily embedded* into $\mathfrak{B}$ whenever

(1) there is a model $\mathfrak{C}$ such that $\mathfrak{A} \cong \mathfrak{C}$ and $\mathfrak{C} \prec \mathfrak{B}$

or

(2) there is a model $\mathfrak{D}$ such that $\mathfrak{A} \prec \mathfrak{D}$ and $\mathfrak{D} \cong \mathfrak{B}$.

Exercise 13. Again, prove that, in fact, (1) and (2) are equivalent.

Theorem 10. *(The diagram lemmas) Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}$.*

(1) $\mathfrak{A}$ *is isomorphically embedded in* $\mathfrak{B}$ *iff* $\mathfrak{B}$ *can be expanded to a model of* $\triangle_{\mathfrak{A}}$.
(2) $\mathfrak{A}$ *is elementarily embedded in* $\mathfrak{B}$ *iff* $\mathfrak{B}$ *can be expanded to a model of* $Th(\mathfrak{A}_{\mathbf{A}})$.

PROOF. We sketch the proof of 1.
($\Rightarrow$) If f is as in 1 of Definition 26 above, then $\langle \mathfrak{B}, f(a)\rangle_{a\in\mathbf{A}} \models \triangle_{\mathfrak{A}}$.
($\Leftarrow$) If $\langle \mathfrak{B}, b_a\rangle_{a\in\mathbf{A}} \models \triangle_{\mathfrak{A}}$, then let $f(a) = b_a$.

□

EXERCISE 14. Give a careful proof of part 2 of the theorem.

We now apply these notions to graph theory and to calculus. The natural language for graph theory has one binary relation symbol which we call E (to suggest the word "edge"). Graph Theory has the following two axioms:

- $(\forall x)(\forall y)E(x,y) \leftrightarrow E(y,x)$
- $(\forall x)\neg E(x,x)$.

A graph is, of course, a model of graph theory.

COROLLARY 2. *Every planar graph can be four coloured.*

PROOF. We will have to use the famous result of Appel and Haken that every *finite* planar graph can be four coloured. Model Theory will take us from the finite to the infinite. We recall that a planar graph is one that can be embedded, or drawn, in the usual Euclidean plane and to be four coloured means that each vertex of the graph can be assigned one of four colours in such a way that no edge has the same colour for both endpoints.

Let $\mathfrak{A}$ be an infinite planar graph. Introduce four new unary relation symbols: R, G, B, Y (for red, green, blue and yellow). We wish to prove that there is some expansion $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \sigma$ where σ is the sentence in the expanded language:

$$
\begin{aligned}
&(\forall x)[R(x) \vee G(x) \vee B(x) \vee Y(x)] \\
&\qquad\qquad\qquad \wedge (\forall x)[R(x) \rightarrow \neg(G(x) \vee B(x) \vee Y(x))] \wedge \ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge (\forall x)(\forall y)\neg(R(x) \wedge R(y) \wedge E(x,y)) \wedge \cdots
\end{aligned}
$$

which will ensure that the interpretations of R, G, B and Y will four colour the graph.

Let $\Sigma = \triangle_{\mathfrak{A}} \cup \{\sigma\}$. Any finite subset of Σ has a model, based upon the appropriate finite subset of $\mathfrak{A}$. By the compactness theorem, we get $\mathfrak{B} \models \Sigma$. Since $\mathfrak{B} \models \sigma$, the interpretations of R, G, B and Y four colour it. By the diagram lemma $\mathfrak{A}$ is isomorphically embedded in the reduct of $\mathfrak{B}$, and this isomorphism delivers the four-colouring of $\mathfrak{A}$.

□

A graph with the property that every pair of vertices is connected with an edge is called *complete*. At the other extreme, a graph with no edges is called *discrete*. A very important theorem in finite combinatorics says that most graphs contain an example of one or the other as a subgraph. A subgraph of a graph is, of course, a submodel of a model of graph theory.

Corollary 3. *(Ramsey's Theorem)*
For each $n \in \mathbb{N}$ there is an $r \in \mathbb{N}$ such that if $\mathfrak{G}$ is any graph with r vertices, then either $\mathfrak{G}$ contains a complete subgraph with n vertices or a discrete subgraph with n vertices.

Proof. We follow F. Ramsey who began by proving an infinite version of the theorem (also called Ramsey's Theorem).

Claim. *Each infinite graph $\mathfrak{G}$ contains either an infinite complete subgraph or an infinite discrete subgraph.*

Proof of Claim. By force of logical necessity, there are two possiblities:

(1) there is an infinite $X \subseteq \mathbf{G}$ such that for all $x \in X$ there is a finite $F_x \subseteq X$ such that $E(x, y)$ for all $y \in X \setminus F_x$,
(2) for all infinite $X \subseteq \mathbf{G}$ there is a $x \in X$ and an infinite $Y \subseteq X$ such that $\neg E(x, y)$ for all $y \in Y$.

If (1) occurs, we recursively pick $x_1 \in X$, $x_2 \in X \setminus F_{x_1}$, $x_3 \in X \setminus (F_{x_1} \cup F_{x_2})$, etc, to obtain an infinite complete subgraph. If (2) occurs we pick $x_0 \in \mathbf{G}$ and $Y_0 \subseteq \mathbf{G}$ with the property and then recursively choose $x_1 \in Y_0$ and $Y_1 \subseteq Y_0$, $x_2 \in Y_1$ and $Y_2 \subseteq Y_1$ and so on, to obtain an infinite discrete subgraph.

We now use Model Theory to go from the infinite to the finite. Let σ be the sentence, of the language of graph theory, asserting that there is no complete subgraph of size n.

$$(\forall x_1 \dots \forall x_n)[\neg E(x_1, x_2) \vee \neg E(x_1, x_3) \vee \cdots \vee \neg E(x_{n-1}, x_n)].$$

Let τ be the sentence asserting that there is no discrete subgraph of size n.

$$(\forall x_1 \dots \forall x_n)[E(x_1, x_2) \vee E(x_1, x_3) \vee \cdots \vee E(x_{n-1}, x_n)].$$

Let $\mathcal{T}$ be the set consisting of σ, τ and the axioms of graph theory.

If there is no r as Ramsey's Theorem states, then $\mathcal{T}$ has arbitrarily large finite models. By Theorem 2, $\mathcal{T}$ has an infinite model, contradicting the claim.

□

The following theorem of A. Robinson finally solved the centuries old problem of infinitesimals in the foundations of calculus.

Theorem 11. *(The Leibniz Principle)*
There is an ordered field ${}^*\mathbb{R}$ called the hyperreals, containing the reals $\mathbb{R}$ and an infinitesimal number such that any statement about the reals which holds in $\mathbb{R}$ also holds in ${}^*\mathbb{R}$.

Proof. Let $\mathfrak{R}$ be $\langle \mathbb{R}, +, \cdot, <, \mathbf{0}, \mathbf{1} \rangle$. We will make the statement of the theorem precise by proving that there is some model $\mathfrak{H}$, in the same language $\mathcal{L}$ as $\mathfrak{R}$ and with the universe called ${}^*\mathbb{R}$, such that $\mathfrak{R} \prec \mathfrak{H}$ and there is $b \in {}^*\mathbb{R}$ such that $0 < b < a$ for each positive $a \in \mathbb{R}$.

For each real number a, we introduce a new constant symbol c_a. In addition, another new constant symbol d is introduced. Let Σ be the set of sentences in the expanded language given by:

$$\mathrm{Th}\mathfrak{R}_{\mathbb{R}} \cup \{0 < d < c_a : a \text{ is a positive real } \}$$

We can obtain a model $\mathfrak{C} \models \Sigma$ by the compactness theorem. Let $\mathfrak{C}'$ be the reduct of $\mathfrak{C}$ to $\mathcal{L}$. By the elementary diagram lemma $\mathfrak{R}$ is elementarily embedded in $\mathfrak{C}'$, and so there is a model $\mathfrak{H}$ for $\mathcal{L}$ such that $\mathfrak{C}' \cong \mathfrak{H}$ and $\mathfrak{R} \prec \mathfrak{H}$.

□

REMARK. This idea is extremely useful in understanding calculus. An element $x \in {}^*\mathbb{R}$ is said to be *infinitesimal* whenever $-r < x < r$ for each positive $r \in \mathbb{R}$. 0 is infinitesimal. Two elements $x, y \in \mathbb{R}$ are said to be *infinitely close*, written $x \approx y$ whenever $x - y$ is infinitesimal. Note: x is infinitesimal iff $x \approx 0$. An element $x \in {}^*\mathbb{R}$ is said to be *finite* whenever $-r < x < r$ for some positive $r \in \mathbb{R}$. Else it is *infinite.*

Each finite $x \in {}^*\mathbb{R}$ is infinitely close to some real number, called the *standard part* of x, written $st(x)$.

To differentiate f, for each $\triangle x \in {}^*\mathbb{R}$ generate $\triangle y = f(x + \triangle x) - f(x)$. Then $f'(x) = st\left(\frac{\triangle y}{\triangle x}\right)$ whenever this exists and is the same for each infinitesimal $\triangle x \neq 0$.

The *increment lemma* states that if $y = f(x)$ is differentiable at x and $\triangle\, x \approx 0$, then $\triangle y = f'(x) \triangle x + \varepsilon \triangle x$ for some infinitesimal ε.

Proofs of the usual theorems of calculus are now easier.

The following theorem is considered one of the most fundamental results of mathematical logic. We give a detailed proof.

THEOREM 12. *(Robinson Consistency Theorem)*
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages with $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are satisfiable theories in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is satisfiable iff there is no sentence σ of $\mathcal{L}$ such that $\mathcal{T}_1 \models \sigma$ and $\mathcal{T}_2 \models \neg\sigma$.

PROOF. The direction $\Rightarrow$ is easy and motivates the whole theorem.

We begin the proof in the $\Leftarrow$ direction. Our goal is to show that $\mathcal{T}_1 \cup \mathcal{T}_2$ is satisfiable. The following claim is a first step.

CLAIM. $\mathcal{T}_1 \cup \{$ *sentences* σ *of* $\mathcal{L} : \mathcal{T}_2 \models \sigma\}$ *is satisfiable.*

PROOF OF CLAIM. Using the compactness theorem and considering conjunctions, it suffices to show that if $\mathcal{T}_1 \models \sigma_1$ and $\mathcal{T}_2 \models \sigma_2$ with σ_2 a sentence of $\mathcal{L}$, then $\{\sigma_1, \sigma_2\}$ is satisfiable. But this is true, since otherwise we would have $\sigma_1 \models \neg\sigma_2$ and hence $\mathcal{T}_1 \models \neg\sigma_2$ and so $\neg\sigma_2$ would be a sentence of $\mathcal{L}$ contradicting our hypothesis.

The basic idea of the proof from now on is as follows. In order to construct a model of $\mathcal{T}_1 \cup \mathcal{T}_2$ we construct models $\mathfrak{A} \models \mathcal{T}_1$ and $\mathfrak{B} \models \mathcal{T}_2$ and an isomorphism $f : \mathfrak{A}|\mathcal{L} \to \mathfrak{B}|\mathcal{L}$ between the reducts of $\mathfrak{A}$ and $\mathfrak{B}$ to the language $\mathcal{L}$, witnessing that $\mathfrak{A}|\mathcal{L} \cong \mathfrak{B}|\mathcal{L}$. We then use f to carry over interpretations of symbols in $\mathcal{L}_1 \setminus \mathcal{L}$ from $\mathfrak{A}$ to $\mathfrak{B}$, giving an expansion $\mathfrak{B}^*$ of $\mathfrak{B}$ to the language $\mathcal{L}_1 \cup \mathcal{L}_2$. Then, since $\mathfrak{B}^*|\mathcal{L}_1 \cong \mathfrak{A}$ and $\mathfrak{B}^*|\mathcal{L}_2 = \mathfrak{B}$ we get $\mathfrak{B}^* \models \mathcal{T}_1 \cup \mathcal{T}_2$.

The remainder of the proof will be devoted to constructing such an $\mathfrak{A}$, $\mathfrak{B}$ and f. $\mathfrak{A}$ and $\mathfrak{B}$ will be constructed as unions of elementary chains of $\mathfrak{A}_n$'s and $\mathfrak{B}_n$'s while f will be the union of $f_n : \mathfrak{A}_n \hookrightarrow \mathfrak{B}_n$.

We begin with $n = 0$, the first link in the elementary chain.

CLAIM. *There are models $\mathfrak{A}_0 \models \mathcal{T}_1$ and $\mathfrak{B}_0 \models \mathcal{T}_2$ with an elementary embedding $f_0 : \mathfrak{A}_0|\mathcal{L} \hookrightarrow \mathfrak{B}_0|\mathcal{L}$.*

Proof of Claim. Using the previous claim, let

$$\mathfrak{A}_0 \models \mathcal{T}_1 \cup \{ \text{ sentences } \sigma \text{ of } \mathcal{L} : \mathcal{T}_2 \models \sigma\}$$

We first wish to show that $\mathrm{Th}(\mathfrak{A}_0|\mathcal{L})_{\mathbf{A}_0} \cup \mathcal{T}_2$ is satisfiable. Using the compactness theorem, it suffices to prove that if $\sigma \in \mathrm{Th}(\mathfrak{A}_0|\mathcal{L})_{\mathbf{A}_0}$ then $\mathcal{T}_2 \cup \{\sigma\}$ is satisfiable. For such a σ let $c_{a_0}, \ldots, c_{a_n}$ be all the constant symbols from $\mathcal{L}_{\mathbf{A}_0} \setminus \mathcal{L}$ which appear in σ. Let φ be the formula of $\mathcal{L}$ obtained by replacing each constant symbol c_{a_i} by a new variable u_i. We have

$$\mathfrak{A}_0|\mathcal{L} \models \varphi[a_0, \ldots, a_n]$$

$$\text{and so } \mathfrak{A}_0|\mathcal{L} \models \exists u_0 \ldots \exists u_n \varphi$$

By the definition of $\mathfrak{A}_0$, it cannot happen that $\mathcal{T}_2 \models \neg\exists u_0 \ldots \exists u_n \varphi$ and so there is some model $\mathfrak{D}$ for $\mathcal{L}_2$ such that $\mathfrak{D} \models \mathcal{T}_2$ and $\mathfrak{D} \models \exists u_0 \ldots \exists u_n \varphi$. So there are elements $d_0, \ldots, d_n$ of $\mathbf{D}$ such that $\mathfrak{D} \models \varphi[d_o, \ldots, d_n]$. Expand $\mathfrak{D}$ to a model $\mathfrak{D}^*$ for $\mathcal{L}_2 \cup \mathcal{L}_{\mathbf{A}_0}$, making sure to interpret each c_{a_i} as d_i. Then $\mathfrak{D}^* \models \sigma$, and so $\mathfrak{D}^* \models \mathcal{T}_2 \cup \{\sigma\}$.

Let $\mathfrak{B}_0^* \models \mathrm{Th}(\mathfrak{A}_0|\mathcal{L})_{\mathbf{A}_0} \cup \mathcal{T}_2$. Let $\mathfrak{B}_0$ be the reduct of $\mathfrak{B}_0^*$ to $\mathcal{L}_2$; clearly $\mathfrak{B}_0 \models \mathcal{T}_2$. Since $\mathfrak{B}_0|\mathcal{L}$ can be expanded to a model of $\mathrm{Th}(\mathfrak{A}_0|\mathcal{L})_{\mathbf{A}_0}$, the Elementary Diagram Lemma gives an elementary embedding

$$f_0 : \mathfrak{A}_0|\mathcal{L} \hookrightarrow \mathfrak{B}_0|\mathcal{L}$$

and finishes the proof of the claim.

The other links in the elementary chain are provided by the following result.

Claim. *For each $n \geq 0$ there are models $\mathfrak{A}_{n+1} \models \mathcal{T}_1$ and $\mathfrak{B}_{n+1} \models \mathcal{T}_2$ with an elementary embedding*

$$f_{n+1} : \mathfrak{A}_{n+1}|\mathcal{L} \hookrightarrow \mathfrak{B}_{n+1}|\mathcal{L}$$

such that

$$\mathfrak{A}_n \prec \mathfrak{A}_{n+1},\ \mathfrak{B}_n \prec \mathfrak{B}_{n+1},\ f_{n+1} \text{ extends } f_n \text{ and } \mathbf{B}_n \subseteq \text{ range of } f_{n+1}.$$

$$\begin{array}{ccccccccccc}
\mathfrak{A}_0 & \prec & \mathfrak{A}_1 & \prec & \cdots & \prec & \mathfrak{A}_n & \prec & \mathfrak{A}_{n+1} & \prec & \cdots \\
\downarrow_{f_0} & & \downarrow_{f_1} & & & & \downarrow_{f_n} & & \downarrow_{f_{n+1}} & & \\
\mathfrak{B}_0 & \prec & \mathfrak{B}_1 & \prec & \cdots & \prec & \mathfrak{B}_n & \prec & \mathfrak{B}_{n+1} & \prec & \cdots
\end{array}$$

The proof of this claim will be discussed shortly. Assuming the claim, let $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$, $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ and $f = \bigcup_{n\in\mathbb{N}} f_n$. The Elementary Chain Theorem gives that $\mathfrak{A} \models \mathcal{T}_1$ and $\mathfrak{B} \models \mathcal{T}_2$. The proof of the theorem is concluded by simply verifying that $f : \mathfrak{A}|\mathcal{L} \to \mathfrak{B}|\mathcal{L}$ is an isomorphism.

The proof of the claim is long and quite technical; it would not be inappropriate to omit it on a first reading. The proof, of course, must proceed by induction on n. The case of a general n is no different from the case $n = 0$ which we state and prove in some detail.

CLAIM. *There are models $\mathfrak{A}_1 \models \mathcal{T}_1$ and $\mathfrak{B}_1 \models \mathcal{T}_2$ with an elementary embedding $f_1 : \mathfrak{A}_1|\mathcal{L} \hookrightarrow \mathfrak{B}_1|\mathcal{L}$ such that $\mathfrak{A}_0 \prec \mathfrak{A}_1$, $\mathfrak{B}_0 \prec \mathfrak{B}_1$, f_1 extends f_0 and $\mathbf{B}_0 \subseteq$ range of f_1.*

$$\begin{array}{ccc} \mathfrak{A}_0 & \prec & \mathfrak{A}_1 \\ \downarrow_{f_0} & & \downarrow_{f_1} \\ \mathfrak{B}_0 & \prec & \mathfrak{B}_1 \end{array}$$

PROOF OF CLAIM. Let $\mathfrak{A}_0^+$ be the expansion of $\mathfrak{A}_0$ to the language $\mathcal{L}_1^+ = \mathcal{L}_1 \cup \{c_a : a \in \mathbf{A}_0\}$ formed by interpreting each c_a as $a \in \mathbf{A}_0$; $\mathfrak{A}_0^+$ is just another notation for $(\mathfrak{A}_0)_{\mathbf{A}_0}$. The elementary diagram of $\mathfrak{A}_0^+$ is $\mathrm{Th}\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+}$. Let $\mathfrak{B}_0^*$ be the expansion of $\mathfrak{B}_0|\mathcal{L}$ to the language

$$\mathcal{L}^* = \mathcal{L} \cup \{c_a : a \in \mathbf{A}_0\} \cup \{c_b : b \in \mathbf{B}_0\}$$

formed by interpreting each c_a as $f_0(a) \in \mathbf{B}_0$ and each c_b as $b \in \mathbf{B}_0$.

We wish to prove that $\mathrm{Th}\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+} \cup \mathrm{Th}\mathfrak{B}_0^*$ is satisfiable. By the compactness theorem it suffices to prove that $\mathrm{Th}\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+} \cup \{\sigma\}$ is satisfiable for each σ in $\mathrm{Th}\mathfrak{B}_0^*$. For such a sentence σ, let $c_{a_0}, \ldots, c_{a_m}, c_{b_0}, \ldots, c_{b_n}$ be all those constant symbols occuring in σ but not in $\mathcal{L}$. Let $\varphi(u_0, \ldots, u_m, w_0, \ldots, w_m)$ be the formula of $\mathcal{L}$ obtained from σ by replacing each constant symbol c_{a_i} by a new variable u_i and each constant symbol c_{b_i} by a new variable w_i. We have $\mathfrak{B}_0^* \models \sigma$ so

$$\mathfrak{B}_0|\mathcal{L} \models \varphi[f_0(a_0), \ldots, f_0(a_m), b_0, \ldots, b_n]$$

$$\text{So } \mathfrak{B}_0|\mathcal{L} \models \exists w_0 \ldots \exists w_n \varphi[f_0(a_0), \ldots, f_0(a_m)]$$

Since f_0 is an elementary embedding we have :

$$\mathfrak{A}_0|\mathcal{L} \models \exists w_0 \ldots \exists w_n \varphi[a_0, \ldots, a_m]$$

Let $\hat{\varphi}(w_0, \ldots, w_n)$ be the formula of $\mathcal{L}_1^+$ obtained by replacing occurences of u_i in $\varphi(u_0, \ldots, u_m, w_0, \ldots, w_n)$ by c_{a_i}; then $\mathfrak{A}_0^+ \models \exists w_0 \ldots \exists w_n \hat{\varphi}$. So, of course,

$$\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+} \models \exists w_0 \ldots \exists w_n \hat{\varphi}$$

and this means that there are $d_0, \ldots, d_n$ in $\mathbf{A}_0^+ = \mathbf{A}_0$ such that

$$(\mathfrak{A}_0^+)_{\mathbf{A}_0^+} \models \hat{\varphi}[d_0, \ldots, d_n].$$

We can now expand $\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+}$ to a model $\mathfrak{D}$ by interpreting each c_{b_i} as d_i to obtain $\mathfrak{D} \models \sigma$ and so $\mathrm{Th}\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+} \cup \{\sigma\}$ is satisfiable.

Let $\mathfrak{E} \models \mathrm{Th}\left(\mathfrak{A}_0^+\right)_{\mathbf{A}_0^+} \cup \mathrm{Th}\mathfrak{B}_0^*$. By the elementary diagram lemma $\mathfrak{A}_0^+$ is elementarily embedded into $\mathfrak{E}|\mathcal{L}_1^+$. So there is a model $\mathfrak{A}_1^+$ for $\mathcal{L}_1^+$ with $\mathfrak{A}_0^+ \prec \mathfrak{A}_1^+$ and an isomorphism $g : \mathfrak{A}_1^+ \to \mathfrak{E}|\mathcal{L}_1^+$. Using g we expand $\mathfrak{A}_1^+$ to a model $\mathfrak{A}_1'$ isomorphic to $\mathfrak{E}$. Let $\mathfrak{A}_1^*$ denote $\mathfrak{A}_1'|\mathcal{L}^*$; we have $\mathfrak{A}_1^* \models \mathrm{Th}\mathfrak{B}_0^*$.

We now wish to prove that $\mathrm{Th}\left(\mathfrak{A}_1^*\right)_{\mathbf{A}_1^*} \cup \mathrm{Th}\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+}$ is satisfiable, where $\mathfrak{B}_0^+$ is the common expansion of $\mathfrak{B}_0$ and $\mathfrak{B}_0^*$ to the language

$$\mathcal{L}_2^+ = \mathcal{L}_2 \cup \{c_a : a \in \mathbf{A}_0\} \cup \{c_b : b \in \mathfrak{B}_0\}.$$

By the compactness theorem, it suffices to show that

$$\mathrm{Th}\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+} \cup \{\sigma\}$$

is satisfiable for each σ in $\mathrm{Th}\,(\mathfrak{A}_1^*)_{\mathbf{A}_1^*}$. Let $c_{x_0}, \ldots, c_{x_n}$ be all those constant symbols which occur in σ but are not in $\mathcal{L}^*$. Let $\psi(u_0, \ldots, u_n)$ be the formula of $\mathcal{L}^*$ obtained from σ by replacing each c_{x_i} with a new variable u_i. Since $(\mathfrak{A}_1^*)_{\mathbf{A}_1^*} \models \sigma$ we have

$$\mathfrak{A}_1^* \models \psi[x_0, \ldots, x_n],$$

and so

$$\mathfrak{A}_1^* \models \exists u_0 \ldots \exists u_n \psi.$$

Also $\mathfrak{A}_1^* \models \mathrm{Th}\mathfrak{B}_0^*$ and $\mathrm{Th}\mathfrak{B}_0^*$ is a complete theory in the language $\mathcal{L}^*$; hence $\exists u_0 \ldots \exists u_n \psi$ is in $\mathrm{Th}\mathfrak{B}_0^*$. Thus

$$\mathfrak{B}_0^* \models \exists u_0 \ldots \exists u_n \psi$$

and so

$$\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+} \models \exists u_0 \ldots \exists u_n \psi$$

and therefore there are $b_0, \ldots, b_n$ in $\mathbf{B}_0^+ = \mathbf{B}_0$ such that

$$\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+} \models \psi[b_0, \ldots, b_n].$$

We can now expand $\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+}$ to a model $\mathfrak{F}$ by interpreting each c_{x_i} as b_i; then $\mathfrak{F} \models \sigma$ and $\mathrm{Th}\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+} \cup \{\sigma\}$ is satisfiable.

Let $\mathfrak{G} \models \mathrm{Th}\,(\mathfrak{A}_1^*)_{\mathbf{A}_1^*} \cup \mathrm{Th}\left(\mathfrak{B}_0^+\right)_{\mathbf{B}_0^+}$. By the elementary diagram lemma $\mathfrak{B}_0^+$ is elementarily embedded into $\mathfrak{G}|\mathcal{L}_2^+$. So there is a model $\mathfrak{B}_1^+$ for $\mathcal{L}_2^+$ with $\mathfrak{B}_0^+ \prec \mathfrak{B}_1^+$ and an isomorphism $h : \mathfrak{B}_1^+ \to \mathfrak{G}|\mathcal{L}_2^+$. Using h we expand $\mathfrak{B}_1^+$ to a model $\mathfrak{B}_1'$ isomorphic to $\mathfrak{G}$. Let $\mathfrak{B}_1^*$ denote $\mathfrak{B}_1'|\mathcal{L}^*$. Again by the elementary diagram lemma $\mathfrak{A}_1^*$ is elementarily embedded into $\mathfrak{B}_1^*$. Let this be denoted by

$$f_1 : \mathfrak{A}_1^* \hookrightarrow \mathfrak{B}_1^*.$$

Let $a \in \mathbf{A}_0$; we will show that $f_0(a) = f_1(a)$. By definition we have $\mathfrak{B}_0^* \models (v_0 = c_a)[f_0(a)]$ and so $\mathfrak{B}_0^+ \models (v_0 = c_a)[f_0(a)]$. Since $\mathfrak{B}_0^+ \prec \mathfrak{B}_1^+$, $\mathfrak{B}_1^+ \models (v_0 = c_a)[f_0(a)]$ and so $\mathfrak{B}_1^* \models (v_0 = c_a)[f_0(a)]$. Now $\mathfrak{A}_0^+ \models (c_a = v_1)[a]$ and $\mathfrak{A}_0^+ \prec \mathfrak{A}_1^+$ so $\mathfrak{A}_1^+ \models (c_a = v_1)[a]$ so $\mathfrak{A}_1^* \models (c_a = v_1)[a]$. Since f_1 is elementary, $\mathfrak{B}_1^* \models (c_a = v_1)[f_1(a)]$ so $\mathfrak{B}_1^* \models (v_0 = v_1)[f_0(a), f_1(a)]$ and so $f_0(a) = f_1(a)$.

Thus f_1 extends f_0.

Let $b \in \mathbf{B}_0$; we will prove that $b = f_1(a)$ for some $a \in \mathbf{A}_1$. By definition we have: $\mathfrak{B}_0^* \models (v_0 = c_b)[b]$ so $\mathfrak{B}_0^+ \models (v_0 = c_b)[b]$. Since $\mathfrak{B}_0^+ \prec \mathfrak{B}_1^+, \mathfrak{B}_1^+ \models (v_0 = c_b)[b]$ so $\mathfrak{B}_1^* \models (v_0 = c_b)[b]$. On the other hand, since $(\exists v_1)(v_1 = c_b)$ is always satisfied, we have: $\mathfrak{A}_1^* \models (\exists v_1)(v_1 = c_b)$ so there is $a \in \mathbf{A}_1$ such that $\mathfrak{A}_1^* \models (v_1 = c_b)[a]$. Since f_1 is elementary, $\mathfrak{B}_1^* \models (v_1 = c_b)[f_1(a)]$ so $\mathfrak{B}_1^* \models (v_0 = v_1)[b, f_1(a)]$ so $b = f_1(a)$.

Thus $\mathfrak{B}_0 \subseteq$ range of f_1.

We now let $\mathfrak{A}_1$ be $\mathfrak{A}_1^+|\mathcal{L}_1$ and let $\mathfrak{B}_1$ be $\mathfrak{B}_1^+|\mathcal{L}_2$. We get $\mathfrak{A}_0 \prec \mathfrak{A}_1$ and $\mathfrak{B}_0 \prec \mathfrak{B}_1$ and $f_1 : \mathfrak{A}_1|\mathcal{L} \to \mathfrak{B}_1|\mathcal{L}$ remains an elementary embedding.

This completes the proof of the claim. □

Exercise 15. The Robinson Consistency Theorem was originally stated as:

> Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be satisfiable theories in languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively and let $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ be a complete theory in the language $\mathcal{L}_1 \cap \mathcal{L}_2$. Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is satisfiable in the language $\mathcal{L}_1 \cup \mathcal{L}_2$.

Show that this is essentially equivalent to our version in Theorem 12 by first proving that this statement follows from Theorem 12 and then also proving that this statement implies Theorem 12. Of course, for this latter argument you are looking for a proof much shorter than our proof of Theorem 12; however it will help to use the first claim of our proof in your own proof.

Theorem 13. *(Craig Interpolation Theorem)*
Let φ and ψ be sentences such that $\varphi \models \psi$. Then there exists a sentence θ, called the interpolant, such that $\varphi \models \theta$ and $\theta \models \psi$ and every relation, function or constant symbol occuring in θ also occurs in both φ and ψ.

Exercise 16. Show that the Craig Interpolation Theorem follows quickly from the Robinson Consistency Theorem. Also, use the Compactness Theorem to show that Theorem 12 follows quickly from Theorem 13.

CHAPTER 4

Model Completeness

The quantifier $\forall$ is sometimes said to be the *universal quantifier* and the quantifier $\exists$ to be the *existential quantifier*.

A formula φ is said to be *quantifier free* whenever no quantifiers occur in φ.

A formula φ is said to be *universal* whenever it is of the form $\forall x_0 \ldots \forall x_k \theta$ where θ is quantifier free.

A formula φ is said to be *existential* whenever it is of the form $\exists x_0 \ldots \exists x_k \theta$ where θ is quantifier free.

A formula φ is said to be *universal-existensial* whenever it is of the form $\forall x_0 \ldots \forall x_k \exists y_0 \ldots \exists y_k \theta$ where θ is quantifier free.

We extend these notions to theories $\mathcal{T}$ whenever each axiom σ of $\mathcal{T}$ has the property.

Remark. Note that each quantifier free formula φ is trivially equialent to the existential formula $\exists v_i \varphi$ where v_i does not occur in φ.

Exercise 17. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for $\mathcal{L}$ with $\mathbf{A} \subseteq \mathbf{B}$. Verify the following four statements:

(i) $\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{B}_\mathbf{A} \models \mathrm{Th}(\mathfrak{A}_\mathbf{A})$ iff $\mathfrak{A}_\mathbf{A} \models \mathrm{Th}(\mathfrak{B}_\mathbf{A})$.
(ii) $\mathfrak{A} \subseteq \mathfrak{B}$ iff $\mathfrak{B}_\mathbf{A} \models \triangle_\mathfrak{A}$ iff $\mathfrak{A}_\mathbf{A} \models \sigma$ for each quantifier free σ of $\mathrm{Th}(\mathfrak{B}_\mathbf{A})$.
(iii) $\mathfrak{A} \subseteq \mathfrak{B}$ iff $\mathfrak{B}_\mathbf{A} \models \sigma$ for each existential σ of $\mathrm{Th}(\mathfrak{A}_\mathbf{A})$.
(iv) $\mathfrak{A} \subseteq \mathfrak{B}$ iff $\mathfrak{A}_\mathbf{A} \models \sigma$ for each universal σ of $\mathrm{Th}(\mathfrak{B}_\mathbf{A})$.

Definition 28. A model $\mathfrak{A}$ of a theory $\mathcal{T}$ is said to be *existentially closed* if whenever $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \models \mathcal{T}$, we have $\mathfrak{A}_\mathbf{A} \models \sigma$ for each existential sentence σ of $\mathrm{Th}(\mathfrak{B}_\mathbf{A})$.

Remark. If $\mathfrak{A}$ is existentially closed and $\mathfrak{A}' \cong \mathfrak{A}$ then $\mathfrak{A}'$ is also existentially closed.

Definition 29. A theory $\mathcal{T}$ is said to be *model complete* whenever $\mathcal{T} \cup \triangle_\mathfrak{A}$ is complete in the language $\mathcal{L}_\mathbf{A}$ for each model $\mathfrak{A}$ of $\mathcal{T}$.

Theorem 14. *(A. Robinson)*
Let $\mathcal{T}$ be a theory in the language $\mathcal{L}$. The following are equivalent:

(1) *$\mathcal{T}$ is model complete,*
(2) *$\mathcal{T}$ is existentially complete, i.e. each model of $\mathcal{T}$ is existentially closed.*
(3) *for each formula $\varphi(v_0, \ldots, v_p)$ of $\mathcal{L}$ there is a universal formula $\psi(v_0, \ldots, v_p)$ such that $\mathcal{T} \models (\forall v_0 \ldots \forall v_p)(\varphi \leftrightarrow \psi)$*
(4) *for all models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{T}$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \prec \mathfrak{B}$.*

Remark. Equivalently, in part (3) of this theorem the phrase "universal formula" could be replaced by "existential formula". We chose the version which makes the proof smoother.

PROOF. (1) $\Rightarrow$ (2):
Let $\mathfrak{A} \models \mathcal{T}$ and $\mathfrak{B} \models \mathcal{T}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Clearly $\mathfrak{A}_{\mathbf{A}} \models \triangle_{\mathfrak{A}}$ and by Exercise 17 we $\mathfrak{B}_{\mathbf{A}} \models \triangle_{\mathfrak{A}}$. Now by (1), $\mathcal{T} \cup \triangle_{\mathfrak{A}}$ is complete and both $\mathfrak{A}_{\mathbf{A}}$ and $\mathfrak{B}_{\mathbf{A}}$ are models of this theory so they are elementarily equivalent.

So let σ be any sentence of $\mathcal{L}_{\mathbf{A}}$ (existential or otherwise). If $\mathfrak{B}_{\mathbf{A}} \models \sigma$ then $\mathfrak{A}_{\mathbf{A}} \models \sigma$ and (2) follows.

(2) $\Rightarrow$ (3):
Lemma 4 shows that it suffices to prove it for formulas φ in prenex normal form. We do this by induction on the prenex rank of φ which is the number of alternations of quantifiers in φ. The first step is prenex rank 0. Where only universal quantifiers are present the result is trivial. The existential formula case is non-trivial; it is the following claim:

CLAIM. *For each existential formula $\varphi(v_0, \ldots, v_p)$ of $\mathcal{L}$ there is a universal formula $\psi(v_0, \ldots, v_p)$ such that*

$$\mathcal{T} \models (\forall v_0) \ldots (\forall v_p)(\varphi \leftrightarrow \psi)$$

PROOF OF CLAIM. Add new constant symbols $c_0, \ldots, c_p$ to $\mathcal{L}$ to form

$$\mathcal{L}^* = \mathcal{L} \cup \{c_0, \ldots, c_p\}$$

and to form a sentence φ^* of $\mathcal{L}^*$ obtained by replacing each free occurrence of v_i in φ with the corresponding c_i; φ^* is an existential sentence. It suffices to prove that there is a universal sentence γ of $\mathcal{L}^*$ such that $\mathcal{T} \models \varphi^* \leftrightarrow \gamma$.

$$\text{Let } \Gamma = \{\text{universal sentences } \gamma \text{ of } \mathcal{L}^* \text{ such that } \mathcal{T} \models \varphi^* \rightarrow \gamma\}$$

We hope to prove that there is some $\gamma \in \Gamma$ such that $\mathcal{T} \models \gamma \rightarrow \varphi^*$. Note, however, that any finite conjunction $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$ of sentences from Γ is equivalent to a sentence γ in Γ which is simply obtained from $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$ by moving all the quantifiers to the front. Thus it suffices to prove that there are finitely many sentences $\gamma_1, \gamma_2, \ldots, \gamma_n$ from Γ such that

$$\mathcal{T} \models \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \rightarrow \varphi^*.$$

If no such finite set of sentences existed, then each

$$\mathcal{T} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \cup \{\neg\varphi^*\}$$

would be satisfiable. By the compactness theorem, $\mathcal{T} \cup \Gamma \cup \{\neg\varphi^*\}$ would be satisfiable. Therefore it just suffices to prove that $\mathcal{T} \cup \Gamma \models \varphi^*$.

In order to prove that $\mathcal{T} \cup \Gamma \models \varphi^*$, let $\mathfrak{A}$ be any model of $\mathcal{T} \cup \Gamma$ for the language $\mathcal{L}^*$. Consider the reduct $\mathfrak{A}|\mathcal{L}$ and form $\mathcal{L}_{\mathbf{A}}$ as usual but ensure that if the interpretation of the new constant c_i in $\mathfrak{A}$ is a then c_a is chosen to be c_i. In this way we conveniently get $\mathcal{L}^* \subseteq \mathcal{L}_{\mathbf{A}}$.

Let

$$\Sigma = \mathcal{T} \cup \{\varphi^*\} \cup \triangle_{(\mathfrak{A}|\mathcal{L})}.$$

be a set of sentences for the language $\mathcal{L}_{\mathbf{A}}$; we wish to show that Σ is satisfiable.

By the compactness theorem it suffices to consider $\mathcal{T} \cup \{\varphi^*, \tau\}$ where τ is a conjunction of finitely many sentences of $\triangle_{(\mathfrak{A}|\mathcal{L})}$. Let θ be the formula obtained from τ by exchanging each constant symbol from $\mathcal{L}_{\mathbf{A}} \setminus \mathcal{L}$ occurring in τ for a new variable u_a. So

$$\mathfrak{A}|\mathcal{L} \models \exists u_{a_0} \ldots \exists u_{a_m} \theta(u_{a_0}, \ldots, u_{a_m}).$$

But then $\mathfrak{A}$ is not a model of the universal sentence $\forall u_{a_0} \dots \forall u_{a_m} \neg\theta(u_{a_0}, \dots, u_{a_m})$. Recalling that $\mathfrak{A} \models \Gamma$, we are forced to conclude that this universal sentence is not in Γ and so not a consequence of $\mathcal{T} \cup \{\varphi^*\}$. Therefore

$$\mathcal{T} \cup \{\varphi^*\} \cup \{\exists u_{a_0} \dots \exists u_{a_m} \theta(u_{a_0}, \dots, u_{a_m})\}$$

must be satisfiable, and any model of this can be expanded to a model of $\mathcal{T} \cup \{\varphi^*, \tau\}$ and so Σ is satisfiable.

Let $\mathfrak{C} \models \Sigma$. By the diagram lemma, there is a model $\mathfrak{B}$ for $\mathcal{L}$ such that $\mathfrak{B}_{\mathbf{A}} \cong \mathfrak{C}$ and $\mathfrak{A}|\mathcal{L} \subseteq \mathfrak{B}$. Now, both $\mathfrak{A}|\mathcal{L}$ and $\mathfrak{B}$ are models of $\mathcal{T}$ and $\mathfrak{B}_{\mathbf{A}} \models \varphi^*$, so by (2) we get that $(\mathfrak{A}|\mathcal{L})_{\mathbf{A}} \models \varphi^*$. So $\mathfrak{A} \models \varphi^*$.

This means $\mathcal{T} \cup \Gamma \models \varphi^*$ and finishes the proof of the claim.

We will now do the general cases for the proof of the induction on prenex rank. There are two cases, corresponding to the two methods available for increasing the number of alternations of quantifiers:

(a) the addition of universal quantifiers
(b) the addition of existential quantifiers.

For the case (a), suppose $\varphi(v_0, \dots, v_p)$ is $\forall w_0 \dots \forall w_m \chi(v_0, \dots, v_p, w_0, \dots, w_m)$ and χ has prenex rank lower than φ so that we have by the inductive hypothesis that there is a quantifier free formula $\theta(v_0, \dots, v_p, w_0, \dots, w_m, x_0, \dots, x_n)$ with new variables $x_0, \dots, x_n$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p \forall w_0 \dots \forall w_m)(\chi \leftrightarrow \forall x_0 \dots \forall x_n \theta)$$

Therefore, case (a) is concluded by noticing that this gives us

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\forall w_0 \dots \forall w_m \chi \leftrightarrow \forall w_0 \dots \forall w_m \forall x_0 \dots \forall x_n \theta).$$

Exercise 18. Check this step using the definition of satisfaction.

For case (b), suppose $\varphi(v_0, \dots, v_p)$ is $\exists w_0 \dots \exists w_n \chi(v_0, \dots, v_p, w_0, \dots, w_m)$ and χ has prenex rank less than φ. Here we will use the inductive hypothesis on $\neg\chi$ which of course also has prenex rank less than φ. We obtain a quantifier free formula $\theta(v_0, \dots, v_p, w_0, \dots, w_m, x_0, \dots, x_n)$ with new variables $x_0, \dots, x_n$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p \forall w_0 \dots \forall w_m)(\neg\chi \leftrightarrow \forall x_0 \dots \forall x_n \theta)$$

$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\forall w_0 \dots \forall w_m \neg\chi \leftrightarrow \forall w_0 \dots \forall w_m \forall x_0 \dots \forall x_n \theta)$$

$$\text{And } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \chi \leftrightarrow \exists w_0 \dots \exists w_m \exists x_0 \dots \exists x_n \neg\theta)$$

Now $\exists w_0 \dots \exists w_m \exists x_0 \dots \exists x_n \neg\theta$ is an existential formula, so by the claim there is a universal formula ψ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \exists x_0 \dots \exists x_n \neg\theta \leftrightarrow \psi).$$

$$\text{Hence } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_n \chi \leftrightarrow \psi)$$

which is the final result which we needed.

(3) $\Rightarrow$ (4)

Let $\mathfrak{A} \models \mathcal{T}$ and $\mathfrak{B} \models \mathcal{T}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Let φ be a formula of $\mathcal{L}$ and let $a_0, \dots, a_p$ be in $\mathbf{A}$ such that

$$\mathfrak{B} \models \varphi[a_0, \dots, a_p]$$

Obtain a universal formula ψ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\varphi \leftrightarrow \psi)$$

so

$$\mathfrak{B} \models \psi[a_0, \ldots, a_p]$$

Since $\mathfrak{A} \subseteq \mathfrak{B}$

$$\mathfrak{A} \models \psi[a_0, \ldots, a_p]$$

and so $\mathfrak{A} \models \varphi[a_0, \ldots, a_p]$ and $\mathfrak{A} \prec \mathfrak{B}$.

(4) $\Rightarrow$ (1):

Let $\mathfrak{A} \models \mathcal{T}$. We will show that $\mathcal{T} \cup \triangle_{\mathfrak{A}}$ is complete by showing that for any model $\mathfrak{B}$ for $\mathcal{L}_{\mathbf{A}}$, $\mathfrak{B} \models \mathcal{T} \cup \triangle_{\mathfrak{A}}$ implies $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_{\mathbf{A}})$.

Letting $\mathfrak{B}$ be such a model, we note that $\mathfrak{B}|\mathcal{L}$, the restriction of $\mathfrak{B}$ to $\mathcal{L}$, can be expanded to $\mathfrak{B}$, a model of $\triangle_{\mathfrak{A}}$. So by the diagram lemma $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}|\mathcal{L}$. Furthermore, by checking the proof of the diagram lemma we can ensure that there is an

$$f : \mathfrak{A} \hookrightarrow \mathfrak{B}|\mathcal{L}$$

such that for each $a \in \mathbf{A}$, $f(a)$ is the interpretation of c_a in $\mathfrak{B}$. (Recall that $\mathcal{L}_{\mathbf{A}} = \mathcal{L} \cup \{c_a : a \in \mathbf{A}\}$). Moreover, as in Exercise 12, there is a model $\mathfrak{D}$ for $\mathcal{L}$ such that $\mathfrak{A} \subseteq \mathfrak{D}$ and an isomorphism $g : \mathfrak{D} \to \mathfrak{B}|\mathcal{L}$ with the property that for each $a \in \mathbf{A}$, $g(a)$ is the interpretation of c_a in $\mathfrak{B}$.

Now let $\sigma \in \mathrm{Th}(\mathfrak{A}_{\mathbf{A}})$, so that $\mathfrak{A}_{\mathbf{A}} \models \sigma$. Let $\varphi(u_0, \ldots, u_k)$ be the formula of $\mathcal{L}$ obtained by replacing each occurence of c_{a_i} in σ by the new variable u_i. We have

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_k]$$

Since $\mathfrak{A} \subseteq \mathfrak{D}$ we can use (4) to get $\mathfrak{A} \prec \mathfrak{D}$ and so we have $\mathfrak{D} \models \varphi[a_0, \ldots, a_k]$. With the isomorphism g we get that

$$\mathfrak{B}|\mathcal{L} \models \varphi[g(a_0), \ldots, g(a_k)]$$

and since $g(a_i)$ is the interpretation of c_{a_i} in $\mathfrak{B}$ we have $\mathfrak{B} \models \sigma$. Thus $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_{\mathbf{A}})$ and this proves (1). $\square$

Example 9. We will see later that the theory ACF is model complete. But ACF is not complete because the characteristic of the algebraically closed field can vary among models of ACF and the assertion that "I have characteristic p" can easily be expressed as a sentence of the language of ACF.

Exercise 19. Suppose that $\mathcal{T}$ is a model complete theory in $\mathcal{L}$ and that either

(1) any two models of $\mathcal{T}$ are isomorphically embedded into a third or
(2) there is a model of $\mathcal{T}$ which is isomorphically embedded in any other.

Then prove that $\mathcal{T}$ is complete.

Example 10. Let $\mathbb{N}$ be the usual natural numbers and $<$ the usual ordering. Let $\mathfrak{B} = \langle \mathbb{N}, < \rangle$ and $\mathfrak{A} = \langle \mathbb{N} \setminus \{0\}, < \rangle$ be models for the language with one binary relation symbol $<$. $\mathrm{Th}\mathfrak{A}$ is, of course, complete, but it is not model complete because it is not existentially complete. In fact the model $\mathfrak{A}$ is not existentially closed because $\mathfrak{B} \models \mathrm{Th}\mathfrak{A}$ and $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}_{\mathbf{A}} \models (\exists v_0)(v_0 < c_1)$ where c_1 is the constant symbol with interpretation 1. But $\mathfrak{A}_{\mathbf{A}}$ does not satisfy this existential sentence.

Theorem 15. *(Lindström's Test)*
Let $\mathcal{T}$ be a theory in a countable language $\mathcal{L}$ such that

(1) *all models of $\mathcal{T}$ are infinite,*
(2) *the union of any chain of models of $\mathcal{T}$ is a model of $\mathcal{T}$, and*

(3) $\mathcal{T}$ *is* κ*-categorical for some infinite cardinal* κ.

Then $\mathcal{T}$ *is model complete.*

Proof. W.L.O.G. we assume that $\mathcal{T}$ is satisfiable. We use conditions (1) and (2) to prove the following:

Claim. $\mathcal{T}$ *has existentially closed models of each infinite size* κ.

Proof of Claim. By the Löwenheim-Skolem Theorems we get $\mathfrak{A}_0 \models \mathcal{T}$ with $|\mathbf{A}_0| = \kappa$. We recursively construct a chain of models of $\mathcal{T}$ of size κ

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \subseteq \cdots$$

with the property that

if $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{A}_{n+1} \subseteq \mathfrak{B}$ and σ is an existential sentence of $\mathrm{Th}(\mathfrak{B}_{\mathbf{A}_n})$, then $(\mathfrak{A}_{n+1})_{\mathbf{A}_n} \models \sigma$.

Suppose $\mathfrak{A}_n$ is already constructed; we will construct $\mathfrak{A}_{n+1}$. Let Σ_n be a maximally large set of existential sentences of $\mathcal{L}_{\mathbf{A}_n}$ such that for each finite $\Sigma' \subseteq \Sigma_n$ there is a model $\mathfrak{C}$ for $\mathcal{L}_{\mathbf{A}_n}$ such that

$$\mathfrak{C} \models \Sigma' \cup \mathcal{T} \cup \triangle_{\mathfrak{A}_n}$$

By compactness $\mathcal{T} \cup \Sigma_n \cup \triangle_{\mathfrak{A}_n}$ has a model $\mathfrak{D}$ and without loss of generosity $\mathfrak{A}_n \subseteq \mathfrak{D}$. By the Downward Löwenheim-Skolem Theorem we get $\mathfrak{E}$ such that $\mathfrak{A}_n \subseteq \mathfrak{E}$, $|\mathfrak{E}| = \kappa$ and $\mathfrak{E} \prec \mathfrak{D}$.

Let $\mathfrak{A}_{n+1} = \mathfrak{E}|\mathcal{L}$; we will show that $\mathfrak{A}_{n+1}$ has the required properties. Since $\mathfrak{E} \equiv \mathfrak{D}$, $\mathfrak{E} \models \mathcal{T} \cup \triangle_{\mathfrak{A}_n}$ and so $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ (See Exercise 17).

Let $\mathfrak{B} \models \mathcal{T}$ with $\mathfrak{A}_{n+1} \subseteq \mathfrak{B}$ and σ be an existential sentence of $\mathrm{Th}(\mathfrak{B}_{\mathbf{A}_n})$; we will show that $(\mathfrak{A}_{n+1})_{\mathbf{A}_n} \models \sigma$. Since Σ_n consists of existential sentences and $\mathfrak{D} \equiv \mathfrak{E} \equiv (\mathfrak{A}_{n+1})_{\mathbf{A}_n} \subseteq \mathfrak{B}_{\mathbf{A}_n}$ we have (see Exercise 17) that $\mathfrak{B}_{A_n} \models \Sigma_n$. The maximal property of Σ_n then forces σ to be in Σ_n because if $\sigma \notin \Sigma_n$ then there must be some finite $\Sigma' \subseteq \Sigma_n$ for which there is no $\mathfrak{C}$ such that $\mathfrak{C} \models \Sigma' \cup \{\sigma\} \cup \mathcal{T} \cup \triangle_{\mathfrak{A}_n}$; but $\mathfrak{B}_{\mathbf{A}_n}$ is such a $\mathfrak{C}$! Now since $\sigma \in \Sigma_n$ and $\mathfrak{E} \equiv \mathfrak{D} \models \Sigma_n$ we must have $\mathfrak{E} = (\mathfrak{A}_{n+1})_{\mathbf{A}_n} \models \sigma$.

Now let $\mathfrak{A}$ be the union of the chain. By hypothesis $\mathfrak{A} \models \mathcal{T}$. It is easy to check that $|\mathbf{A}| = \kappa$. To check that $\mathfrak{A}$ is existentially closed, let $\mathfrak{B} \models \mathcal{T}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and let σ be an existential sentence of $\mathrm{Th}\mathfrak{B}_{\mathbf{A}}$. Since σ can involve only finitely many constant symbols, σ is a sentence of $\mathcal{L}_{\mathbf{A}_n}$ for some $n \in \mathbb{N}$. Thus $\mathfrak{A}_{n+1} \subseteq \mathfrak{A} \subseteq \mathfrak{B}$ gives that $(\mathfrak{A}_{n+1})_{\mathbf{A}_n} \models \sigma$. Since σ is existential (see Exercise 17 again) we get that $\mathfrak{A} \models \sigma$. This completes the proof of the claim.

We now claim that $\mathcal{T}$ is model complete using Theorem 14 by showing that every model $\mathfrak{A}$ of $\mathcal{T}$ is existentially closed. There are three cases to consider:

(1) $|\mathbf{A}| = \kappa$
(2) $|\mathbf{A}| > \kappa$
(3) $|\mathbf{A}| < \kappa$

where $\mathcal{T}$ is κ-categorical.

Case (1). Let $\mathfrak{A}^*$ be an existentially closed model of $\mathcal{T}$ of size κ. Then there is an isomorphism $f : \mathfrak{A} \to \mathfrak{A}^*$. Hence $\mathfrak{A}$ is existentially closed.

Case (2). Let σ be an existential sentence of $\mathcal{L}_{\mathbf{A}}$ and $\mathfrak{B} \models \mathcal{T}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}_{\mathbf{A}} \models \sigma$. Let $X = \{a \in \mathbf{A} : c_a \text{ occurs in } \sigma\}$. By the Downward Löwenheim-Skolem Theorem we can find $\mathfrak{A}'$ such that $\mathfrak{A}' \prec \mathfrak{A}$, $X \subseteq \mathbf{A}'$ and $|\mathbf{A}'| = \kappa$. Now by Case (1) $\mathfrak{A}'$ is existentially closed and we have $\mathfrak{A}' \subseteq \mathfrak{B}$ and σ in $\mathcal{L}_{\mathbf{A}'}$ so $\mathfrak{A}'_{\mathbf{A}'} \models \sigma$. But since $\sigma \in \mathrm{Th}(\mathfrak{A}'_{\mathbf{A}'})$ and $\mathfrak{A}' \prec \mathfrak{A}$ we have $\mathfrak{A}_{\mathbf{A}} \models \sigma$.

Case (3). Let σ and $\mathfrak{B}$ be as in case (2). By the Upward Löwenheim-Skolem Theorem we can find $\mathfrak{A}'$ such that $\mathfrak{A} \prec \mathfrak{A}'$ and $|\mathfrak{A}'| = \kappa$. By case (1) $\mathfrak{A}'$ is existentially closed.

Claim. *There is a model $\mathfrak{B}'$ such that $\mathfrak{A}' \subseteq \mathfrak{B}'$ and $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{B}'_{\mathbf{A}}$.*

Assuming this claim, we have $\mathfrak{B}' \models \mathcal{T}$ and $\mathfrak{B}'_{\mathbf{A}} \models \sigma$ and by the fact that $\mathfrak{A}'$ is existentially closed we have $\mathfrak{A}'_{\mathbf{A}'} \models \sigma$. Since $\mathfrak{A} \prec \mathfrak{A}'$ we have $\mathfrak{A}_{\mathbf{A}} \models \sigma$.

The following lemma implies the claim and completes the proof of the theorem.

□

Lemma 8. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A}'$ be models for $\mathcal{L}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \prec \mathfrak{A}'$. Then there is a model $\mathfrak{B}'$ for $\mathcal{L}$ such that $\mathfrak{A}' \subseteq \mathfrak{B}'$ and $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{B}'_{\mathbf{A}}$.*

Proof. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}'$ and $\mathcal{L}$ be as above. Note that since $\mathfrak{A} \subseteq \mathfrak{B}$ we have $\mathfrak{B}_{\mathbf{A}} \models \triangle_{\mathfrak{A}}$ and so $\mathfrak{A}_{\mathbf{A}} \subseteq \mathfrak{B}_{\mathbf{A}}$.

Let τ be a sentence from $\triangle_{\mathfrak{A}'}$. Let $\{d_j : 0 \leq j \leq m\}$ be the constant symbols from $\mathcal{L}_{\mathbf{A}'} \setminus \mathcal{L}_{\mathbf{A}}$ appearing in τ. Obtain a quantifier free formula $\varphi(u_0, \ldots, u_m)$ of $\mathcal{L}_{\mathbf{A}}$ by exchanging each d_j in τ with a new variable u_i. Since $\mathfrak{A}'_{\mathbf{A}'} \models \tau$ we have $\mathfrak{A}'_{\mathbf{A}} \models \exists u_0 \ldots \exists u_m \varphi$. Since $\mathfrak{A} \prec \mathfrak{A}'$ we have $\mathfrak{A}_{\mathbf{A}} \prec \mathfrak{A}'_{\mathbf{A}}$ and so $\mathfrak{A}_{\mathbf{A}} \models \exists u_0 \ldots \exists u_m \varphi$. Since $\mathfrak{A}_{\mathbf{A}} \subseteq \mathfrak{B}_{\mathbf{A}}$, $\mathfrak{B}_{\mathbf{A}} \models \exists u_0 \ldots \exists u_m \varphi$. Hence for some $b_0, \ldots, b_m$ in $\mathbf{B}$, $\mathfrak{B}_{\mathbf{A}} \models \varphi[b_0, \ldots, b_m]$. Expand $\mathfrak{B}_{\mathbf{A}}$ to be a model $\mathfrak{B}^*_{\mathbf{A}}$ for the w language $\mathcal{L}_{\mathbf{A}} \cup \{d_j : 0 \leq j \leq m\}$ by interpreting each d_j as b_j. Then $\mathfrak{B}^*_{\mathbf{A}} \models \tau$ and so $\mathrm{Th}(\mathfrak{B}_{\mathbf{A}}) \cup \{\tau\}$ is satisfiable.

This shows that $\mathrm{Th}\mathfrak{B}_{\mathbf{A}} \cup \Sigma$ is satisfiable for each finite subset $\Sigma \subseteq \triangle_{\mathfrak{A}'}$. By the Compactness Theorem there is a model $\mathfrak{C} \models \triangle_{\mathfrak{A}'} \cup \mathrm{Th}\mathfrak{B}_{\mathbf{A}}$. Using the Diagram Lemma for the language $\mathcal{L}_{\mathbf{A}}$ we obtain a model $\mathfrak{B}'$ for $\mathcal{L}$ such that $\mathfrak{A}'_{\mathbf{A}} \subseteq \mathfrak{B}'_{\mathbf{A}}$ and $\mathfrak{B}'_{\mathbf{A}} \cong \mathfrak{C}|\mathcal{L}_{\mathbf{A}}$. Hence $\mathfrak{B}'_{\mathbf{A}} \models \mathrm{Th}\mathfrak{B}_{\mathbf{A}}$ and so $\mathfrak{B}'_{\mathbf{A}} \equiv \mathfrak{B}_{\mathbf{A}}$.

□

Exercise 20. Suppose $\mathfrak{A} \prec \mathfrak{A}'$ are models for $\mathcal{L}$. Prove that for each sentence σ of $\mathcal{L}_{\mathbf{A}}$, if $\triangle_{\mathfrak{A}'} \models \sigma$ then $\triangle_{\mathfrak{A}} \models \sigma$.

Exercise 21. Prove that if $\mathcal{T}$ has a universal-existential set of axioms, then the union of a chain of models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Remark. The converse of this last exercise is also true; it is usually called the Chang - Łoś - Suszko Theorem.

Theorem 16. *The following theories are model complete:*

(1) *dense linear orders without endpoints. (DLO)*
(2) *algebraically closed fields. (ACF)*

Proof. (DLO): This theory has a universal existential set of axioms so that it is closed under unions of chains. It is $\aleph_0$-categorical (by Exercise 10) so Lindström's test applies.

(ACF): We first prove that for any fixed characteristic p, the theory of algebraically closed fields of characteristic p is model complete. The proof is similar to that for DLO, with $\aleph_1$-categoricity (Lemma 7).

Let $\mathfrak{A} \subseteq \mathfrak{B}$ be algebraically closed fields. They must have the same characteristic p. Therefore $\mathfrak{A} \prec \mathfrak{B}$.

□

Corollary 4. *Any true statement about the rationals involving only the usual ordering is also true about the reals.*

Proof. Let $\mathfrak{A} = \langle \mathbb{Q}, \mathbf{<_1} \rangle$ and $\mathfrak{B} = \langle \mathbb{R}, \mathbf{<_2} \rangle$ where $\mathbf{<_1}$ and $\mathbf{<_2}$ are the usual orderings. The precise version of this corollary is: $\mathfrak{A} \prec \mathfrak{B}$. This follows from Theorem 14 and Theorem 16 and the easy facts that $\mathfrak{A} \models \mathbf{DLO}$, $\mathfrak{B} \models \mathbf{DLO}$ and $\mathfrak{A} \subseteq \mathfrak{B}$. The reader will appreciate the power of these theorems by trying to prove $\mathfrak{A} \prec \mathfrak{B}$ directly, without using them. □

Corollary 5. *(Hilbert's Nullstellensatz)*
Let Σ be a finite system of polynomial equations and inequations in several variables with coefficients in the field $\mathfrak{A}$. If Σ has a solution in some field extending $\mathfrak{A}$ then Σ has a solution in the algebraic closure of $\mathfrak{A}$.

Proof. Let σ be the existential sentence of the language $\mathcal{L}_{\mathbf{A}}$ which asserts the fact that there is a solution of Σ. Suppose Σ has a solution in a field $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{B}_{\mathbf{A}} \models \sigma$. So $\mathfrak{B}'_{\mathbf{A}} \models \sigma$ where $\mathfrak{B}'$ is the algebraic closure of $\mathfrak{B}$. Let $\mathfrak{A}'$ be the algebraic closure of $\mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, we have $\mathfrak{A}' \subseteq \mathfrak{B}'$.

By Theorem 16, ACF is model complete, so $\mathfrak{A}' \prec \mathfrak{B}'$. Hence $\mathfrak{A}'_{\mathbf{A}} \equiv \mathfrak{B}'_{\mathbf{A}}$ and $\mathfrak{A}'_{\mathbf{A}} \models \sigma$.

□

Remark. We cannot apply Lindström's Test to the theory of real closed ordered fields (RCF) because RCF is not categorical in any infinite cardinal. This is because, as demonstrated in Theorem 11, RCF neither implies nor denies the existence of infinitesimals. Nevertheless, as we shall later prove, RCF is indeed model complete.

Exercise 22. Use Exercise 19 and the fact that RCF is model complete to show that the theory RCF $\cup \triangle_{\mathfrak{Q}}$ is complete, where $\mathfrak{Q}$ is from Example 3. Hint: step 0 – the rationals, step 1 – the algebraic numbers, step 2 …

CHAPTER 5

The Seventeenth Problem

We will give a complete proof later that RCF, the theory of real closed ordered fields, is model complete. However, by assuming this result now, we can give a solution to the seventeenth problem of the list of twenty-three problems of David Hilbert's famous address to the 1900 International Congress of Mathematicians in Paris.

Corollary 6. *(E. Artin)*
Let $q(x_1, \ldots, x_n)$ be a rational function with real coefficients, which is positive definite. i.e.

$$q(a_1, \ldots, a_n) \geq 0 \text{ for all } a_1, \ldots, a_n \in \mathbb{R}$$

Then there are finitely many rational functions with real coefficients $f_1(x_1, \ldots, x_n)$, $\ldots$, $f_m(x_1, \ldots, x_n)$ such that

$$q(x_1, \ldots, x_n) = \sum_{j=1}^{m} (f_j(x_1, \ldots, x_n))^2$$

We give a proof of this theorem after a sequence of lemmas. The first lemma just uses calculus to prove the special case of the theorem in which q is a polynomial in only one variable. This result probably motivated the original question.

Lemma 9. *A positive definite real polynomial is the sum of squares of real polynomials.*

Proof. We prove this by induction on the degree of the polynomial. Let $p(x) \in \mathbb{R}[x]$ with degree $deg(p) \geq 2$ and $p(x) \geq 0$ for all real x. Let $p(a) = \min\{p(x) : x \in \mathbb{R}\}$, so

$$p(x) = (x-a)q(x) + p(a) \text{ and } p'(a) = 0$$

for some polynomial q. But

$$p'(a) = [(x-a)q'(x) + q(x)]\big|_{x=a} = q(a)$$

so $q(a) = 0$ and $q(x) = r(x)(x-a)$ for some polynomial $r(x)$. So

$$p(x) = p(a) + (x-a)^2 r(x).$$

For all real x we have

$$(x-a)^2 r(x) = p(x) - p(a) \geq 0.$$

Since r is continuous, $r(x) \geq 0$ for all real x, and $deg(r) = deg(p) - 2$. So, by induction $r(x) = \sum_{i=1}^{n} (r_i(x))^2$ where each $r_i(x) \in \mathbb{R}[x]$.

$$\text{So } p(x) = p(a) + \sum_{i=1}^{n} (x-a)^2 (r_i(x))^2 .$$

$$\text{i.e. } p(x) = \left[\sqrt{p(a)}\right]^2 + \sum_{i=1}^{n} \left[(x-a)r_i(x)\right]^2 .$$

□

The following lemma shows why we deal with sums of rational functions rather than sums of polynomials.

LEMMA 10. $x^4y^2 + x^2y^4 - x^2y^2 + 1$ *is positive definite, but not the sum of squares of polynomials.*

PROOF. Let the polynomial be $p(x, y)$. A little calculus shows that the minimum value of p is $\frac{26}{27}$ and confirms that p is positive definite.

Suppose

$$p(x, y) = \sum_{i=1}^{l} (q_i(x, y))^2$$

where $q_i(x, y)$ are polynomials, each of which is the sum of terms of the form ax^my^n. First consider powers of x and the largest exponent m which can occur in any of the q_i. Since no term of p contains x^6 or higher powers of x, we see that we must have $m \leq 2$. Considering powers of y similarly gives that each $n \leq 2$. So each $q_i(x, y)$ is of the form:

$$a_ix^2y^2 + b_ix^2y + c_ixy^2 + d_ix^2 + e_iy^2 + f_ixy + g_ix + h_iy + k_i$$

for some coefficients $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i$ and k_i. Comparing coefficients of x^4y^4 in p and the sum of the q_i^2 gives

$$0 = \sum_{i=1}^{l} a_i^2$$

so each $a_i = 0$. Comparing the coefficients of x^4 and y^4 gives that each $d_i = 0 = e_i$. Now comparing the coefficients of x^2 and y^2 gives that each $g_i = 0 = h_i$. Now comparing the coefficients of x^2y^2 gives

$$-1 = \sum_{i=1}^{l} f_i^2$$

which is impossible.

□

The next lemma is easy but useful.

LEMMA 11. *The reciprocal of a sum of squares is a sum of squares.*

PROOF. For example

$$\frac{1}{A^2 + B^2} = \frac{A^2 + B^2}{(A^2 + B^2)^2} = \left[\frac{A}{A^2 + B^2}\right]^2 + \left[\frac{B}{A^2 + B^2}\right]^2$$

□

The following lemma is an algebraic result of E. Artin and O. Schreier, who invented the theory of real closed fields.

LEMMA 12. *Let* $\mathfrak{A} = \langle \mathbf{A}, +, \cdot, <_{\mathbf{A}}, \mathbf{0}, \mathbf{1} \rangle$ *be an ordered field such that each positive element of* $\mathbf{A}$ *is the sum of squares of elements of* $\mathbf{A}$*. Let* $\mathfrak{B}$ *be a field containing the reduct of* $\mathfrak{A}$ *to* $\{+, \cdot, 0, 1\}$ *as a subfield and such that zero is not the sum of non-zero squares in* $\mathfrak{B}$*.*

Let $b \in \mathbf{B} \setminus \mathbf{A}$ *be such that* b *is not the sum of squares of elements of* $\mathbf{B}$*. Then there is an ordering* $<_{\mathbf{B}}$ *on* $\mathbf{B}$ *with* $b <_{\mathbf{B}} 0$ *such that* $\mathfrak{A}$ *is an ordered subfield of* $\langle \mathbf{B}, +, \cdot, <_{\mathbf{B}}, \mathbf{0}, \mathbf{1} \rangle$.

PROOF. It suffices to find a set $P \subseteq \mathbf{B}$ of "positive elements" of $\mathbf{B}$ such that

(1) $-b \in P$
(2) $0 \notin P$
(3) $c^2 \in P$ for each $c \in \mathbf{B} \setminus \{0\}$
(4) P is closed under $+$ and $\cdot$
(5) for any $c \in \mathbf{B} \setminus \{0\}$ either $c \in P$ or $-c \in P$.

Once P has been obtained, we define $<_{\mathbf{B}}$ as follows:

$$c_1 <_{\mathbf{B}} c_2 \text{ iff } c_2 - c_1 \in P.$$

For each $a \in \mathbf{A}$, if $0 <_{\mathbf{A}} a$ then a is a sum of squares and so by (3) and (4) $a \in P$. Thus $<_{\mathbf{B}}$ extends $<_{\mathbf{A}}$.

So that all that remains to do is to construct such a P. The first approximation to P is P_0.

$$\text{Let } P_0 = \left\{ \sum_{i=1}^{l} c_i^2 - \sum_{j=1}^{m} d_j^2 b : l, m \in \mathbb{N}, c_i \in \mathbf{B}, d_j \in \mathbf{B} \text{ not all zero} \right\}$$

We claim that (1), (2), (3) and (4) hold for P_0. (1) and (3) are obvious. In order to verify (2), note that if $\sum_{j=1}^{m} d_j^2 b = \sum_{i=1}^{l} c_i^2$, then by the previous lemma about reciprocals of sums of squares, b would be a sum of squares. Now (4) holds by definition of P_0, noting that $c_i^2(-d_j^2 b) = -(c_i d_j)^2 b$ and $(-d_j^2 b)(-d_k^2 b) = (d_j d_k b)^2$.

We now construct larger and larger versions of P_0 to take care of requirement (5). We do this in the following way. Suppose $P_0 \subseteq P_1$, P_1 satisfies (1), (2), (3) and (4), and $c \notin P_1 \cup \{0\}$. We define P_2 to be:

$$\{p(-c) : p \text{ is a polynomial with coefficients in } P_1\}.$$

It is easy to see that $-c \in P_2$, $P_1 \subseteq P_2$ and that (1), (3) and (4) hold for P_2.

To show that (2) holds for P_2 we suppose that $p(-c) = 0$ and bring forth a contradiction. Considering even and odd exponents we obtain:

$$p(x) = q(x^2) + xr(x^2)$$

for some polynomials q and r with coefficients in P_1.

If $q(c^2) = 0$, then by (3) and (4), q must be the zero polynomial and r could not be. But we would have

$$0 = p(-c) = -cr(c^2)$$

which gives $r(c^2) = 0$ and a contradiction to (3) and (4). Similarly, $r(c^2) \neq 0$. Now

$$0 = p(-c) = q(c^2) - cr(c^2)$$

means that

$$c = q(c^2) \cdot r(c^2) \cdot \left(\frac{1}{r(c^2)}\right)^2$$

and since each of the factors on the right hand side is in P_1 we get a contradiction. $\square$

Now we need:

LEMMA 13. *Every ordered field can be embedded as a submodel of a real closed ordered field.*

PROOF. It suffices to prove that for every ordered field $\mathfrak{A}$ there is an ordered field $\mathfrak{B}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and for each natural number $n \geq 1$, $\mathfrak{B} \models \sigma_n$ where σ_n is the sentence in the language of field theory which formally states:

> If p is a polynomial of degree at most n and $w < y$ such that $p(w) < 0 < p(y)$ then there is an x such that $w < x < y$ and $p(x) = 0$.

Consider the statement called IH(n):

> For any ordered field $\mathfrak{E}$ there is an ordered field $\mathfrak{F}$ such that $\mathfrak{E} \subseteq \mathfrak{F}$ and $\mathfrak{F} \models \sigma_n$.

IH(1) is true since any ordered field $\mathfrak{E} \models \sigma_1$. We will prove below that for each n, IH(n) implies IH($n+1$).

Given our model $\mathfrak{A} \models$ ORF, we will then be able to construct a chain of models:

$$\mathfrak{A} \subseteq \mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \ldots \subseteq \mathfrak{B}_n \subseteq \mathfrak{B}_{n+1} \subseteq \cdots$$

such that each $\mathfrak{B}_n \models \mathrm{ORF} \cup \{\sigma_n\}$. Let $\mathfrak{B}$ be the union of the chain. Since the theory ORF is preserved under unions of chains (see Exercise 21), $\mathfrak{B} \models$ ORF. Furthermore, the nature of the sentences σ_n allows us to conclude that for each n, $\mathfrak{B} \models \sigma_n$ and so $\mathfrak{B} \models$ RCF. All that remains is to prove that for each n, IH(n) implies IH($n+1$). We first make a claim:

CLAIM. *If $\mathfrak{E} \models ORF \cup \{\sigma_n\}$ and p is a polynomial of degree at most $n+1$ with coefficients from $\mathbf{E}$ and $a < d$ are in $\mathbf{E}$ such that $p(a) < 0 < p(d)$ then there is a model $\mathfrak{F}$ such that $\mathfrak{E} \subseteq \mathfrak{F}$, $\mathfrak{F} \models ORF$ and there is $b \in \mathbf{F}$ such that $a < b < d$ and $p(b) = 0$.*

Let us first see how this claim helps us to prove that IH(n) implies IH(n+1). Let $\mathfrak{E} \models ORF$; we will use the claim to build a model $\mathfrak{F}$ such that $\mathfrak{E} \subseteq \mathfrak{F}$ and $\mathfrak{F} \models \sigma_{n+1}$.

We first construct a chain of models of ORF

$$\mathfrak{E} = \mathfrak{E}_0 \subseteq \mathfrak{E}_1 \subseteq \ldots \subseteq \mathfrak{E}_m \subseteq \mathfrak{E}_{m+1} \subseteq \cdots$$

such that for each m and each polynomial p of degree at most $n+1$ with coefficients from $\mathbf{E}_m$ and each pair of a, d of elements of $\mathbf{E}_m$ such that $p(a) < 0 < p(d)$ there is a $b \in \mathbf{E}_{m+1}$ such that $a < b < d$ and $p(b) = 0$.

Suppose $\mathfrak{E}_m$ has been constructed; we construct $\mathfrak{E}_{m+1}$ as follows: let Σ_m be the set of all existential sentences of $\mathcal{L}_{\mathbf{E}_m}$ of the form

$$(\exists x)(c_a < x \wedge x < c_d \wedge p(x) = 0)$$

where p is a polynomial of degree at most $n+1$ and such that c_a, c_d and the coefficients of the polynomial p are constant symbols from $\mathcal{L}_{\mathbf{E}_m}$ and

$$(\mathfrak{E}_m)_{\mathbf{E}_m} \models p(c_a) < 0 \wedge 0 < p(c_d)$$

We claim that

$$\mathrm{ORF} \cup \triangle_{\mathfrak{E}_m} \cup \Sigma_m$$

is satisfiable.

Using the Compactness Theorem, it suffices to find, for each finite subset $\{\tau_1, \ldots, \tau_k\}$ of Σ_m, a model $\mathfrak{C}$ such that $\mathfrak{E}_m \subseteq \mathfrak{C}$ and

$$\mathfrak{C} \models \mathrm{ORF} \cup \{\tau_1, \ldots, \tau_k\}.$$

By IH(n), obtain a model $\mathfrak{F}_1$ such that $\mathfrak{E}_m \subseteq \mathfrak{F}_1$ and $\mathfrak{F}_1 \models \mathrm{ORF} \cup \{\sigma_n\}$. By the claim, obtain a model $\mathfrak{F}_2$ such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $\mathfrak{F}_2 \models \mathrm{ORF} \cup \{\tau_1\}$. Again by IH(n), obtain $\mathfrak{F}_3$ such that $\mathfrak{F}_2 \subseteq \mathfrak{F}_3$ and $\mathfrak{F}_3 \models \mathrm{ORF} \cup \{\sigma_n\}$. Again by the claim, obtain $\mathfrak{F}_4$ such that $\mathfrak{F}_3 \subseteq \mathfrak{F}_4$ and $\mathfrak{F}_4 \models \mathrm{ORF} \cup \{\tau_2\}$. Continue in this manner, getting models of ORF

$$\mathfrak{E}_m \subseteq \mathfrak{F}_1 \subseteq \ldots \subseteq \mathfrak{F}_{2k}$$

with each $\mathfrak{F}_{2j} \models \tau_j$. Since each τ_j is existential, we get that $\mathfrak{F}_{2k}$ is a model of each τ_j (see Exercise 17).

$$\text{Let } \mathfrak{D} \models \mathrm{ORF} \cup \triangle_{\mathfrak{E}_m} \cup \Sigma_m$$

and then use the Diagram Lemma to get $\mathfrak{E}_{m+1}$ such that $\mathfrak{E}_m \subseteq \mathfrak{E}_{m+1}$, $\mathfrak{E}_{m+1} \models \mathrm{ORF}$ and $\mathfrak{E}_{m+1} \models \Sigma_m$, thus satisfying the required property concerning polynomials from $\mathbf{E}_m$.

Let $\mathfrak{F}$ be the union of the chain. Since ORF is a universal-existential theory, $\mathfrak{F} \models \mathrm{ORF}$ (see Exercise 21) and $\mathfrak{F} \models \sigma_{n+1}$ by construction. So IH$(n+1)$ is proved.

We now finish the entire proof by proving the claim.

Proof of Claim. Suppose that $p(x) = q(x) \cdot s(x)$ with the degree of q at most n. Since $\mathfrak{E} \models \sigma_n$ we are guaranteed $c \in \mathbf{E}$ with $a < c < d$ and $q(c) = 0$. Hence $p(c) = 0$ and we can let $\mathfrak{F} = \mathfrak{E}$.

So we can assume that p is irreducible over $\mathbf{E}$. Introduce a new element b to $\mathbf{E}$ where the place of b in the ordering is given by:

$$b < x \text{ iff } p(y) > 0 \text{ for all } y \text{ with } x \leq y \leq d.$$

Note that continuity-style considerations show that $b < d$.

The fact that p is irreducible over $\mathbf{E}$ means that we can extend $\langle \mathbf{E}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$ by quotients of polynomials in b of degree $\leq n$ in the usual way to form a field $\langle \mathbf{F}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$ in which $p(b) = 0$. We leave the details to the reader, but point out that the construction cannot force $q(b) = 0$ for any polynomial $q(x)$ with coefficients from $\mathbf{E}$ of degree $\leq n$. This is because we could take such a $q(x)$ of lowest degree and divide $p(x)$ by it to get

$$p(x) = q(x) \cdot s(x) + r(x)$$

where degree of r is less than the degree of q. This means that $r(x) = 0$ constantly and so p could have been factored over $\mathbf{E}$.

Now we must expand $\langle \mathbf{F}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$ to an ordered field $\mathfrak{F}$ while preserving the order of $\mathfrak{E}$. We are aided in this by the fact that if q is a polynomial of degree at most n with coefficients from $\mathbf{E}$ then there are a_1 and a_2 in $\mathbf{E}$ such that $a_1 < b < a_2$ and q doesn't change sign between a_1 and a_2; this comes from the fact that $\mathfrak{E} \models \sigma_n$.

□

Proof of the Corollary. Using Lemma 11 we see that it suffices to prove the corollary for a polynomial $p(x_1,\ldots,x_n)$ such that $p(a_1,\ldots,a_n) \geq 0$ for all $a_1,\ldots,a_n \in \mathbb{R}$.

Let $\mathfrak{B} = \langle \mathbb{R}(x_1,\ldots,x_n), \mathbf{+}, \cdot, \mathbf{0}, \mathbf{1}\rangle$ be the field of "rational functions". Note that $\mathfrak{B}$ contains the reduct of $\mathfrak{R}$ to $\{+,\cdot,0,1\}$ as a subfield, where $\mathfrak{R}$ is defined as in Example 3 as the usual real numbers.

By Lemma 12, if p is not the sum of squares in $\mathfrak{B}$, then we can find an ordering $<_{\mathbf{B}}$ on $\mathfrak{B}$, extending the ordering on the reals, such that the expansion $\mathfrak{B}'$ of $\mathfrak{B}$ is an ordered field and $p(x_1,\ldots,x_n) <_{\mathbf{B}} 0$.

We now use Lemma 13 to embed $\mathfrak{B}'$ as a submodel of a real closed field $\mathfrak{M}$, $\mathfrak{B}' \subseteq \mathfrak{M}$.

Let $\varphi(v_1,\ldots,v_n)$ be the quantifier free formula which we informally write as $p(v_1,\ldots,v_n) < 0$ where φ involves constant symbols c_{r_i} for the real coefficients r_i of p. Let ψ be the formula of the language of field theory, obtained from φ by substituting a new variable u_i for each c_{r_i}. We have

$$\mathfrak{B}' \models \exists v_1 \ldots \exists v_n \psi[r_1,\ldots,r_k]$$

$$\text{and so } \mathfrak{M} \models \exists v_1 \ldots \exists v_n \psi[r_1,\ldots,r_k]$$

Since RCF is model complete and $\mathfrak{R} \subseteq \mathfrak{B}' \subseteq \mathfrak{M}$, Theorem 14 gives $\mathfrak{R} \prec \mathfrak{M}$ and so

$$\mathfrak{R} \models \exists v_1 \ldots \exists v_n \psi[r_1,\ldots,r_k]$$

i.e. there exist $a_1,\ldots,a_n$ in $\mathbb{R}$ such that $p(a_1,\ldots,a_n) < 0$.

□

Hilbert also asked:

> If the coefficients of a positive definite rational function are rational numbers (i.e. it is an element of $\mathbb{Q}(x_1,\ldots,x_n)$) is it in fact the sum of squares of elements of $\mathbb{Q}(x_1,\ldots,x_n)$?

The answer is "yes" and the proof is very similar. Let $\mathfrak{Q} = \langle \mathbb{Q}, \mathbf{+}\cdot\!, \mathbf{<}, \mathbf{0}, \mathbf{1}\rangle$ be the ordered field of rationals as in Example 3. Lemma 12 holds for $\mathfrak{A} = \mathfrak{Q}$ and $\mathfrak{B} = \langle \mathbb{Q}(x_1,\ldots,x_n), \mathbf{+}, \mathbf{\cdot}, \mathbf{0}, \mathbf{1}\rangle$; by Lemma 11 every positive rational number is the sum of squares since every positive integer is the sum of squares $n = 1 + 1 + \cdots + 1$.

Exercise 23. Finish the answer to Hilbert's question by making any appropriate changes to the proof of the corollary. Hint: create a real closed ordered field into which $\mathfrak{B}'_{\mathbb{Q}}$ and $\mathfrak{R}_{\mathbb{Q}}$ are each isomorphically embedded. Exercise 15 and Exercise 22 may be useful.

CHAPTER 6

Submodel Completeness

Definition 30. A theory $\mathcal{T}$ is said to *admit elimination of quantifiers* in $\mathcal{L}$ whenever for each formula $\varphi(v_0, \dots, v_p)$ of $\mathcal{L}$ there is a quantifier free formula $\psi(v_0, \dots, v_p)$ such that:

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\varphi(v_0, \dots, v_p) \leftrightarrow \psi(v_0, \dots, v_p))$$

Remark. There is a fine point with regard to the above definition. If φ is actually a sentence of $\mathcal{L}$ there are no free variables $v_0, \dots, v_p$. So $\mathcal{T} \models \varphi \leftrightarrow \psi$ for some quantifier free formula with no free variables. But if $\mathcal{L}$ has no constant symbols, *there are no* quantifier free formulas with no free variables. For this reason we assume that $\mathcal{L}$ has at least one constant symbol, or we restrict to those formulas φ with at least one free variable. This will become relevant in the proof of Theorem 17 for (2) $\Rightarrow$ (3).

Exercise 24. If $\mathcal{T}$ admits elimination of quantifiers in $\mathcal{L}$ and $\mathcal{L}$ has no constant symbols, show that for each sentence σ of $\mathcal{L}$ there is a quantifier free formula $\psi(v_0)$ such that

$$\mathcal{T} \models \sigma \leftrightarrow \forall v_0 \psi \leftrightarrow \exists v_0 \psi$$

Definition 31. A theory $\mathcal{T}$ is said to be *submodel complete* whenever $\mathcal{T} \cup \triangle_{\mathfrak{A}}$ is complete in $\mathcal{L}_{\mathbf{A}}$ for each submodel $\mathfrak{A}$ of a model of $\mathcal{T}$.

Exercise 25. Use Theorem 14 and the following theorem to find four proofs that every submodel complete theory is model complete.

Theorem 17. *Let $\mathcal{T}$ be a theory of a language $\mathcal{L}$. The following are equivalent:*

(1) *$\mathcal{T}$ is submodel complete*
(2) *If $\mathfrak{B}$ and $\mathfrak{C}$ are models of $\mathcal{T}$ and $\mathfrak{A}$ is a submodel of both $\mathfrak{B}$ and $\mathfrak{C}$, then every existential sentence which holds in $\mathfrak{B}_{\mathbf{A}}$ also holds in $\mathfrak{C}_{\mathbf{A}}$.*
(3) *$\mathcal{T}$ admits elimination of quantifiers*
(4) *whenever $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{C}$, $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{C} \models \mathcal{T}$ there is a model $\mathfrak{D}$ such that both $\mathfrak{B}_{\mathbf{A}}$ and $\mathfrak{C}_{\mathbf{A}}$ are elementarily embedded in $\mathfrak{D}_{\mathbf{A}}$.*

Proof. (1) $\Rightarrow$ (2)
Let $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{C} \models \mathcal{T}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$. Then $\mathfrak{B}_{\mathbf{A}} \models \mathcal{T} \cup \triangle_{\mathfrak{A}}$ and $\mathfrak{C}_{\mathbf{A}} \models \mathcal{T} \cup \triangle_{\mathfrak{A}}$. So (1) and Lemma 6 give $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$. Thus (2) is in fact proved for all sentences, not just existential ones.

(2) $\Rightarrow$ (3)
Lemma 4 shows that it suffices to prove (3) for formulas in prenex normal form. We do this by induction on the prenex rank of φ. This claim is the first step.

Claim. *For each existential formula $\varphi(v_0, \dots, v_p)$ of $\mathcal{L}$ there is a quantifier free formula $\psi(v_0, \dots, v_p)$ such that*

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\varphi \leftrightarrow \psi)$$

Proof of Claim. Add new constant symbols $c_0, \dots, c_p$ to $\mathcal{L}$ to form

$$\mathcal{L}^* = \mathcal{L} \cup \{c_0, \dots, c_p\}$$

and to form a sentence φ^* of $\mathcal{L}^*$ obtained by replacing each free occurrence of v_i in φ with the corresponding c_i; φ^* is an existential sentence. It suffices to prove that there is a quantifier free sentence σ of $\mathcal{L}^*$ such that

$$\mathcal{T} \models \varphi^* \leftrightarrow \sigma.$$

$$\text{Let } S = \{ \text{ quantifier free sentences } \sigma \text{ of } \mathcal{L}^* : \mathcal{T} \models \varphi^* \to \sigma\}.$$

It suffices to find some σ in S such that $\mathcal{T} \models \sigma \to \varphi^*$. Since a finite conjunction of sentences of S is also in S, it suffices to find $\sigma_1, \dots, \sigma_n$ in S such that

$$\mathcal{T} \models \sigma_1 \wedge \cdots \wedge \sigma_n \to \varphi^*.$$

If no such finite subset $\{\sigma_1, \dots, \sigma_n\}$ of S exists, then each

$$\mathcal{T} \cup \{\sigma_1, \dots, \sigma_n\} \cup \{\neg\varphi^*\}$$

would be satisfiable. So it suffices to prove that $\mathcal{T} \cup S \models \varphi^*$.

Let $\mathfrak{C} \models \mathcal{T} \cup S$ with the intent of proving that $\mathfrak{C} \models \varphi^*$. Let $\mathfrak{A}$ be the least submodel of $\mathfrak{C}$ in the sense of the language $\mathcal{L}^*$. That is, every element of $\mathfrak{A}$ is the interpretation of a constant symbol from $\mathcal{L} \cup \{c_0, \dots, c_p\}$ or built from these using the functions of $\mathfrak{C}$. As in the proof of Theorem 14, we can ensure that $\mathcal{L}^* \subseteq \mathcal{L}_{\mathbf{A}}$.

$$\text{Let } P = \{\sigma \in \triangle_{\mathfrak{A}} : \sigma \text{ is a sentence of } \mathcal{L}^*\}.$$

We wish to show that $\mathcal{T} \cup \{\varphi^*\} \cup P$ is satisfiable. By compactness, it suffices to consider $\mathcal{T} \cup \{\varphi^*, \tau\}$ where τ is a sentence in P. If this set is not satisfiable then $\mathcal{T} \models \varphi^* \to \neg\tau$ so that by definition of S we have $\neg\tau \in S$ and hence $\mathfrak{C} \models \neg\tau$. But this is impossible since $\mathfrak{A} \subseteq \mathfrak{C}$ means that $\mathfrak{C}_{\mathbf{A}} \models \triangle_{\mathfrak{A}}$.

Let $\mathfrak{B}' \models \mathcal{T} \cup \{\varphi^*\} \cup P$. The interpretations of $\{c_0, \dots, c_p\}$ generate a submodel of $\mathfrak{B}'$ isomorphic to $\mathfrak{A}$. So there is a model $\mathfrak{B}$ for $\mathcal{L}^*$ such that $\mathfrak{B} \cong \mathfrak{B}'$ and $\mathfrak{A} \subseteq \mathfrak{B}$.

In order to invoke (2) we use the restrictions $\mathfrak{A}|\mathcal{L}$, $\mathfrak{B}|\mathcal{L}$ and $\mathfrak{C}|\mathcal{L}$ of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ to the language $\mathcal{L}$. We have $\mathfrak{B}|\mathcal{L} \models \mathcal{T}$, $\mathfrak{C}|\mathcal{L} \models \mathcal{T}$, $\mathfrak{A}|\mathcal{L} \subseteq \mathfrak{B}|\mathcal{L}$ and $\mathfrak{A}|\mathcal{L} \subseteq \mathfrak{C}|\mathcal{L}$. φ^* is an existential sentence of $\mathcal{L}^* \subseteq \mathcal{L}_{\mathbf{A}}$ and since $\mathfrak{B}' \models \varphi^*$ we have $(\mathfrak{B}|\mathcal{L})_{\mathbf{A}} \models \varphi^*$. So by (2), $(\mathfrak{C}|\mathcal{L})_{\mathbf{A}} \models \varphi^*$ and finally $\mathfrak{C} \models \varphi^*$ which completes the proof of the claim.

We now do the general cases for the proof of the induction on prenex rank. There are two cases, corresponding to the two methods available for increasing the number of alternations of quantifiers:

(a) the addition of universal quantifiers
(b) the addition of existential quantifiers.

For case (a), suppose $\varphi(v_0, \dots, v_p)$ is $\forall w_0 \dots \forall w_m \chi(v_0, \dots, v_p, w_0, \dots, w_m)$ and χ has prenex rank lower than φ. Then $\neg\chi$ also has prenex rank lower than φ and we can use the inductive hypothesis on $\neg\chi$ to obtain a quantifier free formula $\theta_1(v_0, \dots, v_p, w_0, \dots, w_m)$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\forall w_0 \dots \forall w_m)(\neg\chi \leftrightarrow \theta_1)$$
$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \neg\chi \leftrightarrow \exists w_0 \dots \exists w_m \theta_1)$$

By the claim there is a quantifier free formula $\theta_2(v_0, \dots, v_p)$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \theta_1 \leftrightarrow \theta_2)$$
$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \neg\chi \leftrightarrow \theta_2)$$
$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\forall w_0 \dots \forall w_m \chi \leftrightarrow \neg\theta_2)$$

and so $\neg\theta_2$ is the quantifier free formula equivalent to φ.

For case (b), suppose $\varphi(v_0, \dots, v_p)$ is $\exists w_0 \dots \exists w_m \chi(v_0, \dots, v_p, w_0, \dots, w_m)$ and χ has prenex rank lower than φ. We use the inductive hypothesis on χ to obtain a quantifier free formula $\theta_1(v_0, \dots, v_p, , w_0, \dots, w_m)$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\forall w_0 \dots \forall w_m)(\chi \leftrightarrow \theta_1)$$
$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \chi \leftrightarrow \exists w_0 \dots \exists w_m \theta_1)$$

By the claim there is a quantifier free formula $\theta_2(v_0, \dots, v_p)$ such that

$$\mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \theta_1 \leftrightarrow \theta_2)$$
$$\text{So } \mathcal{T} \models (\forall v_0 \dots \forall v_p)(\exists w_0 \dots \exists w_m \chi \leftrightarrow \theta_2)$$

and so θ_2 is the quantifier free formula equivalent to φ. This completes the proof.

(3) $\Rightarrow$ (4)

Let $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{C}$, $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{C} \models \mathcal{T}$. Using the Elementary Diagram Lemma it will suffice to show that Th($\mathfrak{B}_{\mathbf{B}}$)$\cup$ Th($\mathfrak{C}_{\mathbf{C}}$) is satisfiable. Without loss of generosity, we can ensure that $\mathcal{L}_{\mathbf{B}} \cap \mathcal{L}_{\mathbf{C}} = \mathcal{L}_{\mathbf{A}}$.

By the Robinson Consistency Theorem, it suffices to show that there is no sentence σ of $\mathcal{L}_{\mathbf{A}}$ such that both:

$$\text{Th}(\mathfrak{B}_{\mathbf{B}}) \models \sigma \text{ and } \text{Th}(\mathfrak{C}_{\mathbf{C}}) \models \neg\sigma$$

Suppose σ is such a sentence and let $\{c_{a_0}, \dots, c_{a_p}\}$ be the set of constant symbols from $\mathcal{L}_{\mathbf{A}} \setminus \mathcal{L}$ appearing in σ.

Let $\varphi(u_0, \dots, u_p)$ be obtained from σ by exchanging each c_{a_i} for a new variable u_i. Let $\psi(u_0, \dots, u_p)$ be the quantifier free formula from (3):

$$\mathcal{T} \models (\forall u_0, \dots, \forall u_p)(\varphi \leftrightarrow \psi)$$

Let ψ^* be the result of substituting c_{a_i} for each u_i in ψ. ψ^* is also quantifier free.

Since $\mathfrak{B}_{\mathbf{B}} \models \sigma$, $\mathfrak{B} \models \varphi[a_0, \dots, a_p]$. Since $\mathfrak{B} \models \mathcal{T}$, $\mathfrak{B} \models \psi[a_0, \dots, a_p]$ and so $\mathfrak{B}_{\mathbf{A}} \models \psi^*$. Since ψ^* is quantifier free and $\mathfrak{A}_{\mathbf{A}} \subseteq \mathfrak{B}_{\mathbf{A}}$ we have $\mathfrak{A}_{\mathbf{A}} \models \psi^*$; since $\mathfrak{A}_{\mathbf{A}} \subseteq \mathfrak{C}_{\mathbf{A}}$ we then get that $\mathfrak{C}_{\mathbf{A}} \models \psi^*$. Hence $\mathfrak{C} \models \psi[a_0, \dots, a_p]$ and then since $\mathfrak{C} \models \mathcal{T}$ we then get that $\mathfrak{C} \models \varphi[a_0, \dots, a_p]$. But then this means that $\mathfrak{C}_{\mathbf{A}} \models \sigma$ and so $\mathfrak{C}_{\mathbf{C}} \models \sigma$ so σ is in Th($\mathfrak{C}_{\mathbf{C}}$) and we are done.

(4) $\Rightarrow$ (1)

Let $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{A} \subseteq \mathfrak{B}$; we show that $\mathcal{T} \cup \triangle_{\mathfrak{A}}$ is complete. Noting that $\mathfrak{B}_{\mathbf{A}} \models \mathcal{T} \cup \triangle_{\mathfrak{A}}$, we see that it suffices by Lemma 6 to show that $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}'$ for each $\mathfrak{C}' \models \mathcal{T} \cup \triangle_{\mathfrak{A}}$.

For each such $\mathfrak{C}'$, by the Diagram Lemma, there is a model $\mathfrak{C}$ for $\mathcal{L}$ such that $\mathfrak{A} \subseteq \mathfrak{C}$ and $\mathfrak{C}_{\mathbf{A}} \cong \mathfrak{C}'$. Then $\mathfrak{C} \models \mathcal{T}$ so by (4) there is a $\mathfrak{D}$ into which both $\mathfrak{B}_{\mathbf{A}}$ and $\mathfrak{C}_{\mathbf{A}}$ are elementarily embedded.

In particular $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{D}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$ so we are done.

$\square$

Example 11. (Chang and Keisler)
Let $\mathcal{T}$ be the theory in the language $\mathcal{L} = \{U, V, W, R, S\}$ where U, V and W are unary relation symbols and R and S are binary relation symbols having axioms which state that there are infinitely many things, that $U \cup V \cup W$ is everything, that U, V and W are pairwise disjoint, that R is a one-to-one function from U onto V and that S is a one-to-one function from $U \cup V$ onto W.

Exercise 26. Show that $\mathcal{T}$ above is complete and model complete but not submodel complete.

Hints: For completeness, use the Łoś-Vaught test and for model completeness use Lindström's test. For submodel completeness use (2) of the theorem with $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{A} \subseteq \mathfrak{B}$ where $a \in \mathbf{A} = \{b \in \mathbf{B} : \mathfrak{B} \models W(v_0)[b]\}$ along with the sentence

$$(\exists v_0)(U(v_0) \wedge S(v_0, c_a)).$$

Remark. We will prove in the next chapter that each of the following theories admits elimination of quantifiers:

(1) dense linear orders with no end points (DLO)
(2) algebraically closed fields (ACF)
(3) real closed ordered fields (RCF)

C. H. Langford proved elimination of quantifiers for DLO in 1924. The cases of ACF and RCF were more difficult and were done by A. Tarski. Thus, by Exercise 25, we will have model completeness of RCF which was promised at the beginning of Chapter 5.

Exercise 27. Use the fact that RCF admits elimination of quantifiers to prove that RCF is complete; another result originally due to A. Tarski.

Hint: Show that the standard model of Number Theory of Example 6 can be isomorphically embedded into any real closed field and then use (4) from Theorem 17.

Exercise 28. Let $\mathcal{T}$ be the theory DLO in the language $\mathcal{L} = \{<, c_1, c_2\}$ where c_1 and c_2 are constant symbols. Use the fact that DLO admits elimination of quantifiers in its own language $\{<\}$ to show that $\mathcal{T}$ is submodel complete. But, show also, that $\mathcal{T}$ is not complete.

As an application of quantifier elimination of ACF we have the following:

Corollary 7. *(Tarski)*
The truth value of any algebraic statement about the complex numbers can be determined algebraically in a finite number of steps.

Proof. Let $\mathfrak{C}$ be the complex numbers in the language of field theory $\mathcal{L}$; let σ be a sentence of $\mathcal{L}_{\mathbb{C}}$. Then let $\mathbf{A}$ be the finite subset of $\mathbb{C}$ consisting of those elements of $\mathbb{C}$ (other than 0 or 1) which are mentioned in σ. Let φ be the formula of $\mathcal{L}$ formed by exchanging each c_a for a new variable. Then ACF $\models \forall v_0 \ldots \forall v_p(\varphi \leftrightarrow \psi)$ for some quantifier free ψ. Hence $\mathfrak{C} \models \sigma$ iff $\mathfrak{C} \models \varphi[a_0, \ldots, a_p]$ iff $\mathfrak{C} \models \psi[a_0, \ldots, a_p]$ but checking this last statement amounts to evaluating finitely many polynomials in $a_0, \ldots, a_p$.

$\square$

Remark. In fact Tarski's original proof actually gave an explicit method for finding the quantifier free formulas and this led, via the corollary above, to an effective decision proceedure for determining the truth of elementary algebraic statements about the reals or the complex numbers.

As an application of quantifier elimination of RCF we have:

Corollary 8. *(The Tarski-Seidenberg Theorem)*
The projection of a semi-algebraic set in $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *for* $m < n$ *is also semi-algebraic. The semi-algebraic sets of* $\mathbb{R}^n$ *are defined to be all those subsets of* $\mathbb{R}^n$ *which can be obtained by repeatedly taking unions and intersections of sets of the form*

$$\{\langle x_1, \dots, x_n\rangle \in \mathbb{R}^n : p(x_1, \dots, x_n) = 0\}$$
$$\textit{and } \{\langle x_1, \dots, x_n\rangle \in \mathbb{R}^n : q(x_1, \dots, x_n) < 0\}$$

where p *and* q *are polynomials with real coefficients.*

Proof. We first need two simple results which we state as exercises.
Let $\mathfrak{R} = \langle \mathbb{R}, \mathbf{+}, \mathbf{\cdot}, \mathbf{<}, \mathbf{0}, \mathbf{1}\rangle$ be the usual model of the reals. Let $\mathcal{T}$ be RCF considered as a theory in the language $\mathcal{L}_\mathbb{R}$.

Exercise 29. Since RCF admits elimination of quantifiers, $\mathcal{T}$ admits elimination of quantifiers as a theory in the language $\mathcal{L}_\mathbb{R}$.

Exercise 30. A set $X \subseteq \mathbb{R}^n$ is semi-algebraic iff there is a quantifier free formula $\varphi(v_1, \dots, v_n)$ of $\mathcal{L}_\mathbb{R}$ such that

$$X = \{\langle x_1, \dots, x_n\rangle : \mathfrak{R}_\mathbb{R} \models \varphi[x_1, \dots, x_n]\}.$$

Now, in order to prove the corollary, let $X \subseteq \mathfrak{R}^n$ be semi-algebraic and let φ be its associated quantifier free formula. The projection Y of X into $\mathfrak{R}^m$ is

$$\{\langle x_1, \dots, x_m\rangle : \text{for some } x_{m+1}, \dots, x_n \ \langle x_1, \dots, x_m, x_{m+1}, \dots, x_n\rangle \in X\}$$
$$\text{So } Y = \{\langle x_1, \dots, x_m\rangle : \mathfrak{R}_\mathbb{R} \models \exists v_{m+1} \dots \exists v_n \varphi[x_1, \dots, x_m]\}$$

Since $\mathcal{T}$ admits elimination of quantifiers, there is a quantifier free formula θ of $\mathcal{L}_\mathbb{R}$ such that

$$\mathcal{T} \models (\forall v_1 \dots \forall v_m)(\exists v_{m+1} \dots \exists v_n \varphi \leftrightarrow \theta)$$

So for all $x_1, \dots, x_m$

$$\mathfrak{R}_\mathbb{R} \models \exists v_{m+1} \dots \exists v_n \varphi[x_1, \dots, x_m] \text{ iff } \mathfrak{R}_\mathbb{R} \models \theta[x_1, \dots, x_m]$$
$$\text{So } Y = \{\langle x_1, \dots, x_m\rangle : \mathfrak{R}_\mathbb{R} \models \theta[x_1, \dots, x_m]\}$$

and by the exercise, Y is semi-algebraic.

□

CHAPTER 7

Model Completions

Closely related to the notions of model completeness and submodel completeness is the idea of a model completion.

Definition 32. Let $\mathcal{T} \subseteq \mathcal{T}^*$ be two theories in a language $\mathcal{L}$. $\mathcal{T}^*$ is said to be a *model completion* of $\mathcal{T}$ whenever $\mathcal{T}^* \cup \triangle_{\mathfrak{A}}$ is satisfiable and complete in $\mathcal{L}_{\mathbf{A}}$ for each model $\mathfrak{A}$ of $\mathcal{T}$.

Lemma 14. *Let $\mathcal{T}$ be a theory in a language $\mathcal{L}$.*

(1) *If $\mathcal{T}^*$ is a model completion of $\mathcal{T}$, then for each $\mathfrak{A} \models \mathcal{T}$ there is a $\mathfrak{B} \models \mathcal{T}^*$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.*
(2) *If $\mathcal{T}^*$ is a model completion of $\mathcal{T}$, then $\mathcal{T}^*$ is model complete.*
(3) *If $\mathcal{T}$ is model complete, then it is a model completion of itself.*
(4) *If $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ are both model completions of $\mathcal{T}$, then $\mathcal{T}_1^* \models \mathcal{T}_2^*$ and $\mathcal{T}_2^* \models \mathcal{T}_1^*$.*

Proof. (1) Easy. (2) Easier. (3) Easiest. (4) This needs a proof.

Let $\mathfrak{A} \models \mathcal{T}_2^*$. It will suffice to prove that $\mathfrak{A} \models \mathcal{T}_1^*$.

Let $\mathfrak{A}_0 = \mathfrak{A}$. since $\mathfrak{A}_0 \models \mathcal{T}$ and $\mathcal{T}_1^*$ is a model completion of $\mathcal{T}$ we obtain, from (1), a model $\mathfrak{A}_1 \models \mathcal{T}_1^*$ such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$. Similarly, since $\mathfrak{A}_1 \models \mathcal{T}$ and $\mathcal{T}_2^*$ is a model completion of $\mathcal{T}$ we obtain $\mathfrak{A}_2 \models \mathcal{T}_2^*$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$.

Continuing in this manner we obtain a chain:

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1} \subseteq \cdots$$

Let $\mathfrak{B}$ be the union of the chain, $\cup\{\mathfrak{A}_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ we have $\mathfrak{A}_{2n} \models \mathcal{T}_2^*$. By part (2) of this lemma and by part (4) of Theorem 14 we get that for each n, $\mathfrak{A}_{2n} \prec \mathfrak{A}_{2n+2}$. By the Elementary Chain Theorem $\mathfrak{A}_0 \prec \mathfrak{B}$. Similarly $\mathfrak{A}_1 \prec \mathfrak{B}$. So $\mathfrak{A}_0 \equiv \mathfrak{A}_1$ and hence $\mathfrak{A} \models \mathcal{T}_1^*$.

□

Remark. Part (4) of the above lemma shows that model completions are essentially unique. That is, if model completions $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ of $\mathcal{T}$ are closed theories in the sense of Definition 12 then $\mathcal{T}_1^* = \mathcal{T}_2^*$. Since there is no loss in assuming that model completions are closed theories, we speak of *the* model completion of a theory $\mathcal{T}$.

Theorem 18. *Suppose $\mathcal{T} \subseteq \mathcal{T}^*$ are theories for a language $\mathcal{L}$ such that for each $\mathfrak{A} \models \mathcal{T}$ there is a $\mathfrak{B} \models \mathcal{T}^*$ with $\mathfrak{A} \subseteq \mathfrak{B}$. The following are equivalent:*

(1) *$\mathcal{T}^*$ is the model completion of $\mathcal{T}$.*
(2) *For each $\mathfrak{A} \models \mathcal{T}$, $\mathfrak{B} \models \mathcal{T}^*$ and $\mathfrak{C} \models \mathcal{T}^*$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$ we have a model $\mathfrak{D}$ such that both $\mathfrak{B}_{\mathbf{A}}$ and $\mathfrak{C}_{\mathbf{A}}$ are elementarily embedded into $\mathfrak{D}_{\mathbf{A}}$.*

(3) *For each* $\mathfrak{A} \models \mathcal{T}$, $\mathfrak{B} \models \mathcal{T}^*$ *and* $\mathfrak{C} \models \mathcal{T}^*$ *such that* $\mathfrak{A} \subseteq \mathfrak{B}$ *and* $\mathfrak{A} \subseteq \mathfrak{C}$ *we have a model* $\mathfrak{D}$ *such that* $\mathfrak{B}_{\mathbf{A}} \subseteq \mathfrak{D}_{\mathbf{A}}$ *and* $\mathfrak{C}_{\mathbf{A}}$ *is elementarily embedded into* $\mathfrak{D}_{\mathbf{A}}$.

(4) *For each* $\mathfrak{A} \models \mathcal{T}$, $\mathfrak{B} \models \mathcal{T}^*$ *and* $\mathfrak{C} \models \mathcal{T}^*$ *such that* $\mathfrak{A} \subseteq \mathfrak{B}$ *and* $\mathfrak{A} \subseteq \mathfrak{C}$ *we have a model* $\mathfrak{D}$ *such that* $\mathfrak{B}_{\mathbf{A}}$ *is isomorphically embedded into* $\mathfrak{D}_{\mathbf{A}}$ *and* $\mathfrak{C} \prec \mathfrak{D}$.

Proof. (1) $\Rightarrow$ (2) By the Elementary Diagram Lemma it suffices to prove that the union of the elementary diagrams of $\mathfrak{B}_{\mathbf{A}}$ and $\mathfrak{C}_{\mathbf{A}}$, is satisfiable. By the Robinson Consistency Theorem it suffices to show that there is no sentence σ of $\mathcal{L}_{\mathbf{A}}$ such that $\mathrm{Th}\mathfrak{B}_{\mathbf{A}} \models \sigma$ and $\mathrm{Th}\mathfrak{C}_{\mathbf{A}} \models \neg\sigma$.

By (1) we have that $\mathcal{T}^* \cup \triangle_{\mathfrak{A}}$ is a complete theory in $\mathcal{L}_{\mathbf{A}}$. By assumption and the Diagram Lemma both $\mathfrak{B}_{\mathbf{A}} \models \mathcal{T}^* \cup \triangle_{\mathfrak{A}}$ and $\mathfrak{C}_{\mathbf{A}} \models \mathcal{T}^* \cup \triangle_{\mathfrak{A}}$. Therefore $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$. This means that $\mathrm{Th}\mathfrak{B}_{\mathbf{A}} = \mathrm{Th}\mathfrak{C}_{\mathbf{A}}$ and the result follows.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) easily follow from the definitions.

(4) $\Rightarrow$ (1) We first show that $\mathcal{T}^*$ is model complete using Theorem 14; we show that $\mathcal{T}^*$ is existentially complete. Let $\mathfrak{A} \models \mathcal{T}^*$; we show that $\mathfrak{A}$ is existentially closed. Let $\mathfrak{B} \models \mathcal{T}^*$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and let σ be an existential sentence of $\mathcal{L}_{\mathbf{A}}$ with $\mathfrak{B}_{\mathbf{A}} \models \sigma$; our aim is to prove that $\mathfrak{A}_{\mathbf{A}} \models \sigma$.

We invoke (4) with $\mathfrak{C} = \mathfrak{A}$ to get a model $\mathfrak{D}$ such that $\mathfrak{A} \prec \mathfrak{D}$ and $\mathfrak{B}_{\mathbf{A}}$ is isomorphically embedded into $\mathfrak{D}_{\mathbf{A}}$. Referring to Exercise 12 we get a model $\mathfrak{E}$ for $\mathcal{L}_{\mathbf{A}}$ with $\mathfrak{B}_{\mathbf{A}} \subseteq \mathfrak{E}$ and $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}$. Since σ is existential, By Exercise 17 we have that $\mathfrak{E} \models \sigma$; and by Exercise 7, $\mathfrak{D}_{\mathbf{A}} \models \sigma$. Now $\mathfrak{A} \prec \mathfrak{D}$ implies that $\mathfrak{A}_{\mathbf{A}} \equiv \mathfrak{D}_{\mathbf{A}}$ so $\mathfrak{A}_{\mathbf{A}} \models \sigma$ and $\mathcal{T}^*$ is model complete.

We now show that $\mathcal{T}^*$ is the model completion of $\mathcal{T}$. Let $\mathfrak{A} \models \mathcal{T}$; by the hypothesis on $\mathcal{T}$ and $\mathcal{T}^*$ we have that $\mathcal{T}^* \cup \triangle_{\mathfrak{A}}$ is satisfiable. We show that $\mathcal{T}^* \cup \triangle_{\mathfrak{A}}$ is complete in $\mathcal{L}_{\mathbf{A}}$ by showing that for each $\mathfrak{B} \models \mathcal{T}^*$ and $\mathfrak{C} \models \mathcal{T}^*$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$ we have $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$.

Letting $\mathfrak{B}$ and $\mathfrak{C}$ be as above, we invoke (4) to obtain a model $\mathfrak{D}$ such that $\mathfrak{B}_{\mathbf{A}}$ is isomorphically embedded into $\mathfrak{D}_{\mathbf{A}}$ and $\mathfrak{C} \prec \mathfrak{D}$. $\mathfrak{C} \prec \mathfrak{D}$ gives that $\mathfrak{D} \models \mathcal{T}^*$. The isomorphic embedding gives us a model $\mathfrak{E}$ such that $\mathfrak{B} \subseteq \mathfrak{E}$ and $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$. So $\mathfrak{E} \models \mathcal{T}^*$. Using model completeness of $\mathcal{T}^*$ and Theorem 14 we can conclude that $\mathfrak{B} \prec \mathfrak{E}$. We have:

$$\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{E}_{\mathbf{A}} \equiv \mathfrak{D}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$$

and we are done. □

Let's compare the definitions of model completion and submodel complete. Let $\mathcal{T}^*$ be the model completion of $\mathcal{T}$. Then $\mathcal{T}^*$ will be submodel complete provided that every submodel of a model of $\mathcal{T}^*$ is a model of $\mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}^*$, it would be enough to show that every submodel of a model of $\mathcal{T}$ is again a model of $\mathcal{T}$. And this is indeed the case whenever $\mathcal{T}$ is a universal theory, that is, whenever $\mathcal{T}$ has a set of axioms consisting of universal sentences. Unfortunately, this is not always the case.

Our ultimate aim is to show that DLO, ACF and RCF are submodel complete. We will in fact show that these theories are the model completions of LOR, FEI and ORF respectively. See Example 5 to recall the axioms for these theories. Now LOR is a universal theory but FEI and ORF are not. The culprits are the existence axioms for inverses:

$$\forall x \exists y (x + y = 0) \text{ and } \forall x \exists y ((x \neq 0) \to (y \cdot x = 1))$$

.

In fact, a submodel $\mathfrak{A}$ of a field $\mathfrak{B}$ is only a commutative semi-ring, not necessarily a subfield. Nevertheless, $\mathfrak{A}$ generates a subfield of $\mathfrak{B}$ in a *unique* way. This motivates the following definition.

DEFINITION 33. A theory $\mathcal{T}$ is said to be *almost universal* whenever $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{A} \subseteq \mathfrak{C}$, $\mathfrak{C} \models \mathcal{T}$ imply there are models $\mathfrak{D}$ and $\mathfrak{E}$ such that $\mathfrak{D} \models \mathcal{T}$, $\mathfrak{A} \subseteq \mathfrak{D} \subseteq \mathfrak{B}$ and $\mathfrak{E} \models \mathcal{T}$, $\mathfrak{A} \subseteq \mathfrak{E} \subseteq \mathfrak{C}$ and $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$.

EXAMPLE 12. LOR is almost universal since any universal theory $\mathcal{T}$ is almost universal — just let $\mathfrak{D} = \mathfrak{E} = \mathfrak{A}$ and note $\mathfrak{A} \models \mathcal{T}$.

EXAMPLE 13. FEI is almost universal — just let $\mathfrak{D}$ and $\mathfrak{E}$ be the subfields of $\mathfrak{B}$ and $\mathfrak{C}$, respectively, generated by $\mathbf{A}$. The isomorphism $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$ is the natural one obtained from the identity map on $\mathbf{A}$.

EXAMPLE 14. ORF is almost universal — again just let $\mathfrak{D}$ and $\mathfrak{E}$ be the ordered subfields of $\mathfrak{B}$ and $\mathfrak{C}$, respectively, generated by $\mathbf{A}$. The extension of the identity map on $\mathbf{A}$ to the isomorphism $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$ is aided by the fact that the order placement of the inverse of an element a is completely determined by the order placement of a.

THEOREM 19. *Let $\mathcal{T}$ and $\mathcal{T}^*$ be theories of the language $\mathcal{L}$ such that $\mathcal{T}$ is almost universal and $\mathcal{T}^*$ is the model completion of $\mathcal{T}$. Then $\mathcal{T}^*$ is submodel complete.*

PROOF. We show that condition (2) of Theorem 17 is satisfied. Let $\mathfrak{B}$ and $\mathfrak{C}$ be models of $\mathcal{T}^*$ with $\mathfrak{A}$ a submodel of both $\mathfrak{B}$ and $\mathfrak{C}$. We will show that, in fact, $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$.

Now $\mathcal{T} \subseteq \mathcal{T}^*$ so $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{C} \models \mathcal{T}$. Since $\mathcal{T}$ is almost universal there are models $\mathfrak{D}$ and $\mathfrak{E}$ of $\mathcal{T}$ such that $\mathfrak{A} \subseteq \mathfrak{D} \subseteq \mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{E} \subseteq \mathfrak{C}$ and $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$. So $\mathfrak{B}_{\mathbf{D}} \models \mathcal{T}^* \cup \triangle_{\mathfrak{D}}$ and $\mathfrak{C}_{\mathbf{E}} \models \mathcal{T}^* \cup \triangle_{\mathfrak{E}}$.

Now $\mathfrak{B}_{\mathbf{D}}$ is a model for the language $\mathcal{L}_{\mathbf{D}}$ whereas $\mathfrak{C}_{\mathbf{E}}$ is a model for $\mathcal{L}_{\mathbf{E}}$. We wish to obtain a model $\mathfrak{C}'$ for $\mathcal{L}_{\mathbf{D}}$ which "looks exactly like" $\mathfrak{C}_{\mathbf{E}}$. We just let $\mathbf{C}'$ be $\mathbf{C}$ and in fact let $\mathfrak{C}'|\mathcal{L}_{\mathbf{A}} = \mathfrak{C}_{\mathbf{E}}|\mathcal{L}_{\mathbf{A}}$. The interpretation of a constant symbol $c_d \in \mathcal{L}_{\mathbf{D}} \setminus \mathcal{L}_{\mathbf{A}}$ is the interpretation of $c_e \in \mathcal{L}_{\mathbf{E}} \setminus \mathcal{L}_{\mathbf{A}}$ in $\mathfrak{C}_{\mathbf{E}}$ where the isomorphism $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$ takes d to e.

Now $\mathfrak{D} \models \mathcal{T}$ and since $\mathcal{T}^*$ is the model completion of $\mathcal{T}$, $\mathcal{T}^* \cup \triangle_{\mathfrak{D}}$ is complete. The isomorphism $\mathfrak{D}_{\mathbf{A}} \cong \mathfrak{E}_{\mathbf{A}}$ ensures that $\mathfrak{C}' \models \mathcal{T}^* \cup \triangle_{\mathfrak{D}}$. So $\mathfrak{B}_{\mathbf{D}} \equiv \mathfrak{C}'$. Hence $\mathfrak{B}_{\mathbf{D}}|\mathcal{L}_{\mathbf{A}} \equiv \mathfrak{C}'|\mathcal{L}_{\mathbf{A}}$; that is, $\mathfrak{B}_{\mathbf{A}} \equiv \mathfrak{C}_{\mathbf{A}}$.

□

The way to show that DLO, ACF and RCF admit elimination of quantifiers is now clear: use Theorem 19. This reduces to showing that DLO, ACF and RCF are the model completions of LOR, FEI and ORF respectively. To do this we will use Theorem 18, so we first need to show that each pair of these theories satisfy the general hypothesis of Theorem 18: if $\mathfrak{A} \models \mathcal{T}$ then there is a $\mathfrak{B} \models \mathcal{T}^*$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.

For the case $\mathcal{T}$ = LOR and $\mathcal{T}^*$ = DLO is easy; every linear order can be enlarged to a dense linear order without endpoints by judiciously placing copies of the rationals into the linear order.

The case $\mathcal{T}$ = FEI and $\mathcal{T}^*$ = ACF is just the well known fact that every field has an algebraic closure.

The case $\mathcal{T}$ = ORF and $\mathcal{T}^*$ = RCF is just Lemma 13.

So all that remains of the quest to prove elimination of quantifiers for DLO, ACF and RCF is to verify condition (4) of Theorem 18 in each of these cases. We rephrase this condition slightly as:

> For each $\mathfrak{A} \models \mathcal{T}$, $\mathfrak{B} \models \mathcal{T}^*$ and $\mathfrak{C} \models \mathcal{T}^*$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$ there is a $\mathfrak{D}$ such that $\mathfrak{C} \prec \mathfrak{D}$ and an isomorphic embedding $f : \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $f \restriction \mathbf{A}$ is the identity on A.

At this point the reader may already be able to verify this condition for one or more of the pairs $\mathcal{T}$ = LOR and $\mathcal{T}^*$ = DLO, $\mathcal{T}$ = FEI and $\mathcal{T}^*$ = ACF, or $\mathcal{T}$ = ORF and $\mathcal{T}^*$ = RCF. However the remainder of this chapter is devoted to a uniform method.

Definition 34. Let $\mathcal{L}$ be a language and $\Sigma(v_0)$ a set of formulas of $\mathcal{L}$ in the free variable v_0. A model $\mathfrak{A}$ for $\mathcal{L}$ is said to *realize* $\Sigma(v_0)$ whenever there is some $a \in \mathbf{A}$ such that $\mathfrak{A} \models \varphi[a]$ for each $\varphi(v_0)$ in $\Sigma(v_0)$.

Definition 35. The set of formulas $\Sigma(v_0)$ in the free variable v_0, is said to be a *type* of the model $\mathfrak{A}$ whenever

(1) every finite subset of $\Sigma(v_0)$ is realized by $\mathfrak{A}$
(2) $\Sigma(v_0)$ is maximal with respect to (1).

Remark. Every set of formulas $\Sigma(v_0)$ having property (1) of the definition of type can be enlarged to also have property (2).

Lemma 15. *Suppose $\mathfrak{A}$ is a model for a language $\mathcal{L}$. Let $X \subseteq \mathbf{A}$ and let $\Sigma(v_0)$ be a type of $\mathfrak{A}_X$ in the language $\mathcal{L}_X$. Then there is a $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B}_X$ realizes $\Sigma(v_0)$.*

Proof. Let $\mathcal{T} = \mathrm{Th}\mathfrak{A}_{\mathbf{A}} \cup \Sigma(c)$ where c is a new constant symbol and $\Sigma(c) = \{\varphi(c) : \varphi \in \Sigma(v_0)\}$ and of course $\varphi(c)$ is $\varphi(v_0)$ with c replacing v_0.

By the definition of type, for each finite $\mathcal{T}' \subseteq \mathcal{T}$, there is an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \mathcal{T}'$. The Compactness Theorem and the Elementary Diagram Lemma will complete the proof.

□

Lemma 16. *Suppose $\mathfrak{A}$ is a model for a language $\mathcal{L}$. There is a model $\mathfrak{B}$ for $\mathcal{L}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B}_{\mathbf{A}}$ realizes each type of $\mathfrak{A}_{\mathbf{A}}$ in the language $\mathcal{L}_{\mathbf{A}}$.*

Proof. Let $\{\Sigma_\alpha(v_0) : \alpha \in I\}$ enumerate all types of $\mathfrak{A}_{\mathbf{A}}$ in the language $\mathcal{L}_{\mathbf{A}}$. For each $\alpha \in I$ introduce a new constant symbol c_α and let

$$\Sigma_\alpha(c_\alpha) = \{\varphi(c_\alpha) : \varphi \in \Sigma_\alpha(v_0)\}.$$

Let $\Sigma = \cup\{\Sigma_\alpha(c_\alpha) : \alpha \in I\}$. Let $\Sigma' \subseteq \Sigma$ be any finite subset.

Claim. *$\Sigma' \cup \mathrm{Th}\mathfrak{A}_{\mathbf{A}}$ is satisfiable for the language $\mathcal{L}_{\mathbf{A}} \cup \{c_\alpha : \alpha \in I\}$.*

Proof of Claim. Let $\Sigma_{\alpha_1}(v_0), \dots, \Sigma_{\alpha_n}(v_0)$ be finitely many types such that

$$\Sigma' \subseteq \Sigma_{\alpha_1}(c_0) \cup \Sigma_{\alpha_2}(c_1) \cup \cdots \cup \Sigma_{\alpha_n}(c_{\alpha_n}).$$

By Lemma 15 there is a model $\mathfrak{A}_1$ such that $\mathfrak{A} \prec \mathfrak{A}_1$ and $(\mathfrak{A}_1)_{\mathbf{A}}$ realizes $\Sigma_{\alpha_1}(v_0)$. Using Lemma 15 repeatedly, we can obtain

$$\mathfrak{A} \prec \mathfrak{A}_1 \prec \mathfrak{A}_2 \prec \cdots \prec \mathfrak{A}_n$$

such that each $(\mathfrak{A}_j)_{\mathbf{A}}$ realizes $\Sigma_{\alpha_j}(v_0)$.

Now $\mathfrak{A} \prec \mathfrak{A}_n$ so $(\mathfrak{A}_n)_{\mathbf{A}} \models \mathrm{Th}\mathfrak{A}_{\mathbf{A}}$. It is easy to check that since each $\mathfrak{A}_j \prec \mathfrak{A}_n$, $\mathfrak{A}_n$ realizes each $\Sigma_{\alpha_j}(v_0)$ and furthermore so does $(\mathfrak{A}_n)_{\mathbf{A}}$. So we can expand $(\mathfrak{A}_n)_{\mathbf{A}}$ to the language $\mathcal{L}_{\mathbf{A}} \cup \{c_{\alpha_1}, \ldots, c_{\alpha_n}\}$ to satisfy $\Sigma' \cup \mathrm{Th}\mathfrak{A}_{\mathbf{A}}$.

By the claim and the Compactness Theorem, there is a model $\mathfrak{C} \models \Sigma \cup \mathrm{Th}\mathfrak{A}_{\mathbf{A}}$. By the Elementary Diagram Lemma, $\mathfrak{A}$ is elementarily embedded into $\mathfrak{C}|\mathcal{L}$, the restriction of $\mathfrak{C}$ to the language $\mathcal{L}$. Therefore there is a model $\mathfrak{B}$ for $\mathcal{L}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B}_{\mathbf{A}} \cong \mathfrak{C}|\mathcal{L}_{\mathbf{A}}$. It is now straightforward to check that $\mathfrak{B}_{\mathbf{A}}$ realizes each type $\Sigma_\alpha(v_0)$.

$\square$

For any set X we denote by $|X|$ the cardinality of X.

Definition 36. A model $\mathfrak{A}$ for $\mathcal{L}$ is said to be *κ-saturated* whenever we have that for each $X \subseteq \mathbf{A}$ with $|X| < \kappa$, $\mathfrak{A}_X$ realizes each type of $\mathfrak{A}_X$.

Recall that κ^+ is defined to be the cardinal number just larger than κ. So a model $\mathfrak{A}$ will be κ^+-saturated whenever we have that for each $X \subseteq \mathbf{A}$ with $|X| \leq \kappa$, $\mathfrak{A}_X$ realizes each type of $\mathfrak{A}_X$. In particular, if B is any set, $\mathfrak{A}$ will be $|B|^+$-saturated whenever we have that for each $X \subseteq \mathbf{A}$ with $|X| \leq |B|$, $\mathfrak{A}_X$ realizes each type of $\mathfrak{A}_X$.

Remark. A model $\mathfrak{A}$ is said to be *saturated* whenever it is $|\mathbf{A}|$-saturated, where $|\mathbf{A}|$ is the size of the universe of $\mathfrak{A}$. For example, $\langle \mathbb{Q}, < \rangle$ is saturated; to prove this let X be a finite subset of $\mathbb{Q}$ and let $\Sigma(v_0)$ be a type of $\langle \mathbb{Q}, < \rangle_X$. By Lemma 15 and the Downward Löwenheim-Skolem Theorem get a countable $\mathfrak{B}$ such that $\langle \mathbb{Q}, < \rangle_X \prec \mathfrak{B}_X$ and $\mathfrak{B}$ realizes $\Sigma(v_0)$. Use the hint for Exercise 10 to show that $\langle \mathbb{Q}, < \rangle_X \cong \mathfrak{B}_X$ and then note that this means that $\Sigma(v_0)$ is realized in $\langle \mathbb{Q}, < \rangle_X$.

Lemma 17. *(R. Vaught)*
Suppose $\mathfrak{C}$ is an infinite model for $\mathcal{L}$ and $\mathbf{B}$ is an infinite set. There is a $|\mathbf{B}|^+$-saturated model $\mathfrak{D}$ such that $\mathfrak{C} \prec \mathfrak{D}$.

Proof. We build an elementary chain

$$\mathfrak{C} = \mathfrak{C}_0 \prec \mathfrak{C}_1 \prec \mathfrak{C}_2 \prec \cdots \prec \mathfrak{C}_n \prec \cdots \quad n \in \mathbb{N}$$

such that for each $n \in \mathbb{N}$ $(\mathfrak{C}_{n+1})_{\mathbf{C}_n}$ realizes each type of $(\mathfrak{C}_n)_{\mathbf{C}_n}$. This comes immediately by repeatedly applying Lemma 16. Let $\mathfrak{D}$ be the union of the chain; the Elementary Chain Theorem assures us that $\mathfrak{C} \prec \mathfrak{D}$ and indeed each $\mathfrak{C}_n \prec \mathfrak{D}$.

Let $X \subseteq \mathbf{D}$ with $|X| \leq |\mathbf{B}|$ and let $\Sigma(v_0)$ be a type of $\mathfrak{D}_X$. If $X \subseteq \mathbf{C}_n$ for some n, then $\Sigma(v_0)$ is a type of $(\mathfrak{C}_n)_X$ since $(\mathfrak{C}_n)_X \prec \mathfrak{D}_X$. Now $\Sigma(v_0)$ can be enlarged to a type of $(\mathfrak{C}_n)_{\mathbf{C}_n}$ which is realized in $(\mathfrak{C}_{n+1})_{\mathbf{C}_n}$ and so $\Sigma(v_0)$ is realized in $(\mathfrak{C}_{n+1})_{\mathbf{C}_n}$. Since $(\mathfrak{C}_{n+1})_{\mathbf{C}_n} \prec \mathfrak{D}_{\mathbf{C}_n}$, we can easily check that $\Sigma(v_0)$ is realized in $\mathfrak{D}_{\mathbf{C}_n}$. Since $\Sigma(v_0)$ involves only constant symbols associated with X, we have that $\mathfrak{D}_X$ realizes $\Sigma(v_0)$.

We have *almost* proved that $\mathfrak{D}$ is $|\mathbf{B}|^+$-saturated, but not quite, because there is no guarantee that if

$$X \subseteq \mathbf{D} = \cup\{\mathbf{C}_n : n \in \mathbb{N}\}$$

and $|X| \leq |\mathbf{B}|$, then $X \subseteq \mathbf{C}_n$ for some n. However, there would be no problem if X was finite. The problem with infinite X is that the elementary chain is not long enough to catch X.

The solution is to upgrade the notion of an elementary chain to include chains which are indexed by any well ordered set, not just the natural numbers. We sketch the appropriate generalization of the above argument from the case of $\langle \mathbb{N}, < \rangle$ to the case of an arbitrary well ordered set $\langle I, < \rangle$ with least element 0.

We construct an elementary chain of models

$$\mathfrak{C} = \mathfrak{C}_0 \prec \cdots \prec \mathfrak{C}_\beta \prec \ldots \quad \beta \in I$$

recursively as follows. At stage β, suppose we have already constructed $\mathbf{C}_\alpha$ for each $\alpha \in I$ with $\alpha < \beta$. The union of the chain up to β

$$\mathfrak{E} = \cup\{\mathfrak{C}_\alpha : \alpha \in I \text{ and } \alpha < \beta\}$$

falls under the scope of an upgraded Elementary Chain Theorem (which is proved exactly as Theorem 4) and so $\mathfrak{C}_\alpha \prec \mathfrak{E}$ for each $\alpha \in I$ with $\alpha < \beta$. We now use Lemma 16 as before to get $\mathfrak{C}_\beta$ such that $\mathfrak{E} \prec \mathfrak{C}_\beta$ and $(\mathfrak{C}_\beta)_{\mathbf{E}}$ realizes each type of $\mathfrak{E}_{\mathbf{E}}$.

As before, let $\mathfrak{D} = \cup\{\mathfrak{C}_\alpha : \alpha \in I\}$ be the union of the entire chain and by the upgraded Elementary Chain Theorem $\mathfrak{C} \prec \mathfrak{D}$. Also as before, $\mathfrak{D}_X$ realizes each type of $\mathfrak{D}_X$ for each $X \subseteq \mathbf{D}$ such that $X \subseteq \mathbf{C}_\alpha$ for some $\alpha \in I$.

But now we can complete the proof of the lemma by choosing a well ordered set $\langle I, < \rangle$ large enough so that if

$$X \subseteq \mathbf{D} = \cup\{\mathbf{C}_\alpha : \alpha \in I\}$$

and $|X| \leq |\mathbf{B}|$ then there is some $\alpha \in I$ such that $X \subseteq \mathbf{C}_\alpha$. Such a well ordered set is well known to exist — for example, any ordinal with cofinality $> |\mathbf{B}|$.

□

Definition 37. We say that $\mathfrak{B}$ is a *simple extension* of $\mathfrak{A}$ whenever

(1) $\mathfrak{A} \subseteq \mathfrak{B}$ and
(2) there is some $b \in \mathbf{B}$ such that no properly smaller submodel of $\mathfrak{B}$ contains $\mathbf{A} \cup \{b\}$.

Theorem 20. *(Blum's Test)*
Suppose $\mathcal{T} \subseteq \mathcal{T}^*$ are theories of a language $\mathcal{L}$. Suppose further that:

(1) *$\mathcal{T}$ is an almost universal theory*
(2) *every model of $\mathcal{T}$ can be extended to a model of $\mathcal{T}^*$ and*
(3) *for each $\mathfrak{A} \models \mathcal{T}$ and each simple extension $\mathfrak{B}$ of $\mathfrak{A}$ which is a submodel of a model of $\mathcal{T}$, and for each $\mathfrak{C} \models \mathcal{T}^*$ with $\mathfrak{A} \subseteq \mathfrak{C}$ such that $\mathfrak{C}$ is $|\mathbf{B}|^+$-saturated, there is an isomorphic embedding $f : \mathfrak{B} \to \mathfrak{C}$ such that $f \restriction \mathbf{A}$ is the identity on $\mathbf{A}$.*

Then:

(4) *for each $\mathfrak{A} \models \mathcal{T}$, $\mathfrak{B} \models \mathcal{T}^*$ and $\mathfrak{C} \models \mathcal{T}^*$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$ we have a model $\mathfrak{D}$ such that $\mathfrak{B}_{\mathbf{A}}$ is isomorphically embedded into $\mathfrak{D}_{\mathbf{A}}$ and $\mathfrak{C} \prec \mathfrak{D}$,*
(5) *$\mathcal{T}^*$ is the model completion of $\mathcal{T}$ and*
(6) *$\mathcal{T}^*$ admits elimination of quantifiers.*

Proof. Because of Theorem 17, Theorem 18 and Theorem 19, (5) and (6) follow from (1), (2) and (4). We will therefore only need to prove (4).

Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be as in (4). Using Lemma 17 we obtain a $|\mathbf{B}|^+$-saturated model $\mathfrak{D}$ such that $\mathfrak{C} \prec \mathfrak{D}$.

We wish to prove that $\mathfrak{B}_A$ is isomorphically embedded into $\mathfrak{D}_A$.

Since $\mathfrak{A} \subseteq \mathfrak{D}$, the following collection $\mathcal{E}$ of functions is nonempty.

$$\{e : \text{for some } \mathfrak{E} \subseteq \mathfrak{B} \;\; e\colon \mathfrak{E}_A \hookrightarrow \mathfrak{D}_A \text{ is an isomorphic embedding}\}$$

and so has a maximal member f in the sense that no other $e \in \mathcal{E}$ extends f. From $f\colon \mathfrak{F}_A \hookrightarrow \mathfrak{D}_A$ and Exercise 12 we get $\mathfrak{G}$ with $\mathfrak{F} \subseteq \mathfrak{G}$ and an isomorphism $g\colon \mathfrak{G} \to \mathfrak{D}$ extending f.

Claim. $\mathfrak{F} \models \mathcal{T}$

Proof of Claim. We have both $\mathfrak{F} \subseteq \mathfrak{B}$ and $\mathfrak{F} \subseteq \mathfrak{G}$. By condition (1), there are models $\mathfrak{H}$ and $\mathfrak{J}$ of $\mathcal{T}$ with $\mathfrak{F} \subseteq \mathfrak{H} \subseteq \mathfrak{B}$ and $\mathfrak{F} \subseteq \mathfrak{J} \subseteq \mathfrak{G}$ such that $\mathfrak{H}_{\mathbf{F}} \cong \mathfrak{J}_{\mathbf{F}}$. This gives an isomorphic embedding $h\colon \mathfrak{H} \hookrightarrow \mathfrak{G}$ such that $h \restriction \mathbf{F}$ is the identity on $\mathbf{F}$.

The composition $g \circ h\colon \mathfrak{H} \hookrightarrow \mathfrak{D}$ is an isomorphic embedding with the property that for all $x \in \mathbf{F}$:

$$(g \circ h)(x) = g(x) = f(x).$$

By the maximality of f, $f = g \circ h$. Hence $\mathfrak{F} = \mathfrak{H}$ and $\mathfrak{F} \models \mathcal{T}$, finishing the proof of the claim.

Claim. $\mathfrak{F} = \mathfrak{B}$

Proof of Claim. If not, pick $b \in \mathbf{B} \setminus \mathbf{F}$ and form the simple extension $\mathfrak{F}'$ of $\mathfrak{F}$ by b. Since $\mathfrak{G} \cong \mathfrak{D}$, $\mathfrak{G}$ is also $|\mathbf{B}|^+-$ saturated so that we can apply condition (3) to $\mathfrak{F}$, $\mathfrak{F}'$ and $\mathfrak{G}$. We obtain an isomorphic embedding $f'\colon \mathfrak{F}' \hookrightarrow \mathfrak{G}$ such that $f' \restriction \mathbf{F}$ is the identity on $\mathbf{F}$. But now $g \circ f'$ contradicts the maximality of f and completes the proof of the claim.

Therefore f isomorphically embeds $\mathfrak{B}_{\mathbf{A}}$ into $\mathfrak{D}_{\mathbf{A}}$.

□

The following lemma completes the proofs that each of the theories DLO, ACF and RCF admit elimination of quantifiers.

Lemma 18. *Each of the following three pairs of theories $\mathcal{T}$ and $\mathcal{T}^*$ satisfy condition (3) of Blum's Test.*

(1) *$\mathcal{T}$ = LOR, theory of linear orderings. $\mathcal{T}^*$ = DLO, theory of dense linear orderings without endpoints.*
(2) *$\mathcal{T}$ = FEI, theory of fields. $\mathcal{T}^*$ = ACF, theory of algebraically closed fields.*
(3) *$\mathcal{T}$ = ORF, theory of ordered fields. $\mathcal{T}^*$ = RCF, theory of real closed fields.*

Proof of (1). Let $\mathfrak{A}$ and $\mathfrak{B}$ be linear orders, with $\mathbf{B} = \mathbf{A} \cup \{b\}$ and $\mathfrak{A} \subseteq \mathfrak{B}$.

Let $\mathfrak{C}$ be a $|\mathbf{B}|^+$-saturated dense linear order without endpoints with $\mathfrak{A} \subseteq \mathfrak{C}$. We wish to find an isomorphic embedding $f : \mathfrak{B} \to \mathfrak{C}$ which is the identity on $\mathbf{A}$.

Consider a type of $\mathfrak{C}_{\mathbf{A}}$ containing the following formulas:

$$c_a < v_0 \text{ for each } a \in \mathbf{A} \text{ such that } a < b$$

$$v_0 < c_a \text{ for each } a \in \mathbf{A} \text{ such that } b < a$$

Since $\mathfrak{C}$ is a dense linear order without endpoints each finite subset of the type can be realized in $\mathfrak{C}_{\mathbf{A}}$.

Saturation now gives some $t \in \mathbf{C}$ realizing this type. We set $f(b) = t$ and we are finished.

Proof of (2). Let $\mathfrak{A}$ be a field and $\mathfrak{B}$ a simple extension of $\mathfrak{A}$ witnessed by b such that $\mathfrak{B}$ is a submodel of a field (a commutative ring).

Let $\mathfrak{C}$ be a $|\mathbf{B}|^+$-saturated algebraically closed field such that $\mathfrak{A} \subseteq \mathfrak{C}$. We wish to find an isomorphic embedding $f : \mathfrak{B} \to \mathfrak{C}$ which is the identity on $\mathfrak{A}$.

There are two cases:

(I) b is algebraic over $\mathfrak{A}$,
(II) b is transcedental over $\mathfrak{A}$.

Case(I). Let p be a polynomial with coefficients from $\mathbf{A}$ such that $p(b) = 0$ but b is not the root of any such polynomial of lower degree. Since $\mathfrak{C}$ is algebraically closed there is a $t \in \mathbf{C}$ such that $p(t) = 0$. We extend the identity map f on $\mathbf{A}$ to make $f(b) = t$. We extend f to the rest of $\mathfrak{B}$ by letting $f(r(b)) = r(t)$ for any polynomial r with coefficients from $\mathbf{A}$. It is straightforward to show that f is still a well-defined isomorphic embedding.

Case (II). Let us consider a type of $\mathfrak{C}_{\mathbf{A}}$ containing the following set of formulas:

$$\{\neg(p(v_0) = 0)\}$$

where p is a polynomial with coefficients in $\{c_a : a \in \mathbf{A}\}$.

Since $\mathfrak{C}$ is algebraically closed, it is infinite and hence each finite subset is realized in $\mathfrak{C}_{\mathbf{A}}$. Saturation will now give some $t \in \mathbf{C}$ such that t realizes the type.

We set $f(b) = t$. Since t is transcedental over $\mathfrak{A}$, the extension of f to all of $\mathbf{B}$ comes easily from the fact that every element of $\mathbf{B} \setminus \mathbf{A}$ is the value at b of some polynomial function with coefficients from A.

$\square$

Proof of (3). Let $\mathfrak{A}$ be an ordered field and $\mathfrak{B}$ be a simple extension of $\mathfrak{A}$ witnessed by b such that $\mathfrak{B}$ is a submodel of an ordered field (an ordered commutative ring).

Let $\mathfrak{C}$ be a $|\mathbf{B}|^+$-saturated real closed field such that $\mathfrak{A} \subseteq \mathfrak{C}$. We wish to find an isomorphic embedding $f : \mathfrak{B} \to \mathfrak{C}$ which is the identity on $\mathfrak{A}$.

There are two cases:

(I) b is algebraic over $\mathfrak{A}$.
(II) b is transcedental over $\mathfrak{A}$.

Case (I). Since b is algebraic over $\mathfrak{A}$ we have a polynomial p with coefficients in $\mathbf{A}$ such that $p(b) = 0$. All other elements of the universe of the simple extension $\mathfrak{B}$ are of the form $q(b)$ where q is a polynomial with coefficients in $\mathbf{A}$. Before beginning the main part of the proof we need some algebraic facts.

Claim. *Let $\mathfrak{D}$ be a real closed ordered field and $q(x)$ be a polynomial over $\mathfrak{D}$ of degree n. Then for any $e \in \mathfrak{D}$ we have:*

$$q(x) = \sum_{m=0}^{n} \frac{q^{(m)}(e)}{m!}(x - e)^m$$

where $q^{(m)}$ stands for the polynomial which is the m-th derivative of q.

Proof of Claim. This is Taylor's Theorem from Calculus; unfortunately we cannot use Calculus to prove it because we are in $\mathfrak{D}$, not necessarily the reals $\mathfrak{R}$. However the reader can check that the Binomial Theorem gives the identity for the special cases of $q(x) = x^n$ and that these special cases readily give the full result.

Claim. *Let $\mathfrak{D}$ be a real closed ordered field and $q(x)$ a polynomial over $\mathfrak{D}$ with $e \in \mathbf{D}$ and $q(e) = 0$. If there is an $a < e$ such that $q(x) > 0$ for all $a < x < e$ then $q'(e) \leq 0$. If there is an $a > e$ such that $q(x) > 0$ for all $e < x < a$ then $q'(e) \geq 0$. Here q' is the first derivative of q.*

Proof of Claim. From the previous claim we get

$$\frac{q(x) - q(e)}{x - e} = q'(e) + (x - e)\left(\sum_{m=2}^{n} \frac{q^{(m)}(e)}{m!}(x - e)^{m-2}\right)$$

for any $x \neq e$ in $\mathfrak{D}$. By choosing x close enough to e we can ensure that the entire right hand side has the same sign as $q'(e)$. A proof by contradiction now follows readily.

Claim. *Let $\mathfrak{D}$ be a real closed ordered field and $q(x)$ be a polynomial over $\mathfrak{D}$ with $e \in \mathbf{D}$ and $q(e) = 0$. If w and z are in $\mathbf{D}$ such that $w < e < z$ and $q(w) \cdot q(z) > 0$ then there is a d in $\mathbf{D}$ such that $w < d < z$ and $q'(d) = 0$.*

Proof of Claim. Without loss of generosity $q(w) > 0$ and $q(z) > 0$. Since q has only finitely many roots, we can pick d_1 to be the least x such that $w < x \leq e$ and $q(x) = 0$. Since $q(x) \neq 0$ for all $w < x < d_1$, the Intermediate Value Property of Real Closed Ordered Fields shows that q cannot change sign here and so $q(x) > 0$ for all $w < x < d_1$. By the previous claim, $q'(d_1) \leq 0$. A similar argument with z shows that there is a d_2 such that $e \leq d_2 < z$ and $q'(d_2) \geq 0$. If $d_1 = e = d_2$ take $d = e$. If $d_1 < d_2$ the Intermediate Value Property gives a d with the required properties.

Claim. *Let $\mathfrak{D}$ be a real closed ordered field with an ordered field $\mathfrak{E} \subseteq \mathfrak{D}$. Let $f : \mathfrak{E} \to \mathfrak{C}$ be an isomorphic embedding into a real closed ordered field. Let q be a polynomial with coefficients in $\mathbf{E}$ such that $\{x \in \mathbf{D} : q'(x) = 0\} \subseteq \mathbf{E}$. Let $d \in \mathbf{D} \setminus \mathbf{E}$ be such that $q(d) = 0$ but d is not a root of a polynomial with coefficients from $\mathbf{E}$ which has lower degree. Then f can be extended over the subfield of $\mathfrak{D}$ generated by $\mathbf{E} \cup \{d\}$.*

Proof of Claim. Since the finitely many roots of q' from $\mathbf{D}$ actually lie in $\mathbf{E}$, we can get e_1 and e_2 in $\mathbf{E}$ such that $e_1 < d < e_2$ and $q'(x) \neq 0$ for all x in $\mathbf{D}$ such that $e_1 < x < e_2$. Furthermore for all x in E we have $q(x) \neq 0$. We can now apply the previous claim to get that $q(w) \cdot q(z) < 0$ for all w and z in $\mathbf{E}$ such that $e_1 < w < d < z < e_2$.

We now move to the real closed ordered field $\mathfrak{C}$ and the isomorphic embedding f. For each w and z in $\mathbf{E}$ such that $e_1 < w < d < z < e_2$ we have $f(w) < f(z)$ and $q(f(w)) \cdot q(f(z)) < 0$. By the Intermediate Value property of $\mathfrak{C}$ we get, for each such w and z, a $y \in \mathbf{C}$ such that $f(w) < y < f(z)$ and $q(y) = 0$. Since q has only finitely many roots there is some $t \in \mathbf{C}$ such that $q(t) = 0$, $f(w) < t$ for all $e_1 < w < d$ and $t < f(z)$ for all $d < z < e_2$.

We now extend f by letting $f(d) = t$ and $f(r(d)) = r(t)$ for any polynomial r with coefficients from $\mathbf{E}$. It is straightforward to check that the extension is a well-defined isomorphic embedding of the simple extension of $\mathfrak{E}$ by d into $\mathfrak{C}$. We use the fact that ORF is almost universal to extend the isomorphic embedding to all of the subfield of $\mathfrak{D}$ generated by $\mathbf{E} \cup \{d\}$, since we can rephrase the definition of almost universal as follows:

> Whenever $\mathfrak{C} \models \mathcal{T}$, $\mathfrak{D} \models \mathcal{T}$, $\mathfrak{E}' \subseteq \mathfrak{D}$ and $f : \mathfrak{E}' \hookrightarrow \mathfrak{C}$ is an isomorphic embedding there is a model $\mathfrak{E}'' \models \mathcal{T}$ such that $\mathfrak{E}' \subseteq \mathfrak{E}'' \subseteq \mathfrak{D}$ and f extends over $\mathfrak{E}''$.

It is now time for the main part of the proof of this case. Using Lemma 13, let $\mathfrak{D}$ be a real closed ordered field with $\mathfrak{B} \subseteq \mathfrak{D}$. We have a polynomial p with coefficients from $\mathbf{A}$ such that $p(b) = 0$. By induction on the degree of p, we can show that there is a sequence of elements $d_0, \ldots, d_m = b$ of elements of $\mathfrak{D}$, a sequence of subfields of $\mathfrak{D}$:

$$\mathfrak{A} = \mathfrak{E}_0 \subseteq \mathfrak{E}_1 \subseteq \ldots \subseteq \mathfrak{E}_{m+1}$$

with each $d_j \in \mathbf{E}_{j+1} \setminus \mathbf{E}_j$ and corresponding isomorphic embeddings

$$f_j : \mathfrak{E}_j \to \mathfrak{C}$$

coming from the previous claim and having the property that f_0 is the identity and f_{j+1} extends f_j. In this way we extend the identity map $f_0 : \mathfrak{A}_0 \hookrightarrow \mathfrak{C}$ until we reach $f_{m+1} : \mathfrak{E}_{m+1} \hookrightarrow \mathfrak{C}$. We then note that since $b \in \mathfrak{E}_{m+1}$ we have $\mathfrak{B} \subseteq \mathfrak{E}_{m+1}$ and we are finished.

Case (II). Let us consider a type of $\mathfrak{C}_\mathbf{A}$ containing the following formulas:

$$c_a < v_0 \text{ for all } a \in \mathbf{A} \text{ with } a < b$$

$$v_0 < c_a \text{ for all } a \in \mathbf{A} \text{ with } b < a$$

$$\neg(p(v_0) = 0) \text{ for all polynomials } p \text{ with coefficients in } \{c_a : a \in \mathbf{A}\}$$

Since each interval of $\mathfrak{C}$ is infinite, each finite subset of this type is realized by $\mathfrak{C}_\mathbf{A}$. Saturation now gives $t \in \mathbf{C}$ which realizes this type. We put $f(b) = t$.

We can now extend f on the rest of $\mathbf{B} \setminus \mathbf{A}$, since each such element is the value at b of a polynomial function with coefficients from $\mathbf{A}$.

□

Question. Suppose $\mathfrak{K}$ is the reduct of a real closed ordered field to the language of field theory. Can you show that $\mathfrak{K}[\sqrt{-1}]$ is algebraically closed using Model Theory and the Fundamental Theorem of Algebra?

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