Basic Model Theory

John L. Bell

1. Structures and First-Order Languages

A *structure* is a triple

$$\mathfrak{A} = (A,\ \{R_i\colon i \in I\},\ \{e_j\colon j \in J\}),$$

where A, the *domain* or *universe* of $\mathfrak{A}$, is a *nonempty* set, $\{R_i\colon i \in I\}$ is an indexed family of relations on A and $\{e_j\colon j \in J\})$ is an indexed set of elements —the *designated elements* of A. For each $i \in I$ there is then a natural number $\lambda(i)$ —the *degree* of R_i —such that R_i is a $\lambda(i)$-place relation on A, i.e., $R_i \subseteq A^{\lambda(i)}$. This λ may be regarded as a function from I to the set ω of natural numbers; the pair (λ, J) is called the *type* of **A**. Structures of the same type are said to be *similar*.

Note that since an n-place operation $f\colon A^n \to A$ can be regarded as an $(n+1)$-place relation on A, algebraic structures containing operations such as groups, rings, vector spaces, etc. may be construed as structures in the above sense.

The *cardinality* $\|\mathfrak{A}\|$ of a structure $\mathfrak{A}$ is defined to be the cardinality $|A|$ of its domain A.

The *first-order language* $\mathscr{L}$ of *type* (λ, J) has the following categories of *basic symbols*:

(i) *individual variables:* a denumerable sequence $v_0, v_1, \ldots;$
(ii) *predicate symbols:* for each $i \in I$, a predicate symbol P_i of degree $\lambda(i)$;
(iii) *individual constants*: for each $j \in J$ an individual constant c_j;
(iv) *equality symbol*: the symbol =;
(v) *logical operators:* $\neg$ (negation), $\wedge$ (conjunction);
(vi) *existential quantifier symbol*: $\exists$ ("there exists");
(vii) *punctuation symbols*: e.g. ( ) , [ ].

Predicate and constant symbols are often called *extralogical* symbols; variables and constants are collectively known as *terms:* we shall use symbols t, u, possibly with subscripts, to denote arbitrary terms.

Atomic formulas of $\mathscr{L}$ are finite strings of basic symbols of either of the forms $P_i t_1 \ldots t_{\lambda(i)}$ or $t = u$, where $t_1, \ldots, t_{\lambda(i)}, t, u$ are terms. *Formulas* of $\mathscr{L}$ (or $\mathscr{L}$-*formulas*) are finite strings of basic symbols defined in the following recursive manner:

(a) any atomic formula is a formula;
(b) if φ, ψ are formulas, so also are $\neg\varphi$, $\varphi \wedge \psi$, and $\exists x\varphi$, where x is any variable v_n;
(c) a finite string of symbols is a formula exactly when it follows from finitely many applications of (a) and (b) that it is one.

We write $Form(\mathscr{L})$ for the set of all formulas of $\mathscr{L}$. The *degree* (of complexity) of a formula is defined to be the number of occurrences of logical operators and quantifiers in it.

The symbols $\vee$ (disjunction), $\rightarrow$ (implication) and $\forall$ (universal quantifier) are introduced as *abbreviations*:

$$\varphi \vee \psi \quad \text{for } \neg(\neg\varphi \wedge \neg\psi)$$
$$\varphi \rightarrow \psi \quad \text{for } \neg\varphi \vee \psi$$
$$\varphi \leftrightarrow \psi \quad \text{for } (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$
$$\forall x\varphi \quad \text{for } \neg\exists x\neg\varphi.$$

We also write $\bigwedge_{i=1}^{n} \varphi_i$ for $\varphi_1 \wedge \ldots \wedge \varphi_n$.

It will be assumed that the notions of *free* and *bound* occurrence of a variable in a formula are understood. We write $\varphi(v_0, \ldots, v_n)$ to indicate that the free variables of φ are among $v_0, \ldots, v_n$. We also write $\varphi(x/t)$, or simply $\varphi(t)$, for the result of substituting t at each free occurrence of x in φ. More generally, we write $\varphi(t_0, \ldots, t_n)$ for the result of substituting t_i at each occurrence of v_i , for $i = 0, \ldots, n,$ in $\varphi(v_0, \ldots, v_n)$. An $\mathscr{L}$- *sentence* is an $\mathscr{L}$- formula without free variables. We write $Sent(\mathscr{L})$ for the set of all $\mathscr{L}$-sentences.

The *cardinality* $\|\mathscr{L}\|$ of $\mathscr{L}$ is defined to be the cardinality of its set of basic symbols.

Lemma. $\|\mathscr{L}\| = |Form(\mathscr{L})|$.

Proof. Let $\|\mathscr{L}\| = \kappa$. Since κ is infinite and each formula is a finite string of symbols, $|Form(\mathscr{L}) \leq \kappa$. The fact that κ is infinite also implies that either the set of terms or the set of predicate symbols of $\mathscr{L}$ (or both) must have cardinality κ. In either case the set of atomic formulas of the form $P_i t \ldots t$ has cardinality κ, so that $|Form(\mathscr{L})| \geq \kappa$. The Lemma follows. ■

For $\Sigma \subseteq Sent(\mathscr{L})$ we define $\mathscr{L}_\Sigma$ to be the language whose extralogical symbols are precisely those occurring in at least one sentence of Σ.

Lemma. $\|\mathscr{L}_\Sigma\| = \max(\aleph_0, |\Sigma|)$.

Proof. If Σ is finite, evidently $\|\mathscr{L}_\Sigma\| = \aleph_0$. Now suppose that $|\Sigma| = \kappa \geq \aleph_0$. We have $|\Sigma| \leq Form(\mathscr{L}_\Sigma)| = \|\mathscr{L}_\Sigma\|$ by the previous lemma. For each $\sigma \in \Sigma$ let $S(\sigma)$ be the set of ($\mathscr{L}_\Sigma$-) symbols occurring in σ: then $S(\sigma)$ is finite. Also the set K of terms of $\mathscr{L}_\Sigma$ is included in the union of the sets $S(\sigma)$ for $\sigma \in \Sigma$, so that

$$|K| \leq |\bigcup\{S(\sigma): \sigma \in \Sigma\}| \leq \sum_{\sigma\in\Sigma} |S(\sigma)| \leq |\Sigma|.\aleph_0 = |\Sigma|.$$

Thus $\|\mathscr{L}_\Sigma\| \leq |K| + \aleph_0 + \aleph_0 \leq |\Sigma|$, and hence $\|\mathscr{L}_\Sigma\| = |\Sigma|$ as required. ■

2. Satisfaction, validity, and models.

If $\mathscr{L}$ is a first-order language, a structure having the same type as that of $\mathscr{L}$ is called an $\mathscr{L}$-

structure. Let $\mathfrak{A} = (A, \{R_i\colon i \in I\}, \{e_j\colon j \in J\})$ be an $\mathscr{L}$-structure, where $\mathscr{L}$ has type (λ, J), and let $\boldsymbol{a} = (a_0, a_1, \ldots)$ be a countable sequence of elements of A (such a sequence will be referred to henceforth as an *A-sequence*). For any predicate symbol or term of $\mathscr{L}$, we define its *interpretation under* $(\mathfrak{A},\mathbf{a})$ as follows:

$$P_i^{(\mathfrak{A},\boldsymbol{a})} = R_i \quad c_j^{(\mathfrak{A},\boldsymbol{a})} = e_j \quad v_n^{(\mathfrak{A},\boldsymbol{a})} = a_n.$$

Since $P_i^{(\mathfrak{A},\boldsymbol{a})}$ and $c_j^{(\mathfrak{A},\boldsymbol{a})}$ depend only on $\mathfrak{A}$, we usually just write $P_i^{\mathfrak{A}}$ and $c_j^{\mathfrak{A}}$ for these and call them the *interpretations* of P_i and c_j, respectively, *in* $\mathfrak{A}$.

For $n \in \omega$, $b \in A$ we define

$$[n|b]\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}, b, a_{n+1}, \ldots).$$

For $\varphi \in Form(\mathscr{L})$ we define the relation $\boldsymbol{a}$ *satisfies* φ *in* $\mathfrak{A}$, written

$$\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi,$$

recursively on the degree of φ as follows:

1) for terms t, u,

$$\mathfrak{A} \vDash_{\boldsymbol{a}} t = u \iff t^{(\mathfrak{A},\boldsymbol{a})} = u^{(\mathfrak{A},\boldsymbol{a})};$$

2) for terms $t_1, \ldots, t_{\lambda(i)}$,

$$\mathfrak{A} \vDash_{\boldsymbol{a}} P_i t_1 \ldots t_{\lambda(i)} \iff R_i(t_1^{(\mathfrak{A},\boldsymbol{a})}, \ldots, t_{\lambda(i)}^{(\mathfrak{A},\boldsymbol{a})});$$

3) $\mathfrak{A} \vDash_{\boldsymbol{a}} \neg\varphi \iff$ not $\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$;

4) $\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi \wedge \psi \iff \mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$ and $\mathfrak{A} \vDash_{\boldsymbol{a}} \psi$,

5) $\mathfrak{A} \vDash_{\boldsymbol{a}} \exists v_n \varphi \iff$ for some $b \in A$, $\mathfrak{A} \vDash_{[n|b]\boldsymbol{a}} \varphi$.

The following facts are then easily established:

(a) $\mathfrak{A} \vDash_{\boldsymbol{a}} \forall v_n \varphi \iff$ for all $b \in A$, $\mathfrak{A} \vDash_{[n|b]\boldsymbol{a}} \varphi$;

(b) suppose that $\boldsymbol{a}, \boldsymbol{b}$ are A-sequences such that $a_n = b_n$ whenever v_n occurs free in φ. Then

$$\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi \iff \mathfrak{A} \vDash_{\boldsymbol{b}} \varphi,$$

In view of fact (b), the truth of $\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$ depends only on the interpretations under $(\mathfrak{A},\boldsymbol{a})$ of the free variables of φ, that is, if these are among $v_0, \ldots, v_n$, only on $a_0, \ldots, a_n$. Accordingly, under these conditions we shall often write

$$\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi[a_0, \ldots, a_n] \quad \text{for} \quad \mathfrak{A} \vDash_{\boldsymbol{a}} \varphi.$$

We say that a formula φ is *valid* in $\mathfrak{A}$ if $\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$ for *every A*-sequence $\boldsymbol{a}$ and *satisfiable* in $\mathfrak{A}$ if $\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$ for *some A*-sequence $\boldsymbol{a}$. It follows from (b) above that a sentence σ is satisfiable in a given structure iff it is valid there. If σ is valid in $\mathfrak{A}$ we write

$$\mathfrak{A} \vDash_{\boldsymbol{a}} \varphi$$

and say that $\mathfrak{A}$ is a *model of* σ, or that σ *holds in* $\mathfrak{A}$. If $\Sigma \subseteq Sent(\mathscr{L})$, we say that $\mathfrak{A}$ is a *model of* Σ, and write

$$\mathfrak{A} \vDash \Sigma,$$

if $\mathfrak{A}$ is a model of each member of Σ. If $\varphi \in Form(\mathscr{L})$, we say that Σ *logically entails* φ, and write

$$\Sigma \vDash \varphi,$$

if φ is valid in every model of Σ. In particular, we write

$$\vDash \varphi$$

for $\varnothing \vDash \varphi$; a formula φ satisfying this condition is then valid in every ($\mathscr{L}$-) structure and is called *universally valid.*

Let $\mathscr{L}^*$ be a language which is an *extension* of $\mathscr{L}$, i.e. obtained from $\mathscr{L}$ by adding a set $\{P_i: i \in I^*\}$ of new predicate symbols and a set $\{c_j: j \in J^*\}$ of new constant symbols. Given an $\mathscr{L}^*$-structure

$$\mathfrak{A}^* = (A, \{R_i: i \in I \cup I^*\}, \{e_j: j \in J \cup J^*\}),$$

the $\mathscr{L}$-structure

$$\mathfrak{A} = (A, \{R_i: i \in I\}, \{e_j: j \in J\})$$

is called the *$\mathscr{L}$-reduction* of $\mathfrak{A}^*$. Analogously, $\mathfrak{A}^*$ is called an *$\mathscr{L}^*$-expansion* of $\mathfrak{A}$. Notice that, while an $\mathscr{L}^*$-structure always has a unique $\mathscr{L}$-reduction, an $\mathscr{L}$-structure has in general more than one $\mathscr{L}^*$-expansion. We write $\mathfrak{A}^*|\mathscr{L}$ for the $\mathscr{L}$-reduction of $\mathfrak{A}^*$. It is important to keep in mind the fact that *expanding or reducing has no effect on the domain of a structure; these operations merely add or subtract relations and designated elements.*

The following lemmas are routine. The first is proved by a straightforward induction on the degree of complexity of formulas, the second follows from the definition of $\vDash$.

Expansion lemma. Let $\Sigma \subseteq Sent(\mathscr{L})$, let $\mathscr{L}^*$ be any extension of $\mathscr{L}$, let $\mathfrak{A}$ be any $\mathscr{L}$-structure, and let $\mathfrak{A}^*$ be any $\mathscr{L}^*$-expansion of $\mathfrak{A}$. Then

$$\mathfrak{A} \vDash \Sigma \iff \mathfrak{A}^* \vDash \Sigma . \quad \blacksquare$$

Constants lemma. Let $\mathfrak{A}$ be an $\mathscr{L}$-structure, let $\varphi(v_0, ..., v_n) \in Form(\mathscr{L})$, and let $c_0, ..., c_n$ be constant symbols of $\mathscr{L}$. Then

$$\mathfrak{A} \vDash \varphi(c_0, ..., c_n) \iff \mathfrak{A} \vDash \varphi[c_0^{\mathfrak{A}}, ..., c_n^{\mathfrak{A}}]. \quad \blacksquare$$

3. Review of first-order predicate logic.

Let $\mathscr{L}$ be a first-order language of type (λ, *J*). We specify *axioms* and *rules of inference* for $\mathscr{L}$ as follows. As *axioms* we take

1) all instances of propositional tautologies;

2) *equality axioms:*

$$t = t \quad t = u \to u = t \quad t = u \wedge u = v \to t = v$$
$$(t_1 = u_1 \wedge \ldots \wedge t_{\lambda(i)} = u_{\lambda(i)}) \to [P_i t_1 \ldots t_{\lambda(i)} \to P_i u_1 \ldots u_{\lambda(i)}]$$

3) all formulas of the form

$$\forall x\varphi(x) \to \varphi(t) \qquad \varphi(t) \to \exists x\varphi(x)$$

where, if *t* is a variable, it does not occur bound in φ.

The *rules of inference* of $\mathscr{L}$ are:

1) *modus ponens*:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

2) *quantifier rules*: if x is not free in φ,

$$\frac{\varphi \to \psi(x)}{\varphi \to \forall x\varphi(x)} \qquad\qquad \frac{\psi(x) \to \varphi}{\exists x\psi(x) \to \varphi}$$

A *proof* in $\mathscr{L}$ of φ from a set $\Sigma \subseteq Sent(\mathscr{L})$ is a finite sequence $\psi_1, ..., \psi_n$ of $\mathscr{L}$-formulas, with $\psi_n = \varphi$, each member of which is either an axiom, a member of Σ, or else follows from previous ψ_i by one of the rules of inference. We say that φ is *provable from* Σ, and write

$$\Sigma \vdash \varphi,$$

if there is a proof of φ from Σ. Σ is said to be *consistent* (in $\mathscr{L}$) if for no $\mathscr{L}$-formula φ do we have $\Sigma \vdash \varphi \wedge \neg\varphi$. If $\varnothing \vdash \varphi$, we write $\vdash \varphi$ and say that φ is a *theorem* of $\mathscr{L}$.

We now list a number of basic results concerning these notions. Throughout, Σ denotes an arbitrary set of $\mathscr{L}$-sentences.

Quantifier lemma. If x does not occur free in φ, then

$$\Sigma \vdash \exists x(\varphi \wedge \psi) \leftrightarrow (\varphi \wedge \exists x\psi) \qquad \Sigma \vdash \exists x(\varphi \to \psi) \leftrightarrow (\varphi \to \exists x\psi). \quad \blacksquare$$

Deduction theorem. If $\sigma \in Sent(\mathscr{L})$, then for any formula φ,

$$\Sigma \cup \{\sigma\} \vdash \varphi \iff \Sigma \vdash \varphi \to \psi. \quad \blacksquare$$

Finiteness theorem. If $\Sigma \vdash \varphi$, then $\Sigma_0 \vdash \varphi$ for some finite subset Σ_0 of Σ. $\blacksquare$

Soundness theorem. If $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$. $\blacksquare$

Consistency lemma. (i) Σ is consistent iff $\Sigma \nvdash \varphi$ not for some $\mathscr{L}$-formula φ. (ii) Σ is consistent iff every finite subset of Σ is so. (iii) If $\sigma \in Sent(\mathscr{L})$, $\Sigma \cup \{\sigma\}$ is consistent iff $\Sigma \nvdash \neg\sigma$. $\blacksquare$

Generalization lemma. If $\varphi(v_0, ..., v_n) \in Form(\mathscr{L})$, then

$$\Sigma \vdash \varphi \Rightarrow \Sigma \vdash \forall v_0 \ldots \forall v_n\varphi. \quad \blacksquare$$

4. The completeness and model existence theorems and some of their consequences.

Let $\mathscr{L}$ be a first-order language of type (λ, J). We make the following definitions.

1. An extension $\mathscr{L}^*$ of $\mathscr{L}$ is called a *simple* extension of $\mathscr{L}$ if it is obtained by adding just new constant symbols.

2. Let $\Sigma \subseteq Sent(\mathscr{L})$ and let $\mathscr{L}^*$ be a simple extension of $\mathscr{L}$. A set $\Sigma^* \subseteq Sent(\mathscr{L}^*)$ is called an *$\mathscr{L}$-saturated extension* of Σ in $\mathscr{L}^*$ if $\Sigma \subseteq \Sigma^*$ and, for any $\mathscr{L}$-formula φ with at most one free variable x, there is a constant symbol c of $\mathscr{L}^*$ such that $\Sigma^* \vdash \exists x\varphi(x) \to \varphi(c)$.

3. A set $\Sigma \subseteq Sent(\mathscr{L})$ is *saturated* if for any $\mathscr{L}$-formula φ with at most one free variable x, there is a constant c of $\mathscr{L}$ for which

$$\Sigma \vdash \exists x\varphi(x) \to \varphi(c).$$

If Σ is saturated, then clearly:

$$\Sigma \vdash \exists x\varphi(x) \iff \Sigma \vdash \varphi(c) \text{ for some constant } c \text{ of } \mathscr{L}.$$

Notice also that if some set of $\mathscr{L}$-sentences is saturated, then $\mathscr{L}$ contains at least one constant symbol.

Lemma 1. Suppose that $\Sigma \subseteq Sent(\mathscr{L})$ is consistent. Then there is a consistent $\mathscr{L}$-saturated extension Σ^* in a simple extension $\mathscr{L}^*$ of $\mathscr{L}$ for which $\|\mathscr{L}^*\| = \|\mathscr{L}\|$.

Proof. Let F be the set of $\mathscr{L}$-formulas with at most one free variable (which we shall denote by x). For each $\varphi \in F$ introduce a new constant symbol c_φ in such a way that, if φ and ψ are distinct formulas, then c_φ and c_ψ are distinct constants. In this way we obtain a simple extension $\mathscr{L}^*$ of $\mathscr{L}$: clearly $\|\mathscr{L}^*\| = \|\mathscr{L}\|$.

Now define

$$\Sigma^* = \Sigma \cup \{\exists x\varphi(x) \to \varphi(c_\varphi) \colon \varphi \in F\}.$$

Clearly Σ^* is an $\mathscr{L}$-saturated extension of Σ in $\mathscr{L}^*$. It remains to show that Σ^* is consistent.

Suppose, on the contrary, that Σ^* is inconsistent. Then by the consistency lemma there is a finite subset $\{\varphi_1, \ldots, \varphi_n\}$ of F such that, writing c_i for c_{φ_i}, $\Sigma \cup \{\exists x\varphi_i \to \varphi_i(c_i) \colon i = 1,.., n\}$ is inconsistent. It follows from the consistency lemma that

$$(*) \qquad \Sigma \vdash \neg\bigwedge_{i=1}^{n}[\exists x\varphi_i \to \varphi_i(c_i)]$$

Now choose n distinct variables $x_1,\ldots,x_n$ which do not occur in the proof from Σ of the sentence on the right hand side of the turnstile in (*). If in this proof we change c_i at each of its occurrences to x_i for $i = 1,\ldots,n$, we obtain a proof of the formula $\neg\bigwedge_{i=1}^{n}[\exists x\varphi_i \to \varphi_i(x_i)]$ from Σ, whence

$$\Sigma \vdash \neg\bigwedge_{i=1}^{n}[\exists x\varphi_i \to \varphi_i(x_i)].$$

By the generalization lemma,

$$\Sigma \vdash \forall v_1 \ldots \forall v_n \neg\bigwedge_{i=1}^{n}[\exists x\varphi_i \to \varphi_i(x_i)]$$

so that

$$(**) \qquad \Sigma \vdash \neg\exists v_1 \ldots \exists v_n \bigwedge_{i=1}^{n}[\exists x\varphi_i \to \varphi_i(x_i)].$$

Now the x_i have been chosen in such a way that, if $i \neq j$, then x_i does not occur in $\varphi_j(x_i)$. So it follows from the quantifier lemma that the existential quantifiers on the right hand side of the turnstile in (**) may be moved across the conjunctions and implications to yield

$$\Sigma \vdash \neg \bigwedge_{i=1}^{n} [\exists x \varphi_i \rightarrow \exists x_i \varphi_i(x_i)].$$

But since, clearly, $\vdash \exists x \varphi_i \rightarrow \exists x_i \varphi_i(x_i)$ for each i, it follows that Σ is inconsistent, contradicting assumption. Accordingly Σ^* is consistent and the lemma is proved. ■

A set $\Sigma \subseteq Sent(\mathscr{L})$ is said to be *complete* if, for any $\sigma \in Sent(\mathscr{L})$, we have $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$.

Lemma 2. Suppose that $\Sigma \subseteq Sent(\mathscr{L})$ is consistent. Then there is a complete consistent set $\Sigma' \subseteq Sent(\mathscr{L})$ such that $\Sigma \subseteq \Sigma'$.

Proof. The family of consistent sets of sentences of $\mathscr{L}$ containing Σ, ordered by inclusion, is easily seen to be closed under unions of chains, and so by Zorn's lemma has a maximal member Σ'. If $\sigma \in Sent(\mathscr{L})$ and $\Sigma' \nvdash \sigma$, then $\Sigma' \cup \{\neg\sigma\}$ is consistent by the consistency lemma. Since Σ' is maximal consistent, we must have $\Sigma' \cup \{\neg\sigma\} = \Sigma'$, so *a fortiori* $\Sigma' \vdash \neg\sigma$. Thus Σ' is complete and meets the requirements of the lemma. ■

Theorem 1. Suppose that $\Sigma \subseteq Sent(\mathscr{L})$ is consistent. Then there is a simple extension $\mathscr{L}^+$ of $\mathscr{L}$ such that $\|\mathscr{L}^+\| = \|\mathscr{L}\|$ and a complete saturated consistent set $\Sigma^+ \subseteq Sent(\mathscr{L}^+)$ such that $\Sigma \subseteq \Sigma^+$.

Proof. We construct a sequence $\mathscr{L}_0$, $\mathscr{L}_1$,... of simple extensions of $\mathscr{L}$ and a sequence Σ_0, Σ_1,... of consistent sets of sentences as follows. We begin by putting $\mathscr{L}_0 = \mathscr{L}$ and $\Sigma_0 = \Sigma$. Suppose now that the consistent set $\Sigma_n \subseteq Sent(\mathscr{L}_n)$ has been defined. By Lemma 1 there is a simple extension $\mathscr{L}_n^*$ such that $\|\mathscr{L}_n^*\| = \|\mathscr{L}_n\|$ and a consistent $\mathscr{L}_n$-saturated extension Σ_n^* of Σ_n in $\mathscr{L}_n^*$ And by Lemma 2, there is a complete consistent extension $\Sigma_n^{*\prime}$ of Σ_n in $\mathscr{L}_n^*$: clearly $\Sigma_n^{*\prime}$ is $\mathscr{L}_n$-saturated also. We set $\mathscr{L}_{n+1} = \mathscr{L}_n^*$, $\Sigma_{n+1} = \Sigma_n^{*\prime}$. Then Σ_{n+1} is a complete, consistent $\mathscr{L}_n$-saturated extension of Σ_n in $\mathscr{L}_{n+1}$.

Now we define $\mathscr{L}^+$ to be the union of all the languages $\mathscr{L}_n$ and Σ^+ to be the union of all the sets Σ_n. Since $\|\mathscr{L}_n\| = \|\mathscr{L}_0\| = \|\mathscr{L}\|$ for all n, it follows that $\|\mathscr{L}^+\| = \|\mathscr{L}\|$. Also, $\Sigma^+ \subseteq Sent(\mathscr{L}^+)$, $\Sigma \subseteq \Sigma^+$ and Σ^+, as the union of the chain $\Sigma_0 \subseteq \Sigma_1 \subseteq$... of consistent sets, is itself consistent. For if Σ^+ is inconsistent, let Φ be the finite set of formulas of $\mathscr{L}^+$ in a proof $\mathscr{P}$ of a formula of the form $\varphi \wedge \neg\varphi$ from Σ^+. Then $\Phi \subseteq Form(\mathscr{L}_m)$ for some m, and $\Sigma^+ \cap \Phi \subseteq \Sigma_n$ for some n. Writing q for the larger of m, n, $\mathscr{P}$ is then a proof of $\varphi \wedge \neg\varphi$ from Σ_q in $\mathscr{L}_q$, contradicting the consistency of Σ_q.

Moreover, Σ^+ is complete. for, if $\sigma \in Sent(\mathscr{L}^+)$, then $\sigma \in Sent(\mathscr{L}_n)$, for some n, and so, since Σ_n is complete, either $\Sigma_n \vdash \sigma$ or $\Sigma_n \vdash \neg\sigma$ Since $\Sigma_n \subseteq \Sigma^+$, it follows that $\Sigma^+ \vdash \sigma$ or $\Sigma^+ \vdash \neg\sigma$, proving the claim.

Finally, Σ^+ is saturated. For let $\varphi(x)$ be a formula of $\mathscr{L}^+$ with one free variable x. Then $\varphi(x) \in Form(\mathscr{L}_n)$ for some n. Since Σ_{n+1} is an $\mathscr{L}_n$-saturated extension of Σ_n in $\mathscr{L}_{n+1}$, there is a constant symbol c of $\mathscr{L}_{n+1}$ for which the sentence $\exists x\varphi(x) \to \varphi(c)$ is provable from Σ_{n+1}, and hence also, since $\Sigma_{n+1} \subseteq \Sigma^+$, from Σ^+. Therefore the latter is saturated as claimed. ■

Now let Σ be a fixed consistent set of sentences of $\mathscr{L}$. Let C be the set of constant symbols of $\mathscr{L}$: we shall assume that this set is nonempty. We define the relation $\approx$ on C by

$$c \approx d \iff \Sigma \vdash c = d.$$

It is easy to verify, using the equality axioms in $\mathscr{L}$, that $\approx$ is an equivalence relation. For each $c \in C$ write $\tilde{c}$ for the equivalence class of c with respect to $\approx$; thus

$$\tilde{c} = \{d \in C: \Sigma \vdash c = d\}.$$

Let

$$\widetilde{C} = \{\tilde{c} : c \in C\}$$

be the set of all such equivalence classes. Corresponding to each predicate symbol P_i of $\mathscr{L}$ define the $\lambda(i)$- ary relation R_i on $\widetilde{C}$ by

$$R_i(\widetilde{c_1},\ldots,\widetilde{c_{\lambda(i)}}) \iff \Sigma \vdash P_i c_1\ldots c_{\lambda(i)}.$$

We can now frame the

Definition. The *canonical structure* determined by Σ is the $\mathscr{L}$-structure

$$\mathfrak{A}_\Sigma = (\widetilde{C},\ \{R_i:\ i \in I\},\ \{\widetilde{c_j} : j \in J\}).$$

Observe that $\|\mathfrak{A}_\Sigma\| \leq |C|$.

Theorem 2. Suppose that Σ is complete, consistent and saturated. Then $\mathfrak{A}_\Sigma$ is a model of Σ.

Proof. We show that, for any $\mathscr{L}$-sentence σ,

$$(*)\qquad \mathfrak{A}_\Sigma \models \sigma \iff \Sigma \vdash \sigma.$$

That this holds for atomic sentences is an immediate consequence of the definition of $\mathfrak{A}_\Sigma$. We now argue by induction on the degree of complexity of the sentence σ.

Suppose then that $n > 0$ and that (*) holds for all sentences of degree $< n$. Let σ have degree n; then σ is either a conjunction or a negation of sentences of degree $< n$, or an existentialization of a formula of degree $< n$. Verifying (*) in the first two cases is routine (using the completeness of Σ in the negation case) and we omit the details. In the last case, σ is of the form $\exists x\varphi(x)$, where φ has degree $< n$. We then have

$$\begin{aligned}
\mathfrak{A}_\Sigma \models \sigma &\iff \mathfrak{A}_\Sigma \models \exists x\varphi(x) \\
&\iff \mathfrak{A}_\Sigma \models \varphi[\tilde{c}] \text{ for some } c \in C \\
\text{(by constants lemma)}\quad &\iff \mathfrak{A}_\Sigma \models \varphi(c) \text{ for some } c \in C \\
\text{(by (*))}\quad &\iff \Sigma \vdash \varphi(c) \text{ for some } c \in C \\
\text{(since } \Sigma \text{ is saturated)}\quad &\iff \Sigma \vdash \exists x\varphi(x) \\
&\iff \Sigma \vdash \sigma.
\end{aligned}$$

Therefore σ satisfies (*) and the proof is complete. ■

These results have the following important corollaries.

Model Existence Theorem (Gödel-Henkin). Any consistent set Σ of first-order sentences has a model of cardinality at most $\max(\aleph_0, |\Sigma|)$.

Proof. Let $\kappa = \max(\aleph_0, |\Sigma|)$; then $\kappa = \|\mathscr{L}_\Sigma\|$ by the lemma on p. 3. By Theorem 1 we can extend Σ to a complete consistent saturated set of sentences Φ in a simple extension $\mathscr{L}'$ of $\mathscr{L}_\Sigma$ such that $\|\mathscr{L}'\| = \|\mathscr{L}_\Sigma\| = \kappa$. By Theorem 2, the canonical structure $\mathfrak{A}_\Phi$ is a model of Φ and hence also of Σ. The expansion theorem implies that the $\mathscr{L}_\Sigma$-reduction $\mathfrak{A}'$ of $\mathfrak{A}_\Phi$ is a model of Σ, and that any $\mathscr{L}$-expansion $\mathfrak{A}$ of $\mathfrak{A}'$ is likewise. Moreover, if C is the set of constant symbols of $\mathscr{L}'$, then $\|\mathfrak{A}\| = \|\mathfrak{A}_\Phi\| \leq |C| \leq \|\mathscr{L}_\Sigma\| = \kappa$. The proof is complete. ■

Completeness Theorem. If $\Sigma \subseteq$ *Sent*($\mathscr{L}$) and $\sigma \in$ *Sent*($\mathscr{L}$), then

$$\Sigma \vdash \sigma \Rightarrow \Sigma \models \sigma.$$

Proof. If $\Sigma \nvdash \sigma$, then, by the consistency theorem, $\Sigma \cup \{\sigma\}$ is consistent and so, by the model existence theorem, has a model $\mathfrak{A}$. Since $\mathfrak{A}$ is a model of Σ but not of σ, it follows that $\Sigma \nvDash \sigma$. ■

Compactness Theorem. A set of first-order sentences Σ has a model iff every finite subset of Σ has a model.

Proof. One way round is trivial. If, conversely, every finite subset of Σ has a model, then every finite subset of Σ is consistent and so Σ itself is consistent by the consistency lemma. Therefore Σ has a model by the model existence theorem. ■

Invariance Theorem. Provability and consistency are *invariant with respect to language.* That is, if $\Sigma \subseteq$ *Sent*($\mathscr{L}$) and $\sigma \in$ *Sent*($\mathscr{L}$), and $\mathscr{L}^*$ is an extension of $\mathscr{L}$, then

(a) $\Sigma \vdash \sigma$ in $\mathscr{L} \Leftrightarrow \Sigma \vdash \sigma$ in $\mathscr{L}^*$

(b) Σ is consistent in $\mathscr{L} \Leftrightarrow \Sigma$ is consistent in $\mathscr{L}^*$.

Proof. We prove (a); (b) is an immediate consequence. Clearly $\Sigma \vdash \sigma$ in $\mathscr{L} \Leftrightarrow \Sigma \vdash \sigma$ in $\mathscr{L}^*$.Conversely, if $\Sigma \vdash \sigma$ in $\mathscr{L}^*$, then $\Sigma \models \sigma$ by the completeness theorem, that is, every $\mathscr{L}^*$-structure which is a model of Σ is also a model of σ. If $\mathfrak{A}$ is any $\mathscr{L}$-structure which is a model of Σ, it can be expanded to an $\mathscr{L}^*$-structure $\mathfrak{A}^*$ which, by the expansion lemma, is also a model of $\mathscr{L}$. Then $\mathfrak{A}^*$ is a model of σ, and so, applying the expansion lemma again, $\mathfrak{A}$, as the $\mathscr{L}$-reduction of $\mathfrak{A}^*$, is a model of σ. Therefore, by the completeness theorem, $\Sigma \vdash \sigma$ in $\mathscr{L}$. ■

Löwenheim-Skolem Theorem. If a set Σ of first-order sentences has an infinite model, it has a model of any cardinality $\kappa \geq \max(\aleph_0, |\Sigma|)$.

Proof. For simplicity write $\mathscr{L}$ for $\mathscr{L}_\Sigma$. Let $\mathscr{L}^*$ be the simple extension of $\mathscr{L}$ obtained by adding a set $\{d_j: j \in J\}$ of new constant symbols, where $|J| = \kappa$. Let

$$\Sigma^* = \Sigma \cup \{\neg(d_j = d_k): j, k \in J \,\&\, j \neq k\}.$$

If Σ_0 is any finite subset of Σ^*, only finitely many sentences of the form $\neg(d_j = d_k)$ occur in Σ_0; let $d_{j1}, \ldots, d_{jn}$ be a list of all constant symbols occurring in such sentences in Σ_0. If now $\mathfrak{A}$ is an infinite model of Σ (which we may take to be an $\mathscr{L}$-structure), choose n distinct elements $a_1, \ldots, a_n$ of its domain A. Let $\mathfrak{A}^*$ be the $\mathscr{L}^*$-expansion of $\mathfrak{A}$ in which the interpretation of d_{jp} is a_p for $p = 1, \ldots, n$ and that of d_j is an arbitrary element of A for $j \notin \{j_1, \ldots, j_n\}$. Clearly $\mathfrak{A}^*$ is then a model of Σ_0.

It follows that every finite subset of Σ^* has a model. Thus every finite subset of Σ^* is consistent and so Σ^* is itself consistent. Clearly $|\Sigma^*| = \kappa$, so the model existence theorem implies that Σ^* has a model of cardinality $\leq \kappa$. Since the interpretations of the d_j in any model of Σ^* must be distinct, any such model must have cardinality $\geq \kappa$. So Σ^* has a model of cardinality κ; its $\mathscr{L}$-reduction is a model of Σ of cardinality κ. ■

Overspill Theorem. If a set of first-order sentences has arbitrarily large finite models, it has an infinite model.

Proof. For each $n \in \omega$ let σ_n be a sentence (formulable in any first-order language with equality) asserting that there at least n individuals. Given a set Σ of first-order sentences, let $\Sigma^* = \Sigma \cup \{\sigma_n: n \in \omega\}$. If Σ has arbitrarily large finite models, then each finite subset of Σ^* has a model, so by the compactness theorem Σ^* has a model, which must evidently be an infinite model of Σ. ■

5. Relations between structures.

Let $\mathfrak{A} = (A, \{R_i: i \in I\}, \{e_j: j \in J\})$ and $\mathfrak{B} = (B, \{S_i: i \in I\}, \{d_j: j \in J\})$ be structures of the same type (λ, J). We say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \subseteq B$, $e_j = d_j$ for all $j \in J$, and $R_i = S_i \cap A^{\lambda(i)}$ for all $i \in I$. If C is a nonempty subset of B containing all the designated elements of $\mathfrak{B}$, we define the substructure $\mathfrak{B}|C$ of $\mathfrak{B}$ by

$$\mathfrak{B}|C = (C, \{S_i \cap C^{\lambda(i)}: i \in I\}, \{d_j: j \in J\}).$$

An *embedding* of a structure $\mathfrak{A}$ into a structure $\mathfrak{B}$ is an injective map $f: A \to B$ such that $f(e_j) = d_j$ for all $j \in J$, and for all $i \in I$ and $a_1, \ldots, a_{\lambda(i)} \in A$, we have

$$R_i(a_1, \ldots, a_{\lambda(i)}) \Leftrightarrow S_i(fa_1, \ldots, fa_{\lambda(i)}).$$

If there exists an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we say that $\mathfrak{A}$ is *embeddable* into $\mathfrak{B}$ and write $\mathfrak{A} \sqsubseteq \mathfrak{B}$. If f is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we write $f[\mathfrak{A}]$ for the structure $\mathfrak{B}|f[A]$. A surjective embedding is called an *isomorphism.* If there exists an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, they are said to be *isomorphic* and we write $\mathfrak{A} \cong \mathfrak{B}$.

Let $\mathscr{L}$ be the first-order language of type (λ, J). We say that the $\mathscr{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent,* and write $\mathfrak{A} \equiv \mathfrak{B}$, if $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{B} \models \sigma$ for any $\mathscr{L}$–sentence σ. It is easily shown that isomorphic structures are elementarily equivalent, but the Löwenheim-Skolem theorem implies that the converse fails.

The $\mathscr{L}$-structure $\mathfrak{A}$ is said to be an *elementary substructure* of the $\mathscr{L}$-structure $\mathfrak{B}$, and $\mathfrak{B}$ an *elementary extension* of $\mathfrak{A}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and, for any $\mathscr{L}$–formula $\varphi(v_0, \ldots, v_n)$ and any $a_0, \ldots, a_n \in A$, we have

$$\mathfrak{A} \vDash \varphi[a_0, ..., a_n] \iff \mathfrak{B} \vDash \varphi[a_0, ..., a_n].$$

In this situation we write $\mathfrak{A} \prec \mathfrak{B}$. Evidently $\mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$, but the converse is easily seen to be false.

An embedding f of $\mathfrak{A}$ into $\mathfrak{B}$ is called an *elementary embedding* if for any $\mathscr{L}$-formula $\varphi(v_0, ..., v_n)$ and any $a_0, ..., a_n \in A$ we have

$$\mathfrak{A} \vDash \varphi[a_0, ..., a_n] \iff \mathfrak{B} \vDash [fa_0, ... ,fa_n].$$

In this situation we write $f\colon \mathfrak{A} \prec \mathfrak{B}$. If such an f exists, we write $\mathfrak{A} \precsim \mathfrak{B}$. Clearly $\mathfrak{A} \precsim \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$. It is also easily shown that *any isomorphism is an elementary embedding.*

Tarski-Vaught Lemma. If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}$-structures, then $\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and, for any $\mathscr{L}$-formula $\varphi(v_0, ..., v_n)$ and any $a_0, ..., a_{n-1} \in A$,

(*) if $\mathfrak{B} \vDash \exists v_n\, \varphi[a_0, ..., a_{n-1}]$, then, for some $a \in A$, $\mathfrak{A} \vDash \varphi[a_0, ..., a_{n-1}, a]$.

Proof. One direction is trivial. Conversely, suppose that (*) holds. We prove by induction on the degree of φ that, for any n, any $\mathscr{L}$–formula $\varphi(v_0, ..., v_n)$ and any $a_0, ..., a_n \in A$,

(**) $$\mathfrak{A} \vDash \varphi[a_0, ..., a_n] \iff \mathfrak{B} \vDash \varphi[a_0, ..., a_n].$$

That (**) holds for atomic formulas is obvious, as are the induction steps for $\neg$ and $\wedge$. It remains to show that, if it holds for φ, it also holds for $\exists v_k\, \varphi$. Without loss of generality we may assume that n is greater than the index of every variable (free or bound) occurring in φ, and then, by making a suitable change of variable in φ (i.e., by substituting v_n for v_k), that $k = n$.

If $\mathfrak{A} \vDash \exists v_n \varphi[a_0, ..., a_{n-1}]$, then $\mathfrak{A} \vDash \varphi[a_0, ..., a_{n-1}, a]$ for some $a \in A$, and it follows from (**) for φ that $\mathfrak{B} \vDash \varphi[a_0, ..., a_{n-1}, a]$, whence $\mathfrak{B} \vDash \exists v_n \varphi[a_0, ..., a_{n-1}]$. Conversely, if $\mathfrak{B} \vDash \exists v_n \varphi[a_0, ..., a_{n-1}]$, then, by (*), $\mathfrak{B} \vDash \varphi[a_0, ..., a_{n-1}, a]$ for some $a \in A$, whence $\mathfrak{A} \vDash \varphi[a_0, ..., a_{n-1}, a]$ by (**), so that $\mathfrak{A} \vDash \exists v_n \varphi[a_0, ..., a_{n-1}]$. This completes the induction step and the proof. ■

Corollary. Write $\mathbf{Q}$ and $\mathbb{R}$ for the sets of rational and real numbers. Then

$$(\mathbf{Q}, \leq) \prec (\mathbb{R}, \leq).$$

Proof. We show that the Tarski-Vaught lemma applies. Suppose that, for a formula $\varphi(v_0, ..., v_n)$ of the appropriate language, and $a_0 < ... < a_{n-1} \in \mathbf{Q}$, we have $(\mathbb{R}, \leq) \vDash \exists v_n \varphi[a_0, ..., a_{n-1}]$. Then there is $b \in \mathbb{R}$ such that $(\mathbb{R}, \leq) \vDash \varphi[a_0, ..., a_{n-1}, b]$. Say $a_i < b < a_{i+1}$ (the cases $b <$ or $>$ all a_i being similar). Choose a to be any rational such that $a_i < a < a_{i+1}$. It is easy to construct an isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(a_j) = a_j$ for $0 \leq j \leq n-1$ and $f(b) = a$. This f is also an elementary embedding. Hence $(\mathbb{R}, \leq) \vDash \varphi[fa_0, ...,fa_{n-1}, b]$, i.e. $(\mathbb{R}, \leq) \vDash \varphi[a_0, ..., a_{n-1}, a]$. Since $a \in \mathbf{Q}$, the Tarski-Vaught lemma applies to yield the required conclusion. ■

Given a set X, let $\mathscr{L}_X$ be the simple extension of $\mathscr{L}$ obtained by adding a set $\{c_x: x \in X\}$ of distinct new constant symbols indexed by X. If $\mathfrak{A}$ is an $\mathscr{L}$-structure and X is a subset of its domain A, we write $(\mathfrak{A}, X)$ for the $\mathscr{L}_X$-expansion of $\mathfrak{A}$ in which the interpretation of each c_x is x. If f is a mapping of X into the domain B of an $\mathscr{L}$-structure $\mathfrak{B}$, we write $(\mathfrak{B}, f[X])$ for the $\mathscr{L}_X$-expansion of $\mathfrak{B}$ in which the interpretation of each c_x is $f(x)$.

The *diagram* of $\mathfrak{A}$, $\Delta(\mathfrak{A})$, is the set of atomic and negated atomic sentences that hold in $(\mathfrak{A}, A)$. The *complete diagram* of $\mathfrak{A}$, $\Gamma(\mathfrak{A})$, is the set of all sentences of $\mathscr{L}_A$ that hold in $(\mathfrak{A}, A)$. The proof of the following lemma is then straightforward.

Diagram lemma. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathscr{L}$-structures. Then:

(i) $\mathfrak{A} \sqsubseteq \mathfrak{B}$ iff $\mathfrak{B}$ can be expanded to a model of $\Delta(\mathfrak{A})$;

(ii) $\mathfrak{A} \precsim \mathfrak{B}$ iff $\mathfrak{B}$ can be expanded to a model of $\Gamma(\mathfrak{A})$;

(iii) if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$ iff $(\mathfrak{B}, A) \models \Gamma(\mathfrak{A})$;

(iv) an embedding f of $\mathfrak{A}$ into $\mathfrak{B}$ is an elementary embedding iff $(\mathfrak{A}, A) \equiv (\mathfrak{B}, f[A])$. ■

We now show that infinite structures have elementary substructures and extensions of most cardinalities.

Theorem. Let $\mathfrak{A}$ be an infinite $\mathscr{L}$-structure.

(i) If $X \subseteq A$, then for any cardinal satisfying $\max(|X|, \|\mathscr{L}\|) \leq \kappa \leq |A|$, there is an elementary substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $|B| = \kappa$ and $X \subseteq B$.

(ii) $\mathfrak{A}$ has an elementary extension of any cardinality $\geq \max(|X|, \|\mathscr{L}\|)$.

Proof. (i) Let $<$ be some fixed well-ordering of A. We define a sequence $B_0, B_1, \ldots$ of subsets of A recursively as follows. Choose B_0 to be any subset of A such that $|B_0| = \kappa$ and $X \subseteq B_0$. If B_n has been defined, put

$B_{n+1} = \{b$: for some $\mathscr{L}$-formula $\varphi(v_0, \ldots, v_m)$ and some $b_0, \ldots, b_{m-1} \in B_n$, b is the $<$-least element of A such that $\mathfrak{A} \models \varphi[b_0, \ldots, b_{m-1}, b]\}$.

It is easy to check that $B_n \subseteq B_{n+1}$ and that $|B_{n+1}| = \kappa$. Now define B to be the union of the B_n and $\mathfrak{B} = \mathfrak{A}|B$. Then $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ of cardinality κ and it is easy to apply the Tarski-Vaught lemma to conclude that $\mathfrak{B} \prec \mathfrak{A}$.

(ii) Let Γ be the complete diagram of $\mathfrak{A}$**.** Then $|\Gamma| = \max(|X|, \|\mathscr{L}\|)$. Since Γ is evidently consistent, the model existence theorem implies that it has a model of any cardinality $\kappa \geq |\Gamma| = \max(|A|, \|\mathscr{L}\|)$. The result now follows from the diagram lemma. ■

6. Ultraproducts

A *filter* over a set I is a family $\mathscr{F}$ of subsets of I such that (i) $X, Y \in \mathscr{F} \Leftrightarrow X \cap Y \in \mathscr{F}$, (ii) $\varnothing \notin \mathscr{F}$. It follows immediately from (i) that any filter $\mathscr{F}$ over I satisfies; $X \in \mathscr{F}$ and $X \subseteq Y \in \mathscr{F} \Rightarrow Y \in \mathscr{F}$. An *ultrafilter* over I is a filter $\mathscr{U}$ over I satisfying the condition: for any $X \in \mathscr{U}$, either $X \in \mathscr{U}$ or $I - X \in \mathscr{U}$. In particular, for any $i \in I$, $\mathscr{U}_i = \{X \subseteq I : i \in X\}$ is an ultrafilter over I called the *principal* ultrafilter generated by i. It is easily shown that an ultrafilter is precisely a filter that is maximal in the sense that it is included in no filter apart from itself. A straightforward application of Zorn's Lemma shows that a family $\mathscr{A}$ of subsets of I is included in an ultrafilter over I if and only if it has the *finite intersection property:* that is, for any finite subfamily $\mathscr{B}$ of $\mathscr{A}$ we have $\bigcap\mathscr{B} \neq \varnothing$.

For ease of exposition we confine our attention throughout this section to structures consisting of a nonempty set and a single binary relation on that set. The appropriate language $\mathscr{L}$ for such structures thus has a single predicate symbol of degree 2, say P_0. The type of these structures,a nd of $\mathscr{L}$, is then $((0, 2), \varnothing)$. It should be clear that everything we do can be extended to arbitrary structures merely by complicating the notation.

Now let I be some arbitrary fixed index set, and for each $i \in I$ let $\mathfrak{A}_i = (A_i, R_i)$ be an $\mathscr{L}$-structure. Let ΠA_i be the Cartesian product of the sets A_i: we use letters f, g, h, f', g', h' to denote elements of ΠA_i.

Given a family $\mathscr{F}$ of subsets of I, we define the relation $\sim_{\mathscr{F}}$ on ΠA_i by

$$f \sim_{\mathscr{F}} g \Leftrightarrow \{i \in I : f(i) = g(i)\} \in \mathscr{F}.$$

It is easily shown that, if $\mathscr{F}$ is a filter over I, then $\sim_{\mathscr{F}}$ is an equivalence relation on ΠA_i. From here on we shall suppose that $\mathscr{F}$ *is* a filter over I. For each $f \in \Pi A_i$ we write $f/\mathscr{F}$ for the $\sim_{\mathscr{F}}$-equivalence class of f, and we define

$$\Pi A_i/\mathscr{F} = \{f/\mathscr{F} : f \in \Pi A_i\}.$$

We define the relation R on ΠA_i by:

$$(f, g) \in R \Leftrightarrow \{i \in I : (f(i), g(i)) \in R_i\} \in \mathscr{F}.$$

It is not difficult to show that R is compatible with $\sim_{\mathscr{F}}$ in the sense that, if $f \sim_{\mathscr{F}} f'$ and $g \sim_{\mathscr{F}} g'$, then $fRg \Rightarrow f'Rg'$. That being the case, the relation R on ΠA_i induces the relation R_F on $\Pi A_i/\mathscr{F}$ given by

$$(f/\mathscr{F}, g/\mathscr{F}) \in R_F \Leftrightarrow fRg.$$

The $\mathscr{L}$-structure $\Pi\mathfrak{A}_i/\mathscr{F} = (\Pi A_i/\mathscr{F}, R_F)$ is called the *reduced product* of the family $\{\mathfrak{A}_i : i \in I\}$ over the filter $\mathscr{F}$: if $\mathscr{F}$ is an ultrafilter, the reduced product over $\mathscr{F}$ is called an *ultraproduct.* If, for each $i \in I$, $\mathfrak{A}_i$ is a fixed structure $\mathfrak{A}$, the reduced product is denoted by $\mathfrak{A}^I/\mathscr{F}$ and is called the *reduced power* of $\mathfrak{A}$ over $\mathscr{F}$. When $\mathscr{F}$ is an ultrafilter the reduced power is called an *ultrapower.*

Observe that if $\mathscr{F}$ is the filter $\{I\}$, the reduced power $\Pi\mathfrak{A}_i/\mathscr{F}$ is isomorphic to $(\Pi A_i, R)$, and that, for $k \in I$, the ultraproduct $\Pi\mathfrak{A}_i/\mathscr{U}_k$ is isomorphic to $\mathfrak{A}_k$.

If $\boldsymbol{f} = (f_0, f_1,\ldots)$ is a sequence of elements of ΠA_i, that is, if $\boldsymbol{f} \in (\Pi A_i)^\omega$, we write $\boldsymbol{f}(i)$ for the sequence $(f_0(i), f_1(i),\ldots) \in A_i^\omega$ and, if $\mathscr{U}$ is an ultrafilter over I, $\boldsymbol{f}/\mathscr{U}$ for the sequence

$(f_0/\mathscr{U}, f_1/\mathscr{U}, \ldots) \in (\Pi A_i/\mathscr{U})^\omega$.

We now prove the fundamental theorem on ultraproducts, viz.,

Łoś's Theorem. If $\mathscr{U}$ is an ultrafilter over I, φ a formula of $\mathscr{L}$ and $\boldsymbol{f}$ a sequence of elements of ΠA_i, then

(*) $$\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{\boldsymbol{f}/\mathscr{U}} \varphi \iff \{i \in I: \mathfrak{A}_i \vDash_{\boldsymbol{f}(i)} \varphi\} \in \mathscr{U}.$$

Proof. The proof goes by induction on the complexity of φ. That (*) holds for atomic φ is a straightforward consequence of the definitions of $\sim_{\mathscr{F}}$ and R_F. The induction steps for $\wedge$ and $\neg$ follow easily from the defining properties of ultrafilters. Now suppose that (*) holds for φ (and arbitrary $\boldsymbol{f}$); we show that it holds for $\exists v_n\varphi$.

Define

$$D = \{i \in I: \mathfrak{A}_i \vDash_{\boldsymbol{f}(i)} \exists v_n\varphi\}.$$

We have to show that

$$\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{\boldsymbol{f}/\mathscr{U}} \exists v_n\varphi \iff D \in \mathscr{U}.$$

Suppose that $\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{\boldsymbol{f}/\mathscr{U}} \exists v_n\varphi$. Then there is some $b \in \Pi A_i$ for which $\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{[n|b]\boldsymbol{f}/\mathscr{U}} \varphi$. Let $E = \{i \in I: \mathfrak{A}_i \vDash_{([n|b]\boldsymbol{f})(i)} \varphi\}$. Then by the induction hypothesis $E \in F$. And since $([n|b]\boldsymbol{f})(i) = [n|b(i)]\boldsymbol{f}(i)$, it follows that $E \subseteq D$, and so because $\mathscr{U}$ is a filter, $D \in \mathscr{U}$.

Conversely suppose that $D \in \mathscr{U}$. If $i \in D$, then there is some $b_i \in A_i$ such that $\mathfrak{A}_i \vDash_{[n/b_i]\boldsymbol{f}(i)} \varphi$. By the axiom of choice there is $c \in \Pi A_i$ for which $c(i) = b_i$ for every $i \in D$, and is an arbitrary element of A_i otherwise. Defining

$$C = \{i \in I: \mathfrak{A}_i \vDash_{([n|c]\boldsymbol{f})(i)} \varphi\},$$

we have $D \subseteq C$ so that $C \in \mathscr{U}$. It now follows from the induction hypothesis that

$$\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{([n|c]\boldsymbol{f})/\mathscr{U}} \varphi,$$

i.e., since $([n|c]\boldsymbol{f})/\mathscr{U} = [n|c/\mathscr{U}]\,\boldsymbol{f}/\mathscr{U}$,

$$\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{[n|c/\mathscr{U}]\,\boldsymbol{f}/\mathscr{U}} \varphi.$$

Therefore

$$\Pi\mathfrak{A}_i/\mathscr{U} \vDash_{\boldsymbol{f}/\mathscr{U}} \exists v_n\varphi,$$

completing the proof of the theorem. ■

As an immediate consequence we have the

Corollary. For any $\mathscr{L}$– sentence σ we have

$$\Pi\mathfrak{A}_i/\mathscr{U} \vDash \sigma \iff \{i \in I: \mathfrak{A}_i \vDash \sigma\} \in \mathscr{U}. \quad ■$$

Let $\mathfrak{A}$ be a structure and let $\mathscr{U}$ be an ultrafilter on the set I. For each $a \in A$ let $\hat{a} \in A^I$ be the function given by $\hat{a}(i) = a$ for all $i \in I$. The *canonical embedding* of $\mathfrak{A}$ into $\mathfrak{A}^I/\mathscr{U}$ is the map $d: A \to A^I/\mathscr{U}$ defined by $d(a) = \hat{a}/\mathscr{U}$. It is a straightforward consequence of Łoś's theorem that d is an elementary embedding.

Łoś's theorem may also be used to provide a simple direct proof of the compactness theorem, avoiding the use of the completeness theorem. To wit, suppose that each finite subset Δ of a given set Σ of sentences has a model $\mathfrak{A}_\Delta$; for simplicity write I for the family of all finite subsets of Σ. For each $\Delta \in I$ let $\widetilde{\Delta} = \{\Phi \in I : \Delta \subseteq \Phi\}$. For any members $\Delta_1, \ldots, \Delta_n$ of I, we have

$$\Delta_1 \cup \ldots \cup \Delta_n \in \widetilde{\Delta_1} \cap \ldots \cap \widetilde{\Delta_n},$$

and so the collection $\{\widetilde{\Delta} : \Delta \in I\}$ has the finite intersection property. It can therefore be extended to an ultrafilter $\mathscr{U}$ over I. The ultraproduct $\prod_{\Delta \in I} \mathfrak{A}_\Delta / \mathscr{U}$ is then a model of Σ. For if $\sigma \in \Sigma$, then $\{\sigma\} \in \Delta$, and $\mathfrak{A}_{\{\sigma\}} \models \sigma$; moreover, $\mathfrak{A}_\Delta \models \sigma$ whenever $\sigma \in \Delta$. Hence

$$\widetilde{\{\sigma\}} = \{\Delta \in I : \sigma \in \Delta\} \subseteq \{\Delta \in I : \mathfrak{A}_\Delta \models \sigma\}.$$

Since $\widetilde{\{\sigma\}} \in \mathscr{U}$, $\{\Delta \in I : \mathfrak{A}_\Delta \models \sigma\} \in \mathscr{U}$ and therefore, by Łoś's theorem, $\prod_{\Delta \in I} \mathfrak{A}_\Delta / \mathscr{U} \models \sigma$. The proof is complete.

7. Completeness and categoricity

For simplicity, throughout this section we let $\mathscr{L}$ be a *countable* first-order language. By a *theory* in $\mathscr{L}$ we shall mean a set Σ of $\mathscr{L}$-sentences which is closed under provability, i.e such that, for each $\mathscr{L}$–sentence σ, if $\Sigma \vdash \sigma$, then $\sigma \in \Sigma$. A subset Γ of a theory Σ is called a *set of postulates* for Σ if $\Gamma \vdash \sigma$ for every $\sigma \in \Sigma$. Clearly each set Γ of $\mathscr{L}$-sentences is a set of postulates for a unique theory Σ, namely $\Sigma = \{\sigma \in Sent(\mathscr{L}) : \Gamma \vdash \sigma\}$. For each $\mathscr{L}$-structure $\mathfrak{A}$ let $\Theta(\mathfrak{A})$, the *theory* of $\mathfrak{A}$, be the set of all $\mathscr{L}$-sentences holding in $\mathfrak{A}$. Clearly $\Theta(\mathfrak{A})$ is a complete theory.

The following lemma is a straightforward consequence of the completeness theorem.

Lemma. The following conditions on a consistent theory Σ in $\mathscr{L}$ are equivalent:
(i) Σ is complete;
(ii) any pair of models of Σ are elementarily equivalent;
(iii) $\Sigma = \Theta(\mathfrak{A})$ for some $\mathscr{L}$-structure $\mathfrak{A}$. ■

Let κ be an infinite cardinal. A theory Σ is said to be κ-*categorical* if any pair of models of Σ of cardinality κ are isomorphic.

Examples. (i) Let $\mathscr{L}$ have no extralogical symbols and let Σ be the set of all $\mathscr{L}$-sentences which hold in every $\mathscr{L}$-structure. Then Σ is κ-*categorical for every infinite* κ.

(ii) Let $\mathscr{L}$ have just one unary predicate symbol P and let Σ be the set of $\mathscr{L}$-sentences which hold in every $\mathscr{L}$-structure. Then Σ is *not* κ-*categorical for any infinite* κ.

(iii) Let $\mathscr{L}$ be as in (ii) and for each matural number m let σ_m be the first-order sentence which asserts that there are at least m individuals having the property P and at least m individuals not having P. Let Σ be the theory with the set of all σ_m as postulates. Then Σ is $\aleph_0$-*categorical* *but not* κ*-categorical for any* $\kappa > \aleph_0$.

(iv) Let $\mathscr{L}$ be the language whose sole extralogical symbols are countably many constants $c_0, c_1, \ldots$ and let Σ be the theory with postulates $\{\neg(c_m = c_n): m \neq n\}$. Then Σ is κ*-categorical for every* $\kappa > \aleph_0$ *but not* $\aleph_0$ *-categorical.*

One of the deepest results in model theory is *Morley's theorem* (whose proof is too difficult to be included here) which asserts that the four possibilities above are *exhaustive*, that is, if a theory in a countable language is κ-categorical for *some* $\kappa > \aleph_0$, it is κ-categorical for *all* $\kappa > \aleph_0$.

The next result provides a simple, but useful, sufficient condition for completeness.

Theorem. (Vaught's test.) Let Σ be a consistent theory with no finite models and which is κ-categorical for some infinite κ. Then Σ is complete.

Proof. If Σ is not complete, then there is a sentence σ such that neither σ nor $\neg\sigma$ are provable from Σ. So both $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$ are consistent and hence have models, which must be infinite since Σ was assumed to have no finite models. Therefore, by Löwenheim-Skolem, both $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$ have models of cardinality κ. Since σ holds in one of these models but not in the other, Σ is not κ-categorical. ■

This theorem may be applied to establish the completeness of various theories.

UDO — the theory of *unbounded dense linear orderings* — is formulated in a language with just one binary predicate symbol R and has the following postulates (where we write $x \neq y$ for $\neg\,(x = y)$):

(i) $\forall x Rxx \wedge \forall x\forall y[Rxy \wedge Ryx \to x = y] \wedge \forall x\forall y\forall z[Rxy \wedge Ryz \to Rxz]$
$\wedge\ \forall x\forall y[Rxy \vee Ryx]$

(ii) $\forall x\forall y[Rxy \wedge x \neq y \to \exists x[x \neq z \wedge y \wedge z \wedge Rxz \wedge Rzy]]$

(iii) $\forall x\exists y\exists z[x \neq y \wedge x \neq z \wedge Ryx \wedge Rxz]$

Postulate (i) asserts that R is a linear ordering, (ii) that it is dense, and (iii) that it is unbounded below and above. Natural examples of models of **UDO** are $(\mathbf{Q}, \leq)$ and $(\mathbb{R}, \leq)$.

Theorem. **UDO** is $\aleph_0$-categorical and so, by Vaught's test, complete.

Proof. Let $(A, \leq)$ and $(B, \leq)$ be denumerable models of **UDO**. Thus each is an unbounded dense linearly ordered set. Let $A = \{a_n: n \in \omega\}$ and $B = \{b_n: n \in \omega\}$. We define two new sequences $\{a_n^*: n \in \omega\}$ and $\{b_n^*: n \in \omega\}$ as follows. First, put $a_0^* = a_0$ and $b_0^* = b_0$. Now suppose $k > 0$; we consider two cases.

(i) $k = 2m$ is even. In this case we put $a_k^* = a_m$. If, for some $j < k$, $a_k^* = a_j^*$, we put $b_k^* = b_j^*$. Otherwise we let b_k^* be some element of B bearing the same order relations to $b_0^*, \ldots, b_{k-1}^*$ as does a_k^* to $a_0^*, \ldots, a_{k-1}^*$; that is, for each $j < k$, if $a_k^* >$ or $< a_j^*$, then $b_k^* >$ or $< b_j^*$. Since $(B, \leq)$ is a dense unbounded linearly ordered set, it is clear that such an element can always be found.

(ii) $k = 2m + 1$ is odd. In this case we put $b_k^* = b_m$. If $b_k^* = b_j$ for some $j < k$, put $a_k^* = a_j^*$. Otherwise we choose a_k^* to be some element of A bearing the same order relations to $a_0^*, \ldots, a_{k-1}^*$ as does b_k^* to $b_0^*, \ldots, b_{k-1}^*$. Again such an element can always be found.

This completes our recursive definition. We now define $h: A \to B$ by putting $h(a_n^*) = b_n^*$ for each $n \in \omega$. Clearly h is an isomorphism between $(A, \leq)$ and $(B, \leq)$. ■

The theory we consider next is most naturally formulated in a language with *operation symbols*: all our previous results extend naturally to theories in such languages.

The *language* $\mathscr{F}$ *for fields* is a first-order language with constant symbols 0, 1 and binary operation symbols $+, \cdot$. The *theory* **FT** *of fields* has the following postulates (where we write xy for $x \cdot y$):

$$\forall x \forall y[(x + y) + z = x + (y + z)]$$
$$\forall x\, [x + 0 = x]$$
$$\forall x \forall y[x + y = y + x]$$
$$\forall x \exists y[x + y = 0]$$
$$\forall x \forall y \forall z[(xy)z = x(yz)]$$
$$\forall x[1x = x]$$
$$\forall x \forall y[xy = yx]$$
$$\forall x \forall y \forall x[(y + z) = xy + xz]$$
$$\neg(0 = 1).$$

For $p \in \omega$, write $p1$ for $1 + 1 + \ldots + 1$ with p summands. If to the postulates of **FT** we add the infinite set of sentences

$$\{\neg(p1 = 0): p \in \omega\},$$

we get the *theory* $\mathbf{FT_0}$ *of fields of characteristic 0*. (Natural examples are the fields of rationals and reals.)

We now write x^n for the expression $x \cdot (x \cdot (\ldots \cdot (x \cdot x)\ldots)$ with n factors. The infinite list of sentences, for $n \geq 1$,

$$\forall x_0 \ldots \forall x_n[\neg(x_n = 0) \to \exists y(x_n y^n + x_{n-1} y^{n-1} + \ldots + x_1 y + x_0 = 0)]$$

when added to the postulates of $\mathbf{FT_0}$, yields the *theory* $\mathbf{ACF_0}$ *of algebraically closed fields of characteristic* 0. Each new postulate asserts that all polynomials of a given degree n has a zero.

We observe that $\mathbf{ACF_0}$ is *not* $\aleph_0$-categorical. For the field **F** of algebraic numbers and the algebraic closure of the field $\mathbf{F}[\pi]$ obtained by adjoining the transcendental π to **F** are countable nonisomorphic models of $\mathbf{ACF_0}$. On the other hand, a classical theorem of Steinitz asserts that $\mathbf{ACF_0}$ is κ -categorical for any *uncountable* κ, so we conclude from Vaught's test that $\mathbf{ACF_0}$ *is complete* Since the field $\mathbb{C}$ of complex numbers is a model of $\mathbf{ACF_0}$, it follows that $\mathbf{ACF_0}$ *is a set of postulates for the theory of* $\mathbb{C}$.

8. The elementary chain theorem and some of its consequences.

Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots$ be a chain of $\mathscr{L}$-structures: in particular the $\mathfrak{A}_i$ all have the same designated elements. The *union* of the chain is the structure $\mathfrak{A} = \bigcup_{n \in \omega} \mathfrak{A}_n$ defined as follows. The domain of $\mathfrak{A}$ is the set $A = \bigcup_{n\in\omega} A_n$. For $i \in I$, the i^{th} relation R_i of $\mathfrak{A}$ is the union of the corresponding i^{th} relations of the A_n. The designated elements of $\mathfrak{A}$ are the designated elements of the $\mathfrak{A}_n$. Clearly each $\mathfrak{A}_n$ is a substructure of $\mathfrak{A}$.

A chain of structures $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq ...$ in which each $\mathfrak{A}_n$ is an elementary substructure of $\mathfrak{A}_{n+1}$ is called an *elementary chain*. In this case we write $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec ...$.

Elementary Chain Theorem. Each member of an elementary chain of structures is an elementary substructure of the union of the chain.

Proof. Let $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec ...$ be an elementary chain, and let $\mathfrak{A}$ be its union. We prove the following assertion by induction on the degree of a formula: for any $\mathscr{L}$–formula $\varphi(v_0, ..., v_n)$, any $n \in \omega$ and any $a_0, ..., a_m \in A_n$,

(*) $$\mathfrak{A}_n \vDash \varphi[a_0, ..., a_m] \Leftrightarrow \mathfrak{A} \vDash \varphi[a_0, ..., a_m].$$

The proof is routine for atomic formulas, and the induction steps for $\neg$ and $\wedge$ are easy. Now suppose that φ is existential; without loss of generality we may assume that φ is $\exists v_n\psi$, and that ψ satisfies (*).

If $a_0, ..., a_{m-1} \in A_n$ and $\mathfrak{A}_n \vDash \varphi[a_0, ..., a_{m-1}]$, then for some $a \in A_n$ we have $\mathfrak{A}_n \vDash \psi[a_0, ..., a_{m-1}, a]$. So by (*) $\mathfrak{A} \vDash \psi[a_0, ..., a_{m-1}, a]$ whence $\mathfrak{A} \vDash \varphi[a_0, ..., a_{m-1}]$.

Conversely, suppose that $\mathfrak{A} \vDash \varphi[a_0, ..., a_{m-1}]$. Then $\mathfrak{A} \vDash \psi[a_0, ..., a_{m-1}, a]$ for some $a \in A$. For some k, $a \in A_k$. Let ℓ be the larger of k and n. Then $a_0, ..., a_{m-1}, a \in A_\ell$ and so, by (*), $\mathfrak{A}_\ell \vDash \psi[a_0, ..., a_{m-1}, a]$, whence $\mathfrak{A}_\ell \vDash \varphi[a_0, ..., a_{m-1}]$. But $n \leq \ell$ and so, since $\mathfrak{A}_n \prec \mathfrak{A}_\ell$, we conclude that $\mathfrak{A}_n \vDash \varphi[a_0, ..., a_{m-1}]$. ∎

We use this in the proof of the

Joint Consistency Theorem. Let Σ and Π be theories in $\mathscr{L}$, and let $\mathscr{E}$ be the language whose extralogical symbols are those common to $\mathscr{L}_\Sigma$ and $\mathscr{L}_\Pi$. Then the following are equivalent:

(i) $\Sigma \cup \Pi$ is consistent.;

(ii) for no $\mathscr{E}$–sentence σ do we have $\Sigma \vdash \sigma$ and $\Pi \vdash \neg\sigma$;

(iii) for some complete (consistent) theory Δ in $\mathscr{E}$, both $\Sigma \cup \Delta$ and $\Pi \cup \Delta$ are consistent;

(iv) there is an $\mathscr{E}$-structure which can be expanded both to a model of Σ and to a model of Π.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Assume (ii) and let $\Sigma^* = \{\sigma \in Sent(\mathscr{E}): \Sigma \vdash \sigma\}$. It follows easily from (ii) that $\Pi \cup \Sigma^*$ is consistent and so has a model $\mathfrak{A}$. Let Δ be the theory of the $\mathscr{E}$-structure $\mathfrak{A} | \mathscr{E}$. Since $\mathfrak{A} \vDash \Pi \cup \Delta$, $\Pi \cup \Delta$ is consistent. If $\Sigma \cup \Delta$ is inconsistent, there is $\sigma \in \Delta$ such that $\Sigma \vdash \neg\sigma$, i.e. $\neg\sigma \in \Sigma^*$. But then $\mathfrak{A} \vDash \neg\sigma$, whence $\neg\sigma \in \Delta$, a contradiction. Hence $\Sigma \cup \Delta$ is consistent.

(iii) $\Rightarrow$ (iv). Assume (iii), and let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be models of $\Sigma \cup \Delta$ and $\Pi \cup \Delta$, respectively. Then since $\mathfrak{A}_0 | \mathscr{E}$ and $\mathfrak{B}_0 | \mathscr{E}$ are both models of the complete theory Δ, they are elementarily equivalent. It follows easily from this that the union Γ of the complete diagram Γ^* of $\mathfrak{A}_0 | \mathscr{E}$ with the complete diagram Γ^{**} of $\mathfrak{B}_0$ is consistent. (Observe that each finite subset of Γ^* is interpretable in $\mathfrak{B}_0$.) Let $\mathfrak{B}^*$ be a model of Γ and let $\mathfrak{B}_1$ be its $\mathscr{L}$-reduction. Then since $\mathfrak{B}^*$ is a model of both Γ^* and Γ^{**} it follows from the diagram lemma that $\mathfrak{A}_0 \,|\, \mathscr{E} \precsim \mathfrak{B}_1 \,|\, \mathscr{E}$ and $\mathfrak{B}_0 \precsim \mathfrak{B}_1$. Identifying $\mathfrak{B}_0$ with its image in $\mathfrak{B}_1$ makes the former an elementary substructure of the latter. Let f_1 be an elementary embedding of $\mathfrak{A}_0 \,|\, \mathscr{E}$ into $\mathfrak{B}_1 \,|\, \mathscr{E}$.

Passing to the extended language $\mathscr{E}_{A_0}$, the diagram lemma implies that the structures $(\mathfrak{A}_0 \,|\, \mathscr{E},\ A_0) = (\mathfrak{A}_0,\ A_0) \,|\, \mathscr{E}_{A_0}$ and $(\mathfrak{B}_1 \,|\, \mathscr{E},\ f_1[A_0]) = (\mathfrak{B}_1,\ f_1[A_0])$ are elementarily equivalent. Repeating the above construction in the other direction, this time with the $\mathscr{L}_{A_0}$-structures $(\mathfrak{A}_0, A_0)$ and $(\mathfrak{B}_1, f_1[A_0])$ in place of $\mathfrak{A}_0$, $\mathfrak{B}_0$, respectively, we obtain an elementary extension $\mathfrak{A}_1$ of $\mathfrak{A}_0$ and an elementary embedding g_1 of $(\mathfrak{B}_1, f_1[A_0]) \,|\, \mathscr{E}_{A_0}$ into $(\mathfrak{A}_1, A_0) | \mathscr{E}_{A_0}$ Then $g \circ f_1$ is the identity on A_0, so that $f_1 \subseteq g_1^{-1}$.

Iterating this construction yields a diagram

$$\begin{tikzcd}
\mathfrak{A}_0 \arrow[r, phantom, "\prec"] \arrow[dr, "f_1"'] & \mathfrak{A}_1 \arrow[r, phantom, "\prec"] \arrow[dr, "f_2"'] & \mathfrak{A}_2 \arrow[r, phantom, "\prec"] & \ldots \\
\mathfrak{B}_0 \arrow[r, phantom, "\prec"] & \mathfrak{B}_1 \arrow[u, "g_1"] \arrow[r, phantom, "\prec"] & \mathfrak{B}_2 \arrow[u, "g_2"'] \arrow[r, phantom, "\prec"] & \ldots
\end{tikzcd}$$

such that, for each m, f_m is an elementary embedding of $\mathfrak{A}_{m-1} \,|\, \mathscr{E}$ into $\mathfrak{B}_m \,|\, \mathscr{E}$, g_m is an elementary embedding of $\mathfrak{B}_m \,|\, \mathscr{E}$ into $\mathfrak{A}_m \,|\, \mathscr{E}$, and $f_m \subseteq g_m^{-1} \subseteq f_{m+1}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the unions of the elementary chains $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \ldots$ and $\mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \ldots$ respectively. Then, by the elementary chain theorem, $\mathfrak{A}$ is a model of Σ and $\mathfrak{B}$ is a model of Π. Moreover, $\bigcup\limits_{m \in \varpi} f_m$ is an isomorphism of $\mathfrak{A} \,|\, \mathscr{E}$ and $\mathfrak{B} \,|\, \mathscr{E}$ (since, by construction, it has inverse $\bigcup\limits_{m \in \varpi} g_m$. It follows that $\mathfrak{B}$ is isomorphic to a structure $\mathfrak{B}'$ such that $\mathfrak{A} \,|\, \mathscr{E} = \mathfrak{B}' \,|\, \mathscr{E}$. Accordingly the $\mathscr{E}$-structure $\mathfrak{A} \,|\, \mathscr{E}$ can be expanded both to the model $\mathfrak{A}$ of Σ and to the model $\mathfrak{B}'$ of Π.

(iv) $\Rightarrow$ (i). Let $\mathfrak{A}$ be an $\mathscr{E}$-structure expandable both to a model $\mathfrak{B}$ of Σ and to a model $\mathfrak{C}$ of Π. Define the $\mathscr{L}$-structure $\mathfrak{D}$ as follows: the domain of $\mathfrak{D}$ is that of $\mathfrak{A}$; if s is any extralogical symbol of $\mathscr{L}$, then

$$s^{\mathfrak{D}} = \begin{cases} s^{\mathfrak{A}} & \text{if } s \in \mathscr{E} \\ s^{\mathfrak{B}} & \text{if } s \in \mathscr{L} - \mathscr{L}_\Pi \\ s^{\mathfrak{C}} & \text{if } s \in \mathscr{L}_\Pi \end{cases}$$

Clearly $\mathfrak{D} \,|\, \mathscr{L}_\Sigma = \mathfrak{B}$, so $\mathfrak{D} \models \Sigma$. Also, $\mathfrak{D} \,|\, \mathscr{L}_\Pi = \mathfrak{C}$, so $\mathfrak{D} \models \Pi$. Therefore $\mathfrak{D}$ is a model of $\Sigma \cup \Pi$, so the latter is consistent. ■

From this we deduce

Craig's Interpolation Theorem. Suppose σ, τ are $\mathscr{L}$-sentences and $\vdash \sigma \to \tau$. Then there is a sentence θ such that $\vdash \sigma \to \theta$, $\vdash \theta \to \tau$, and every extralogical symbol occurring in θ occurs in both σ and τ.

Proof. Let $\mathscr{E}$ be the language whose extralogical symbols are exactly those occurring in both σ and τ. If $\vdash \sigma \to \tau$, then $\{\sigma, \neg\tau\}$ is inconsistent, so by (ii) of the joint consistency theorem there is an $\mathscr{E}$–sentence θ such that $\sigma \vdash \theta$ and $\neg\tau \vdash \neg\theta$. The result now follows immediately. ■

Suppose that $\Sigma \subseteq Sent(\mathscr{L})$ contains the n-ary predicate symbol P. P is said to be *explicitly definable* from Σ if there is an $\mathscr{L}$-formula $\varphi(x_1, ..., x_n)$, *in which P does not occur*, such that

$$\Sigma \vdash \forall x_1 ... \forall x_n [Px_1 ... x_n \leftrightarrow \varphi].$$

Now let P^* be an n-ary predicate symbol *not* belonging to $\mathscr{L}$, and let Σ^* be the set of sentences obtained from Σ by replacing all occurrences of P by P^*. Then P is said to be *implicitly definable* from Σ if

$$\Sigma \cup \Sigma^* \vdash \forall x_1 ... \forall x_n [Px_1 ... x_n \leftrightarrow P^*x_1 ... x_n].$$

Semantically speaking, this means that any pair of $\mathscr{L}$-structures which are both models of Σ, have the same domain and agree on the interpretation of all extralogical symbols apart possibly from P, must also agree on the interpretation of P.

Clearly, if P is explicitly definable from Σ, it is implicitly definable from Σ. Conversely, we have

Beth's Definability Theorem. If P is implicitly definable from Σ, it is explicitly definable from Σ.

Proof. Suppose P is implicitly definable from Σ. Without loss of generality we may assume Σ to be finite, and we can then replace Σ by the conjunction of all its sentences. So we may assume that Σ consists of a single sentence σ. Let σ* be the result of replacing each occurrence of P in σ by P^*. Then we have

(1) $$\{\sigma, \sigma^*\} \vdash \forall x_1 ... \forall x_n [Px_1 ... x_n \to P^*x_1 ... x_n].$$

Now add new constant symbols $c_1,...,c_n$ to $\mathscr{L}$. Then, by (1),

$$\{\sigma, \sigma^*\} \vdash Pc_1 ... c_n \to P^*c_1 ... c_n.$$

So

$$\vdash \sigma \wedge Pc_1 ... c_n \to (\sigma^* \to P^*c_1 ... c_n).$$

By Craig's theorem, there is a sentence θ whose extralogical symbols are common to both $\sigma \wedge Pc_1 ... c_n$ and $\sigma^* \to P^*c_1 ... c_n$, hence, in particular, not containing P or P^* such that $\vdash \sigma \wedge Pc_1 ... c_n \to \theta$ and $\vdash \theta \to (\sigma^* \to P^*c_1 ... c_n)$.

Therefore

(2) $$\sigma \vdash Pc_1 ... c_n \to \theta$$

and

(3) $$\sigma^* \vdash \theta \to P^*c_1 ... c_n.$$

If we replace P^* by P in (3), σ* becomes σ and θ is unchanged. So

(4) $$\sigma \vdash \theta \to Pc_1 ... c_n.$$

(2) and (4) now give

(5) $$\Sigma \vdash \theta \leftrightarrow Pc_1 ... c_n.$$

But θ is $\varphi(c_1,...c_n)$ for some $\mathscr{L}$–formula $\varphi(x_1, ..., x_n)$ in which P does not occur. Since $c_1, ..., c_n$

are not in $\mathscr{L}$, the result of replacing c_i by x_i ($i = 1, \ldots, n$) in the proof from Σ of $\theta \leftrightarrow Pc_1 \ldots c_n$ yields a proof from Σ of $\varphi \leftrightarrow Px_1 \ldots x_n$. Applying the generalization lemma gives

$$\Sigma \vdash \forall x_1 \ldots \forall x_n[\varphi \leftrightarrow Px_1 \ldots x_n]$$

and so P is explicitly definable from Σ. ■