Math 588, Fall 2001 **Problem Set 1** **Due Sept. 18, 2001**

1. [Burris-Sanka. 1.1.9] Let $\langle A, \leq \rangle$ be a be a finite poset. Show that there is a total (i.e., linear) order $\leq'$ on A such that $\leq \subseteq \leq'$, i.e., $a \leq b$ implies $a \leq' b$.

   Hint: consider the set of all partial orders $\preccurlyeq$ on A such $\leq \subseteq \preccurlyeq$. Show that there must be a maximal one and that any maximal one is a total order. The result holds also for infinite posets, but Zorn's lemma must be used in this case.]

2. [Burris-Sanka. 1.1.10] Let $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ be a lattice. An element $a \in A$ is *join irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$. If $\boldsymbol{A}$ is a finite lattice, show that every element is of the form $a_1 \vee \cdots \vee a_n$, where each a_i is join irreducible.

3. [Burris-Sanka. 1.2.4] Let $\boldsymbol{A} = \langle A, \leq \rangle$ be a poset. A subset S of A is a *lower segment* of $\boldsymbol{A}$ if every element of A that is less than or equal to some element of S is in S, i.e., for all $a \in A$ and $s \in S$, $a \leq s$ implies $a \in S$. Show that the lower segments of $\boldsymbol{A}$ form a lattice with operations under $\cup$ and $\cap$ (the set-theoretical join and meet). If $\boldsymbol{A}$ has a least element, show that the set $L(\boldsymbol{A})$ of non-empty lower segments of $\boldsymbol{A}$ forms a lattice.

4. [Burris-Sanka. 1.2.5 and 1.3.2] If $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ is a lattice, then an *ideal* of $\boldsymbol{A}$ is a nonempty lower segment that is closed under $\vee$. Show that the set $I(\boldsymbol{A})$ of ideals of $\boldsymbol{A}$ forms a lattice under $\subseteq$.

   If $\boldsymbol{A}$ is distributive, show that $\langle I(\boldsymbol{A}), \subseteq \rangle$ is distributive.

5. Let $\boldsymbol{A}$ be a bounded lattice (a lattice is *bounded* if it has a least element 0 and a greatest element 1). Let $\mathrm{Sub}(\boldsymbol{A})$ be the set of all sublattices of $\boldsymbol{A}$ that include 0 and 1. Show that $\mathbf{Sub}(\boldsymbol{A}) = \langle \mathrm{Sub}(\boldsymbol{A}), \subseteq \rangle$ is a complete lattice.

   Show that, if $\boldsymbol{A}$ is distributive, then for all $H, K \in \mathrm{Sub}(\boldsymbol{A})$, $H \vee K$ consists of all elements of $\boldsymbol{A}$ of the form $(h_1 \wedge k_1) \vee \cdots \vee (h_n \wedge k_n)$, with $1 \leq n \in \omega$, $h_1, \ldots, h_n \in H$ and $k_1, \ldots, k_n \in K$.

Math 588, Fall 2001 **Problem Set 2** **Due Oct, 9, 2001**

(1) [Burris-Sanka. 1.4.6] Let $\boldsymbol{L} = \langle L, \leq \rangle$ be an algebraic lattice and D an upward directed subset of L, i.e., for all $d_1, d_2 \in D$ there exists a $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$. Prove that, for every $a \in L$, $a \wedge \bigvee D = \bigvee_{d \in D}(a \wedge d)$.
[*Hint*: Show that, for every compact element c of $\boldsymbol{L}$, $c \leq a \wedge \bigvee D$ iff $c \leq \bigvee_{d \in D}(a \wedge d)$.]

(2) [based on Burris-Sanka. 1.5.7] Let Σ be an arbitrary signature and let $\boldsymbol{A} = \langle A, \{\sigma^A : \sigma \in \Sigma\}\rangle$ be a finitely generated Σ-algebra, i.e., there exists a finite $X \subseteq A$ such that $A = \mathrm{Sg}^{\boldsymbol{A}}(X)$. Prove that, for every set Y that generates $\boldsymbol{A}$, there is a finite $Y' \subseteq Y$ such that Y' generates $\boldsymbol{A}$.

(3) Let $\langle A, \mathcal{C}\rangle$ be a closed-set system, and let $\mathcal{C}_\omega$ be the set of all finitely generated closed sets, i.e., $\mathcal{C}_\omega = \{\, \mathrm{Cl}_{\mathcal{C}}(X) : X \subseteq_\omega A \,\}$. $\langle A, \mathcal{C}_\omega\rangle$ may or may not be a closed-set system, i.e., the set of finitely generated sets may or may not be closed under intersection.
  (a) Prove that, for any set A, $\langle A, \mathrm{Eq}(A)_\omega\rangle$, i.e., the set of all finitely generated equivalence relations, is a closed-set system.
  [*Hint*: Show that every subequivalence relation of a finitely generated equivalence relation is finitely generated. Look at the associated partitions.]
  (b) Let $\Sigma = \{f, g\}$ where f is unary and g is binary. Let $\boldsymbol{A} = \langle \mathbb{Z}, f, g\rangle$ be the Σ-algebra whose universe is the set of integers and f and g are defined as follows. $f(n) = n + 1$ if $0 \leq n$; otherwise $f(n) = n$. $g(n, m) = n - m$ if $0 \leq n < m$; otherwise $g(n, m) = n$. ("+", "−", and "≤" are the usual addition, substraction, and order of the additive group of integers.) Show that $\langle \mathbb{Z}, \mathrm{Sub}(\boldsymbol{A})_\omega\rangle$ is not a closed-set system.

(4) (a) Prove that every infinite mono-unary algebra has a proper subuniverse (i.e, a subuniverse different from both $\emptyset$ and the universe of the algebra).
  (b) Construct an infinite *bi-unary algebra* (i.e., two unary operations) that has no proper subuniverse.

(5) The subuniverses of $\langle \mathbb{Z}, +\rangle$ different from $\langle \mathbb{Z}, +, -, 0\rangle$ and are much harder to characterize. Show that they are all finitely generated.
[*Hint*: Show that the problem can be reduced to showing that every subuniverse of $\langle \omega, +\rangle$ is finitely generated. A subuniverse A of $\langle \omega, +\rangle$ is *eventually periodic* if there exists $n, m \in \omega$ with $m > 0$ such that every $x \in A$ with $x \geq m$ is of the form $m + kn$ with $k \in \omega$. For example, if A is the subalgebra of $\langle \omega, +\rangle$ generated by 3 and 5, then A is eventually periodic with $m = 8$ and $n = 1$. Prove that every subuniverse of $\langle \omega, +\rangle$ is eventually periodic. Use this fact to prove that every subuniverse is finitely generated.]

Math 588, Fall 2001 **Problem Set 3** **Due Oct 23, 2001**

1. [Burris-Sanka. II.3.1] Let $\boldsymbol{A}$ be a Σ-algebra and $X \subseteq A$. Define a infinite sequence $E_0(X) \subseteq E_1(X) \subseteq E_2(X) \subseteq \cdots \subseteq A$ be recursion as follows. $E_0(X) = X$ and $E_{n+1}(X) = E_n(X) \cup \{\, \sigma^{\boldsymbol{A}}(a_1, \dots, a_m) : m \in \omega, \sigma \in \Sigma_m, a_1, \dots, a_m \in E_n(X) \,\}$. Prove that $\mathrm{Sg}^{\boldsymbol{A}}(X) = \bigcup_{n \in \omega} E_n(X)$.

2. Let Σ be the signature of groupoid, i.e., a single binary operation. Consider the binary relations of subalgebra ($\subseteq$) and homomorphic image ($\preccurlyeq$) on the class of all Σ-algebras $\mathsf{Alg}(\Sigma)$. Prove that the $\subseteq ; \succcurlyeq \;=\; \succcurlyeq ; \subseteq$, i.e., prove that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Alg}(\Sigma)$, if there exists a $\boldsymbol{C} \in \mathsf{Alg}(\Sigma)$ such that $\boldsymbol{A} \subseteq \boldsymbol{C} \succcurlyeq \boldsymbol{B}$, then there exists a $\boldsymbol{D} \in \mathsf{Alg}(\Sigma)$ such that $\boldsymbol{A} \succcurlyeq \boldsymbol{D} \subseteq \boldsymbol{B}$, and vice versa.

   [*Hint*: The "vice versa" part is the harder to prove. Under the assumptions that $\boldsymbol{D} \subseteq \boldsymbol{B}$ and there exists an epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{D}$, you have to construct a "superalgebra" $\boldsymbol{C}$ of $\boldsymbol{A}$ (i.e., $\boldsymbol{A} \subseteq \boldsymbol{C}$) and an epimorphism $g\colon \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}$. It is helpful to draw pictures.

   For simplicity you can assume that in this case Σ is a groupoid signature, i.e., a single binary operation (written in infix notation). Without loss of generality we assume that A and B are disjoint (otherwise $\boldsymbol{B}$ may first be replaced with an isomorphic image $\boldsymbol{B}'$ and then at the end the epimorphism $g\colon \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}'$ can be composed with the isomorphism from $\boldsymbol{B}'$ to $\boldsymbol{B}$). Let $C = A \cup (B \setminus D)$. Define $g\colon C \to B$ so that $g(c) = h(c)$ if $c \in A$ and $g(c) = c$ if $c \in B \setminus D$. Then define the operation $\cdot^{\boldsymbol{C}}$ on C so that it agrees with $\cdot^{\boldsymbol{A}}$ on A and the map g is a homomorphism from $\boldsymbol{C}$ to $\boldsymbol{B}$. The definition of $c \cdot^{\boldsymbol{C}} c'$ will require the consideration of several cases depending on whether or not c and c' are in A.]

3. Let $\boldsymbol{A}$ be a groupoid. define a binary operation on the set A^A of all mappings of A into itself as follows. For all $f, g \in A^A$, $f \cdot g$ is the mapping from A to itself such that, for all $a \in A$, $(f \cdot g)(a) = f(a) \cdot^{\boldsymbol{A}} g(a)$. Prove that the set $\mathrm{End}(\boldsymbol{A})$ of endomorphisms of $\boldsymbol{A}$ is closed under this operation iff $\boldsymbol{A}$ satisfies the following *entropic law*. $(x \cdot y) \cdot (z \cdot w) \approx (x \cdot z) \cdot (y \cdot w)$. Prove that if $\boldsymbol{A}$ has an identity element (i.e., an element e such that $e \cdot^{\boldsymbol{A}} a = a = a \cdot^{\boldsymbol{A}} e$ for all $a \in A$), then $\mathrm{End}(\boldsymbol{A})$ is closed under $\cdot$ iff $\boldsymbol{A}$ is a commutative semigroup, i.e., satisfies the commutative law $x \cdot y \approx y \cdot x$ and the associative law $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.

4. A semigroup with identity $\langle A, \cdot, e \rangle$ is called a *monoid*. Prove that every cyclic monoid is commutative. Prove that $\mathbf{H}(\langle \omega, +, 0 \rangle)$, the class of all homomorphic images of $\langle \omega, +, 0 \rangle$, is the class of all cyclic (commutative) monoids. Use this result and the First Isomorphism Theorem to obtain a characterization of $\mathrm{Co}(\langle \omega, +, 0 \rangle)$.

   [*hint*: Show that for every monoid $\boldsymbol{A} = \langle A, \cdot, e \rangle$ and every $a \in A$, there exists a (unique) homomorphism $h\colon \langle w, +, 0 \rangle \to \boldsymbol{A}$ such that $h(1) = e$.]

(over)

5. A *Boolean algebra* is a algebra $\boldsymbol{B} = \langle B, \vee, \wedge, -, 0, 1\rangle$ such that $\langle B, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $-$ is the *complement* operation, i.e., $\boldsymbol{B}$ satisfies the identities $-x \vee x \approx 1$ and $-x \wedge x \approx 0$.

   (a) Prove that the complement of an element is unique, i.e., if $b \vee a = 1$ and $b \vee a = 0$, then $b = -a$. Prove the *law of double negation* $--x \approx x$, and the two *DeMorgan laws*: $-(x \vee y) \approx -x \wedge -y$ and $-(x \wedge y) \approx -x \vee -y)$.

   (b) Let I be an ideal of $\boldsymbol{B}$ (in the sense of Problem #4 of Problem Set 1), and define a binary relation α on B by $a \,\alpha\, b$ if $a - b, b - a \in I$, equivalently (since I is and ideal), if $(a - b) \vee (b - a) \in I$. Prove that α is a congruence of $\boldsymbol{B}$ and that $0/\alpha = I$.

   (c) Let α be any congruence $\boldsymbol{B}$. Prove that $0/\alpha = \{\, b \in B : b \,\alpha\, a \,\}$ is an ideal of $\boldsymbol{B}$, and that, for all $a, b \in B$, $a \,\alpha\, b$ iff $a - b, b - a \in 0/\alpha$.

   Thus there is a bijection between the ideals and congruences of $\boldsymbol{B}$ that clearly preserves $\subseteq$.

Math 588, Fall 2001 **Problem Set 4** **Due Nov 13, 2001**

1. [based on Burris-Sanka. 1.5.10,1.5.11.] A Σ-algebra $\boldsymbol{A}$ has the *principal congruence extension property* (PCEP) if, for every $\boldsymbol{B} \subseteq \boldsymbol{A}$ and all $b, b' \in B$, $\Theta_{\boldsymbol{B}}(b, b') = \Theta_{\boldsymbol{A}}(b, b') \cap B^2$. A class K of Σ-algebras has the PCEP if every algebra in the class has the PCEP.

   (a) If $\boldsymbol{A}$ is an Abelian group and $a, b, c, d \in A$, show that $\langle a, b\rangle \in \Theta_{\boldsymbol{A}}(c, d)$ iff $a - b = n(c - d)$ for some $n \in \mathbb{Z}$ (i.e., $a - b$ is in the cyclic subgroup generated by $c - d$). Use this to show that the the class of Abelian groups has the PCEP.

   (b) If $\boldsymbol{L}$ is a distributive lattice and $a, b, c, d \in L$, show that $\langle a, b\rangle \in \Theta(c, d)$ iff $c \wedge d \wedge a = c \wedge d \wedge b$ and $c \vee d \vee a = c \vee d \vee b$. Use this to show that the class of distributive lattices has the PCEP.

   Note: A Σ-algebra $\boldsymbol{A}$ has the *congruence extension property* (PCEP) if, for every $\boldsymbol{B} \subseteq \boldsymbol{A}$ and every $\beta \in \mathrm{Co}(\boldsymbol{B})$ there is a $\alpha \in \mathrm{Co}(\boldsymbol{A})$ such that $\beta = \alpha \cap B^2$. A class K of Σ-algebras has the CEP if every algebra in the class has the CEP.

   It is easy to see that the CEP implies the PCEP. We shall see later that the converse holds.

2. Let $\boldsymbol{A}$ be a Σ-algebra.

   (a) Let $\boldsymbol{A}$ be a nontrivial Σ-algebra. Prove that if ∇_A is finitely generated as a congruence, then $\boldsymbol{A}$ has at least one simple homomorphic image.

   (b) Prove that if Σ is finite (i.e., has only a finite number of operation symbols) and A is finitely generated as a subuniverse of itself, then ∇_A is finitely generated as a congruence. Hence any finitely generated nontrivial algebra over a finite language type has a simple homomorphic image.

   (c) Show that any nontrivial ring with unit $\langle R, +, \cdot, -, 0, 1\rangle$ has a field as a homomorphic image.

   [*Hint:* Use Zorn's lemma for the first part.

   For any nonempty subset X of A choose an fixed but arbitrary $a \in X$ and let

   $$Y = \big(X \times \{a\}\big) \cup \big(\{\, \sigma^{\boldsymbol{A}}(a, a, \cdots, a) : \sigma \in \Sigma \,\} \times \{a\}.$$

   For the second part show that $A = \mathrm{Sub}^{\boldsymbol{A}}(X)$ implies $\Theta_{\boldsymbol{A}}(Y) = \nabla_A$.

   For the third part, use part (a). You can use the fact that any simple ring is a field.]

3. Let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be system of Σ-algebras. Let $\boldsymbol{B}$ be a Σ-algebra and $\langle\, h_i : i \in I \,\rangle \in \prod_{i\in I} \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A}_i)$. $\langle\, h_i : i \in i \,\rangle$ is said to *separate points* if, for all distinct $b, b' \in B$ there exists an $i \in I$ such that $h_i(b) \neq h_i(b')$.

   Prove that $\boldsymbol{B} \cong \mathbin{;} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ iff there exists a $\langle\, h_i : i \in I \,\rangle \in \prod_{i\in I} \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A}_i)$ that separates points.

   [*Hint:* One of the two implications is an immediate corollary of Theorem 7.15 of Burris and Sankappanavar.]

4. Let $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle$ be a finite system of congruences on $\boldsymbol{A}$. For each $i \leq n$ let

$$\hat{\alpha}_i = \alpha_1 \cap \cdots \cap \alpha_{i-1} \cap \alpha_{i+1} \cap \cdots \cap \alpha_n.$$

   Prove that $\vec{\alpha}$ is a factor congruence system for $\boldsymbol{A}$ iff

   (a) $\alpha_1 \cap \cdots \cap \alpha_n = \Delta_A$, and
   (b) $\alpha_i \mathbin{;} \hat{\alpha}_i = \nabla_A$, for each $i \leq n$.

   [*Hint:* Prove by induction on n that (b) holds iff $\vec{\alpha}$ has the Chinese Remainder Property.]

5. Let $\boldsymbol{A}$ be a Σ-algebra.

   (a) Prove that $\boldsymbol{A}$ is directly irreducible iff, for every system $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$ of Σ-algebras, if $A \cong \prod_{i\in I} \boldsymbol{B}_i$, then there is an $i \in I$ such that $\boldsymbol{B}_j$ is trivial for all $j \in I \setminus \{i\}$.
   (b) Prove that if $\boldsymbol{A}$ is finite, then $\boldsymbol{A}$ is directly irreducible iff, for every system $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$ of Σ-algebras, if $A \cong \prod_{i\in I} \boldsymbol{B}_i$, then there is an $i \in I$ such that $\boldsymbol{A} \cong \boldsymbol{B}_i$.

   [*Hint:* Recall that we define an algebra to be directly irreducible if its only factor congruences are Δ and ∇. Burris and Sankappanavar take the condition in (a) in the special case $I = \{1, 2\}$ to be the definition of direct irreducibility. Thus (a) in this case is their Corollary 7.7. They claim it is a corollary of their Theorems 7.3 and 7.5, but some work is still needed to get it.]

Hint on Problem #1

Lets look at the first part. Let Φ be the set of all pairs of integers $\langle a, b\rangle$ such that $a - b = n(c - d)$ for some $n \in \mathbb{Z}$. You must show that $\Phi = \Theta_{\boldsymbol{A}}(a, b)$, i.e., Φ is the smallest congruence α on $\boldsymbol{A}$ such that $a \,\alpha\, b$. First verify by calculation that Φ is a congruence, i.e., an equivalence relation with the substitution property. For example, lets verify it has the substitution property wrt the operation $+$. I.e, if $a_1 \,\Phi\, b_1$ and $a_2 \,\Phi\, b_2$, then $(a_1+a_2) \,\Phi\, (a_1+b_2)$. So assume $a_1 - b_1 = n_1(c - d)$ and $a_2 - b_2 = n_2(c - d)$. Then $(a_1 + a_2) - (b_1 + b_2) = (a_1-b_1)+(a_2-b_2) = (n_1+n_2)(c-d)$. Clearly, $c \,\Phi\, d$. This shows that $\Theta_{\boldsymbol{A}}(c, d) \subseteq \Phi$. To show the inclusion in the other direction we must show that if $a \,\Phi\, b$, then $a \,\Theta_{\boldsymbol{A}}(c, d)\, b$. Suppose $a-b = n(c-d)$. Then $a = b+n(c-d)$. Since $c \,\Theta_{\boldsymbol{A}}(c, d)\, d$ and $d \,\Theta_{\boldsymbol{A}}(c, d)\, d$ because $\Theta_{\boldsymbol{A}}(c, d)$ is an equivalence relation, we have by the substitution property that $(c-d) \,\Theta_{\boldsymbol{A}}(c, d)\, (d-d) = 0$. So again by the substitution property, $a = (b + n(c - d)) \,\Theta_{\boldsymbol{A}}(c, d)\, (b + n(0)) = b$. So $\Phi \subseteq \Theta_{\boldsymbol{A}}(c, d)$.

Use the same method in part (b). Define a binary relation Φ on the lattice by the given condition. Prove it is a congruence that relates c and d and then prove it is included in $\Theta_{\boldsymbol{L}}(c, d)$

Math 588, Fall 2001 **Problem Set 5** **Due Dec. 13, 2001**

1. [based on Burris-Sanka. II.8.3] Prove that the sudirectly irreducible cyclic (i.e., one-generated) mono-unary algebras are exactly the finite cyclic algebras with empty tail whose cycle is of length a positive power of a prime, or the finite cyclic algebras with nonempty tail whose cycle is of length 1.

   [*Hint:* Start by showing that every finite cyclic mono-unary algebra $\boldsymbol{A}$ with nonempty tail and a cycle of length > 1 fails to be subdirectly irreducible. This is done by finding two congruences that are greater than Δ_A but intersect in Δ_A. Show that there is an endomorphism of $\boldsymbol{A}$ that maps all of $\boldsymbol{A}$ onto its cycle by "wrapping" its tail around its cycle and leaving the cycle itself fixed, and then take the first congruence to be the relation kernel of this map. Show that the equivalence relation that collapses the cycle to one point an leaves the tail alone is a congruence, and take this to be the second congruence. This shows that every subdirectly irreducible finite cyclic algebra is either a cycle of length $>$ 1 with empty tail, or a cycle of length 1 with nonempty tail. You still have to show that algebras of the second kind are SI and that those of the first kind are SI iff the length of the cycle is a power of a prime. Finally, you have to show that the infinite cyclic algebra, i.e., $\langle \omega, s\rangle$ where s is the successor function, is not subdirectly irreducible.

   Extra Credit Problem: Describe all SI mono-unary algebras.]

2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras.

   (a) Let $\boldsymbol{C} \subseteq \boldsymbol{A}$ and $\boldsymbol{D} \subseteq \boldsymbol{B}$. Show that $\boldsymbol{C} \times \boldsymbol{D} \subseteq \boldsymbol{A} \times \boldsymbol{B}$; show by example that in general, not every subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ need be of this form.

   (b) Assume that there is a binary term $t(x_1, x_2)$ such that $t^{\boldsymbol{A}}(a, a') = a$ for all $a, a' \in A$ and $t^{\boldsymbol{B}}(b, b') = b'$ for all $b, b' \in B$. Prove that in this case, every subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$ is of the form $\boldsymbol{C} \times \boldsymbol{D}$ with $\boldsymbol{C} \subseteq \boldsymbol{A}$ and $\boldsymbol{D} \subseteq \boldsymbol{B}$.

   (c) Let $\boldsymbol{A} = \langle A, \cdot, ^{-1}, e\rangle$ and $\boldsymbol{B} = \langle B, \cdot, ^{-1}, e\rangle$ be finite groups such that $|A|$ and $|B|$ are relatively prime. Prove that there exists a binary term t satisfying the condition of part (a). Hence every subgroup of $\boldsymbol{A} \times \boldsymbol{B}$ is a product of subgroups of $\boldsymbol{A}$ and $\boldsymbol{B}$.

   [*Hint for part (b):* Let $\boldsymbol{E} \subseteq \boldsymbol{A} \times \boldsymbol{B}$. Let $C = \{\, a \in A : \exists\, b \in B\, (\langle a, b\rangle \in E)\,\}$ and let $D = \{\, b \in B : \exists\, a \in A\, (\langle a, b\rangle \in E)\,\}$. Prove in general (i.e., without the assumption about the existence of the term t) that $C \in \mathrm{Sub}(\boldsymbol{A})$ and $D \in \mathrm{Sub}(\boldsymbol{B})$ and that $\boldsymbol{E} \subseteq \boldsymbol{C} \times \boldsymbol{D}$. Now prove that $\boldsymbol{E} = \boldsymbol{C} \times \boldsymbol{D}$; here is where the term t is used.

   Prove for every n-ary term $s(x_1, \ldots, x_n)$ and for all $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in A \times B$ that

   $$s^{\boldsymbol{A}\times\boldsymbol{B}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle) = \langle s^{\boldsymbol{A}}(a_1, \ldots, a_n),\ s^{\boldsymbol{B}}(b_1, \ldots, b_n)\rangle.$$

   Use this together with the assumption on t to show that if $\langle a, b'\rangle, \langle a', b\rangle \in E$, then $\langle a, b\rangle \in E$.]

(over)

3. Let $\boldsymbol{A}$ be a Σ-algebra, Let $\langle\, \alpha_i : i \in I\,\rangle \in \mathrm{Co}(\boldsymbol{A})^I$, and $\beta = \bigcap_{i\in I} \alpha_i$. Prove that $\boldsymbol{A}/\beta \cong ; \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{A}/\alpha_i$.

   [*Hint:* Prove that in the lattice $\mathbf{Co}[\boldsymbol{A}/\beta]$, $\bigcap_{i\in I} \alpha_i/\beta = \Delta_{A/\beta}$; for this you can use the Correspondence Theorem.]

4. Let Σ be an arbitrary signature. Prove that the algebra $\mathbf{Te}_\Sigma(X)$ has the unique parsing property wrt X.

   [*Hint:* We represent strings over $\Sigma \cup X$ by the Greek letters α, β, γ, possibly with sub- or superscripts. $\alpha\beta$ will denote the concatenation of α and β. α is an *initial segment* of β, in symbols $\alpha \preccurlyeq \beta$, if there exists a string γ such that $\beta = \alpha\gamma$. If $\gamma \neq \varepsilon$ (the empty string), then γ is a *proper initial segment* of β.

   Define $f\colon (\Sigma \cup X) \to \mathbb{Z}$ by setting $f(x) = -1$ for each $x \in X$, and $f(\sigma) = n - 1$ for each $\sigma \in \Sigma_n$. For every string $\alpha = a_1 \dots a_n$ let $\bar{f}(\alpha) = f(a_1) + \cdots + f(a_n)$; note $\bar{f}(\varepsilon) = 0$.

   Prove the following lemmas.

   Lemma 1. *A string α is a Σ-term iff $\bar{f}(\alpha) = -1$ and $\bar{f}(\beta) \geq 0$ for every proper initial segment of α.*

   For example

$t:$	$\wedge$	$-$	$\vee$	x_1	0	$\wedge$	x_2	1
$f:$	1	0	1	-1	-1	1	-1	-1
$\bar{f}:$	1	1	2	1	0	1	0	-1

   Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ be two sequences of Σ-terms. $\alpha_1 \dots \alpha_n = \beta_1 \dots \beta_m$ iff $n = m$ and $\alpha_i = \beta_i$ for all $i \leq n$.*

   Use this last lemma to prove that $\mathbf{Te}_\Sigma(X)$ has the unique parsing property wrt to X.]

5. (a) Let K be a variety. Assume that K contains a nontrivial finite algebra. Prove that, for all cardinals λ and κ, if $\mathbf{Fr}_\lambda(\mathsf{K}) \cong \mathbf{Fr}_\kappa(\mathsf{K})$ iff $\lambda = \kappa$

   (b) Let $\Sigma = \{\cdot, \pi_1, \pi_2\}$ with $\cdot$ a binary operation and π_2 and π_2 unary operations. Let K be the variety of Σ-algebras defined by the identities $\pi_1(x_1 \cdot x_2) \approx x_2$, $\pi_2(x_1 \cdot x_2) \approx x_2$, and $\pi_1(x) \cdot \pi_2(x) \approx x$. Prove that $\mathbf{Fr}_n(\mathsf{K}) \cong \mathbf{Fr}_m(\mathsf{K})$ for all $n, m \in \omega$ such that $n, m \geq 2$.

   [*Hint:* Part (a). Count the number of homomorphisms from $\mathbf{Fr}_\lambda(\mathsf{K})$ and $\mathbf{Fr}_\kappa(\mathsf{K})$ into some nontrivial finite algebra. Part (b): Prove by induction on n that $\mathbf{Fr}_n(\mathsf{K}) \cong \mathbf{Fr}_{n+1}(\mathsf{K})$. For the base step, suppose $\boldsymbol{F}$ is a free algebra over K with a single free generator. From this free generator construct two elements such that $\boldsymbol{F}$ has the UMP over K wrt these two elements.]

Math 589, Spring 2002 **Problem Set 1** **Due Feb. 6, 2002**

Corrected Mat. 1, 2002

1. Let V be any variety. Prove that for any cardinal κ, the free algebra of V with κ free generators is isomorphic to the coproduct of κ copies of the free algebra with one free generator, i.e., $\mathbf{Fr}_\kappa(V) \cong \coprod_{\xi<\kappa} \boldsymbol{F}_\xi$, where $\boldsymbol{F}_\xi = \mathbf{Fr}_1(V)$ for each $\xi < \kappa$.

   [*note:* This generalizes the well known result that every free Abelian group is isomorphic to a direct sum of $\mathbb{Z}$.]

2. Let $\mathcal{K} \subseteq \mathcal{P}(I)$. Then $\mathcal{K}$ is included in a proper filter and hence an ultrafilter iff, for all $n \in \omega$ and all $K_1, \dots, K_n \in \mathcal{K}$, $K_1 \cap \cdots \cap K_n \neq \emptyset$.

3. Prove that a principal filter $[X)$ on a set I is an ultrafilter iff $|X| = 1$.

4. Let I be an infinite set, and let $\mathcal{F}$ be a proper filter on I. Prove the following.

   (a) If $\mathcal{C}f \subseteq \mathcal{F}$, then $\mathcal{F}$ is nonprincipal.

   (b) If $\mathcal{F}$ is a nonprincipal ultrafilter, then $\mathcal{C}f \subseteq \mathcal{F}$.

   [*Note:* The condition that $\mathcal{F}$ is an ultrafilter is essential in part (b). Here is a counterexample. Let $I = \omega$ and let $\mathcal{F} = \{\, X \in \mathcal{C}f : 0 \in X \,\}$. Clearly $\mathcal{C}f \nsubseteq \mathcal{F}$, but it is easy to see that $\mathcal{F}$ is proper and nonprincipal.]

5. Let $N \in \omega$ and let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be a system of Σ-algebras whose cardinalities are bounded above by N, i.e., $|A_i| \leq N$ for all $i \in I$. let $\mathcal{U}$ be an ultrafilter on I. Prove that $|\prod_{i\in I} \boldsymbol{A}_i/\Phi(\mathcal{U})| \leq N$.

   [*Hint:* You can use Problem 4, part (b).]

1. Let $\langle \boldsymbol{A}_i : i \in I \rangle$ be a system of Σ-algebras, and let $\mathcal{U}$ be an ultrafilter on I. Let φ be a formula of the form ε_1 **or** $\cdots$ **or** ε_n **or** (**not** δ_1) **or** $\cdots$ **or** (**not** δ_m), where $\varepsilon_1, \dots, \varepsilon_n, \delta_1, \dots, \delta_m$ are Σ-equations. Prove that

$$\Big(\prod_{i \in I} \boldsymbol{A}_i\Big)/\Phi(\mathcal{U}) \vDash \varphi \quad \text{iff} \quad \{\, i \in I : \boldsymbol{A}_i \vDash \varphi \,\} \in \mathcal{U}.$$

[*Hint:* The solution cannot be easily obtained simply by generalizing the proof of Lemma 3.38. First of all the class of models of an equational clause is not necessarily closed under homomorphic images or direct products (integral domains, for example). A somewhat different argument is required. Here is how I suggest you proceed. A big part of the problem is to choose the right kind of simplifying notation.

Let I be an index set, which is normally infinite. We use vector notation to represent a "I-dimensional" vector over an arbitrary I-indexed system of sets $\langle A_i : i \in I \rangle$. Thus $\vec{a} = \langle a_i : i \in I \rangle \in \prod_{i \in I} A_i$. On the other hand, we will use "hats" to denote arbitrary finite sequences of elements of A, for example, $\hat{a} = a_0, \dots, a_{k-1} \in A^k$. Thus $\widehat{\vec{a}} = \vec{a}_0, \dots, \vec{a}_{k-1}$ is a finite sequence of I-vectors, i.e., a "I by k" matrix of elements of the A_i. For each $j < k$, $\vec{a}_j$ is the j-th column of the matrix, and for each $i \in I$, $\hat{a}_i = a_{i0}, \dots, a_{i(k-1)}$ is the i-th row.

Let $\varphi(\hat{x})$ be the equational clause

$$\big(t_1(\hat{x}) \approx s_1(\hat{x})\big)\mathbf{or}\cdots\mathbf{or}\big(t_n(\hat{x}) \approx s_n(\hat{x})\big)\mathbf{or}\big(\mathbf{not}\big(u_1(\hat{x}) \approx v_1(\hat{x})\big)\big)\mathbf{or}\cdots\mathbf{or}\big(\mathbf{not}\big(u_m(\hat{x}) \approx v_m(\hat{x})\big)\big),$$

where $\hat{x} = x_0, \dots, x_{k-1}$ is a list of all the variables that occur in φ.

The intermediate notion of *satisfaction* is useful here. Let $\boldsymbol{A}$ be a Σ-algebra and $\hat{a} = a_0, \dots, a_{k-1} \in A^k$. We say that $\hat{a}$ *satifies* $\varphi(\hat{x})$ *in* $\boldsymbol{A}$, in symbols $\langle \boldsymbol{A}, \hat{a} \rangle \vDash \varphi(\hat{x})$, if

$$t_1^{\boldsymbol{A}}(\hat{a}) = s_1^{\boldsymbol{A}}(\hat{a}) \quad \text{or} \cdots \text{or} \quad t_n^{\boldsymbol{A}}(\hat{a}) = s_n(\hat{a}) \quad \text{or} \quad u_1^{\boldsymbol{A}}(\hat{a}) \neq v_1^{\boldsymbol{A}}(\hat{a}) \quad \text{or} \cdots \text{or} \quad u_m^{\boldsymbol{A}}(\hat{a}) \neq v_m^{\boldsymbol{A}}(\hat{a}).$$

Thus $\varphi(\hat{x})$ is universally valid in $\boldsymbol{A}$ if, for every $\hat{a} \in A^k$, $\hat{a}$ satisfies $\varphi(\hat{x})$ in $\boldsymbol{A}$.

Let $\prod_{i \in I} \boldsymbol{A}_i$ be a system of Σ-algebras and $\widehat{\vec{a}} = \vec{a}_0, \dots, \vec{a}_{k-1} \in \big(\prod_{i \in I} A_i\big)^k$. Let $\mathcal{U}$ be an ultrafilter on I, and let $\widehat{\vec{a}}/\Phi(\mathcal{U}) = \vec{a}_0/\Phi(\mathcal{U}), \dots, \vec{a}_{k-1}/\Phi(\mathcal{U})$. Prove the following two lemmas

Lemma 1. $\big\langle \big(\prod_{i \in I} \boldsymbol{A}_i\big)/\Phi(\mathcal{U}),\, \widehat{\vec{a}}/\Phi(\mathcal{U}) \big\rangle \vDash \varphi(\hat{x})$ *iff*

$$\mathrm{EQ}\big(t_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), s_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big) \in \mathcal{U} \quad \textit{or} \cdots \textit{or} \quad \mathrm{EQ}\big(t_n^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), s_n^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big) \in \mathcal{U}$$
$$\textit{or} \quad \mathrm{EQ}\big(u_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), v_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big) \notin \mathcal{U} \quad \textit{or} \cdots \textit{or} \quad \mathrm{EQ}\big(u_m^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), v_m^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big) \notin \mathcal{U}.$$

Lemma 2. $\{\, i \in I : \langle \boldsymbol{A}_i, \hat{a}_i \rangle \vDash \varphi(\hat{x}) \,\} =$

$$\mathrm{EQ}\big(t_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), s_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big) \cup \cdots \cup \mathrm{EQ}\big(t_n^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), s_n^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big)$$
$$\cup\ \overline{\mathrm{EQ}\big(u_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), v_1^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big)} \cup \cdots \cup \overline{\mathrm{EQ}\big(u_m^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}}), v_m^{\prod \boldsymbol{A}_i}(\widehat{\vec{a}})\big)}.$$

(over)

Now for the solution of Problem 1. For the proof of the implication from right to left, use that fact that, for all $\widehat{\vec{a}} \in (\prod_{i\in I} A_i)$, $\{\, i \in I : \boldsymbol{A}_i \vDash \varphi(\hat{x}) \,\} \subseteq \{\, i \in I : \langle \boldsymbol{A}_i, \hat{a}_i \rangle \vDash \varphi(\hat{x}) \,\}$. For the implication in the opposite direction, proof the contrapositive.

Assume $\{\, i \in I : \boldsymbol{A}_i \vDash \varphi(\hat{x}) \,\} \notin \mathcal{U}$, and show that there is a $\widehat{\vec{a}} \in (\prod_{i\in I} A_i)$ such that $\left(\prod_{i\in I} \boldsymbol{A}_i\right)/\Phi(\mathcal{U}) \nvDash \varphi(\hat{x})$.]

2. Let E and Γ be sets of Σ-equations such that $E \vdash \gamma$ for every $\gamma \in \Gamma$. Prove that, for every Σ-equation ε, if $E \cup \Gamma \vdash \varepsilon$, then $E \vdash \varepsilon$.

   [*Note:* The equations of Γ can be viewed as "lemmas" that are used in the "proof" of the "theorem" ε from the "hypotheses" E. To prove this directly one has to show that, given any proof $\delta_1, \dots, \delta_m$ of ε from $E \cup \Gamma$ (in the precise sense of Definition 4.2), one can replace each occurrence of a substitution instance γ' of an equation γ of Γ by a proof of γ' from E, thus obtaining a, generally much longer, proof of ε from E alone. A shorter, indirect proof can be obtained using the soundness and completeness theorems of equational logic, and the fact the logical consequence operation Cn is a closure operator (Theorem 4.7).]

3. Let E be the axioms of groups (of Type II). Prove that $E \cup \{x \cdot x \approx e\} \vdash x \cdot y \approx y \cdot x$.

   [*Hint:* You have to prove that a proof of $x \cdot y \approx y \cdot x$ (in the sense of Definition 4.2) exists. One way to do this is to just write it down like we did in class (or at least started to do) for $(x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$. But this will be very long. It is better to first prove some lemmas and then use Problem #2. You might even want to use lemmas in the proof of a lemma. But for the lowest level lemmas you have to write out formal proofs in the sense of Definition 4.2. I suggest that you give an informal proof of $x \cdot y \approx y \cdot x$ from $E \cup \{x \cdot x \approx e\}$ as you would do in a beginning algebra course and then convert it to a formal proof. You can assume that $E \vdash (x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$, and hence use $(x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$ as a lemma.]

4. Recall that a set E is Σ-equations is *inconsistent* if it has only trivial models.

   (a) Prove that E is inconsistent iff $E \vdash x \approx y$, where x and y are distinct variables.

   (b) Use part (a) to obtain another proof of the compactness theorem for equational logic that does not use reduced products.

   [*Hint:* For part (b) look at the proof that the closure relation Cn is finitary.]

5. Let A and B be sets and $R \subseteq A \times B$. Let $H\colon \mathcal{P}(A) \to \mathcal{P}(B)$ and $G\colon \mathcal{P}(B) \to \mathcal{P}(A)$ be the Galois connection defined by R. Let $\mathcal{C}_A = \{\, C \subseteq A : (G \circ H)(C) = C \,\}$, the closed subsets of A under $G \circ H$. Let $\mathcal{C}_B = \{\, C \subseteq B : (H \circ G)(C) = C \,\}$, the closed subsets of B under $H \circ G$. Prove that the complete lattices $\langle \mathcal{C}_A, \subseteq \rangle$ and $\langle \mathcal{C}_B, \subseteq \rangle$ are dually isomorphic under H. Specifically, prove that H is a bijection between $\mathcal{C}_A$ and $\mathcal{C}_B$ such that, for all $C, C' \in \mathcal{C}_A$, $C \subseteq C'$ iff $H(C) \supseteq H(C')$.

Math 589, Spring 2002 **Problem Set 3** **Due March 13 , 2002**

Corrected May 7, 2002

1. [Pixley] A class of Σ-algebras K is said to be *arithmetical* if it is both congruence-permutable and congruence-distributive. Prove that for any variety V the following are equivalent.
   (a) V is arithmetical.
   (b) There is a $t(x, y, z) \in \mathrm{Te}_\Sigma(x, y, z)$ such the following identities hold in V.

$$t(x, y, x) \approx x, \qquad t(x, x, y) \approx t(y, x, x) \approx y.$$

2. A Σ-algebra $\boldsymbol{A}$ is *primal* if every operation on the universe A of $\boldsymbol{A}$ is a term function, i.e., for every $n \in \omega$ and every $h\colon A^n \to A$, there is a Σ-term $t(x_0, \dots, x_{n-1})$ such that, for all $a_0, \dots, a_{n-1} \in A^n$, $h(a_0, \dots, a_{n-1}) = t^{\boldsymbol{A}}(a_0, \dots, a_{n-1})$.
   (a) Prove that the variety generated by a primal algebra is arithmetical.
   (b) Prove that, for any prime p, the prime field $\boldsymbol{\mathbb{Z}}_p = \langle \mathbb{Z}_p, +, \cdot, -, 0, 1\rangle$ is primal.

3. Assume $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and $\alpha \in \mathrm{Co}(\boldsymbol{A})$. Prove that $h(\alpha) \in \mathrm{Co}(\boldsymbol{B})$ iff

$$\alpha \mathbin{;} \mathrm{rker}(h) \mathbin{;} \alpha \subseteq \mathrm{rker}(h) \mathbin{;} \alpha \mathbin{;} \mathrm{rker}(h).$$

   [*Note:* One of the two implications was proved in class. Prove the other one.]

4. Let $\boldsymbol{A}$ be a Σ-algebra. Prove that the following are equivalent.
   (a) For every Σ-algebra $\boldsymbol{B}$, for every $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and for every $\alpha \in \mathrm{Co}(\boldsymbol{A})$, $h(\alpha) \in \mathrm{Co}(\boldsymbol{B})$.
   (b) $\boldsymbol{A}$ is congruence 3-permutable, i.e., for all $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$, $\alpha \mathbin{;} \beta \mathbin{;} \alpha = \beta \mathbin{;} \alpha \mathbin{;} \beta$.

5. Assume V is a locally finite, congruence distributive variety, and that K and L are subvarieties of V. Let $\mathsf{K} \vee \mathsf{L}$ be the join of K and L in the lattice of subvarieties of V, i.e., $\mathsf{K} \vee \mathsf{L} = \mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K} \cup \mathsf{L})$. Prove that every finite, subdirectly irreducible member of $\mathsf{K} \vee \mathsf{L}$ is either in K or in L.

Math 589, Spring 2002 **Problem Set 4** **Due April 15, 2002**

Corrected May 7, 2002

1. Let V be a variety with EDPM and let $\boldsymbol{A} \in V$.

   (a) Prove that, for all $a, b, c, d, e, f \in A$ and every $\alpha \in \mathrm{Co}(\boldsymbol{A})$,

   $$\Theta_{\boldsymbol{A}}(c,d) \subseteq \Theta_{\boldsymbol{A}}(a,b) \vee \alpha \quad \text{implies} \quad \big(\Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f)\big) \subseteq \big(\Theta_{\boldsymbol{A}}(a,b) \cap \Theta_{\boldsymbol{A}}(e,f)\big) \vee \alpha.$$

   (b) Prove that, for all $a_1, b_1, \ldots, a_n, b_n, c, d, e, f \in A$,

   $$\Theta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i \le n} \Theta_{\boldsymbol{A}}(a_i, b_i) \quad \text{implies} \quad \Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i \le n} \big(\Theta_{\boldsymbol{A}}(a_i, b_i) \cap \Theta_{\boldsymbol{A}}(e,f)\big).$$

   [*Hint:* Part (a): Let $\boldsymbol{B} = \boldsymbol{A}/\alpha$ and $\Delta_\alpha \colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ be that natural map; let $\bar{x} = x/\alpha$ for each $x \in A$. Show that $\Delta^*_\alpha\big(\Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f)\big) = \Theta_{\boldsymbol{B}}(\bar{c}, \bar{d}) \cap \Theta_{\boldsymbol{B}}(\bar{e}, \bar{f})$. Use this to show that $\Theta_{\boldsymbol{B}}(\bar{c}, \bar{d}) \cap \Theta_{\boldsymbol{B}}(\bar{e}, \bar{f}) \subseteq \Theta_{\boldsymbol{B}}(\bar{a}, \bar{b}) \cap \Theta_{\boldsymbol{B}}(\bar{e}, \bar{f})$. Now apply $(\Delta^*_\alpha)^{-1}$ and use Lemma 5.14. Prove part (b) by induction on n using part (a).]

2. Prove that every variety with EDPM is congruence distributive.

   [*Hint:* Let V a variety with EDPM. It suffices to prove that, for every $\boldsymbol{A} \in V$ and all $\alpha, \beta, \gamma \in \mathrm{Co}(\boldsymbol{A})$,

   $$\alpha \cap (\beta \vee \gamma) \subseteq (\alpha \cap \beta) \vee (\alpha \cap \gamma). \tag{1}$$

   Use the first problem to prove that if $\{\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle\}$ is a finite subset β and $\{\langle e_1, f_1\rangle, \ldots, \langle e_m, f_m\rangle\}$ is a finite subset γ, then for all $a, b \in A$,

   $$\begin{aligned} &\Theta_{\boldsymbol{A}}(a,b) \subseteq \Big(\bigvee_{i \le n} \Theta_{\boldsymbol{A}}(c_i, d_i)\Big) \vee \Big(\bigvee_{j \le m} \Theta_{\boldsymbol{A}}(e_j, f_j)\Big) \\ &\quad \text{implies} \quad \Theta_{\boldsymbol{A}}(a,b) \subseteq \Big(\bigvee_{i \le n} \big(\Theta_{\boldsymbol{A}}(c_i, d_i) \cap \Theta_{\boldsymbol{A}}(a,b)\big)\Big) \vee \Big(\bigvee_{j \le m} \big(\Theta_{\boldsymbol{A}}(e_j, f_j) \cap \Theta_{\boldsymbol{A}}(a,b)\big)\Big). \end{aligned}$$

   Use this to prove (1).]

3. Let Φ be a set of UDE's. Recall that $\mathbf{P}_{\mathrm{U}}(\mathsf{K})$ is the class of all isomorphic images of ultraproducts of systems of algebras in K.

   (a) Prove that $\mathbf{P}_{\mathrm{U}}\,\mathsf{Mod}(\Phi) = \mathsf{Mod}(\Phi)$.

   (b) Use part (a) to obtain short proof of Theorem 6.4 from Joństson's Lemma (Theorem 5.25). Recall that Theorem 6.4 says that, for any congruence-distributive variety V,

   $$\mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{Mod}(\Phi) \cap \mathsf{V}) = \mathsf{Mod}_{\mathrm{prim}}(\Phi) \cap \mathsf{V}.$$

[*Hint:* Part (a) is a very easy consequence of the first problem on Problem Set #2. The key to part (b) is to show, as in the proof of Theorem 6.4, that $\mathsf{Mod}_{\text{prim}}(\Phi) \cap \mathsf{V}$ is variety. Jonsson's Lemma can be used to get a simpler proof of this than the one used in the proof of Theorem 6.4.]

(over)

4. An equation is said to be *absorbing* if it is of the form $t(x_0,\dots,x_{n-1}) \approx x_i$ for some $i<n$. For example, $(x\cdot y)\cdot y^{-1} \approx x$ is an absorbing equation. Prove that if a set E has no absorbing equations, then E is consistent.

   [*Hint:* You can either argue directly that E has a nontrivial model, or you can obtain this indirectly by showing the $E \nvdash x \approx y$; to show the latter you may find it convenient to use the relation $\equiv^*_E$, in particular Theorem 4.15 from the notes for week 4 on the class webpage.]

5. Let V be a variety with a finite signature Σ. For each $n \in \omega$, let $\mathrm{Id}_n(V)$ be the set of all identities of V that contain at most n distinct variables, i.e., $\mathrm{Id}_n(V) = \mathrm{Id}(V) \cap \mathrm{Te}_\Sigma(\{x_0,\dots,x_{n-1}\})^2$. Let $V_n = \mathsf{Mod}\big(\mathrm{Id}_n(V)\big)$.

   (a) Prove that, if V is locally finite, then V_n is finitely based for every $n \in \omega$

   Since $\mathrm{Id}_0(V) \subseteq \mathrm{Id}_1(V) \subseteq \mathrm{Id}_2(V) \subseteq \cdots \subseteq \mathrm{Id}(V)$ and $\mathrm{Id}(V) = \bigcup_{n\in\omega} \mathrm{Id}_n(V)$, we have $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V$ and $V = \bigcap_{n\in\omega} V_n$.

   (b) Prove that if V is finitely based, then $V = V_n$ for some $n \in \omega$, and that the converse holds if V is locally finite.

   [*Hint:* Part (a): Let $\hat{x} = \langle x_0,\dots,x_{n-1}\rangle$. Show that there is a system $t_1(\hat{x}),\dots,t_m(\hat{x})$ of terms in n-variables such that for every term $s(\hat{x})$ there is an $i \le m$ such that $s(\hat{x}) \approx t_i(\hat{x})$ is an identity of V_n (i.e., of V); without loss of generality one can assume that the first n terms in this sequence are the variables $x_0,\dots,x_{n-1}$. To show such a sequence exists use the fact that, since V is locally finite, the free algebra $\mathbf{Fr}_n(V)$ over V is finite. It follows that, for every $\sigma \in \Sigma_k$ and every sequence $t_{i_1}(\hat{x}),\dots,t_{i_k}(\hat{x})$ there exists a $t_j(\hat{x})$ such that $\sigma\big(t_{i_1}(\hat{x}),\dots,t_{i_k}(\hat{x})\big) \approx t_j(\hat{x})$ is an identity of V_n. Show that the set of all such equations is a finite base for V_n.]

Math 589, Spring 2002 **Problem Set 4 (Addtional Hints)**

1. (a) Use the following exercise that was given after Lemma 5.14 and eventually proved in class: For any epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and any $X \subseteq A^2$,

$$h^*\big(\Theta_{\boldsymbol{A}}(X)\big) = \Theta_{\boldsymbol{B}}\big(h(X)\big).$$

You will also have to use part (3) of Lemma 5.14 (I said 5.15 in the original hint, but this was a mistake) 5.14(3) says that for any epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and every $\alpha \in \mathrm{Co}(\boldsymbol{A})$,

$$h^{-1}h^*(\alpha) = \alpha \vee \mathrm{rker}(h).$$

(b) This part is proved by induction on n, but it is not a straightforward induction. First of all you do have to prove the stronger result

$$\Theta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i\le n} \Theta_{\boldsymbol{A}}(a_i,b_i)\vee\alpha \quad \text{implies} \quad \Theta_{\boldsymbol{A}}(c,d)\cap\Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\le n}\big(\Theta_{\boldsymbol{A}}(a_i,b_i)\cap\Theta_{\boldsymbol{A}}(e,f)\big)\vee\alpha.$$

By the induction hypothesis we get

$$\Theta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i\le n-1} \Theta_{\boldsymbol{A}}(a_i,b_i) \vee \big(\Theta_{\boldsymbol{A}}(a_n,b_n)\vee\alpha\big)$$

$$\text{implies} \quad \Theta_{\boldsymbol{A}}(c,d)\cap\Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\le n-1} \big(\Theta_{\boldsymbol{A}}(a_i,b_i)\cap\Theta_{\boldsymbol{A}}(e,f)\big) \vee \big(\Theta_{\boldsymbol{A}}(a_n,b_n)\vee\alpha\big).$$

Now consider any set $X \subset A^2$ such that $\Theta_{\boldsymbol{A}}(X) = \Theta_{\boldsymbol{A}}(c,d)\cap\Theta_{\boldsymbol{A}}(e,f)$ (can take X to be all of $\Theta_{\boldsymbol{A}}(c,d)\cap\Theta_{\boldsymbol{A}}(e,f)$). Use part (a) to prove that, for each $\langle x,y\rangle \in X$,

$$\Theta(x,y) \subseteq \bigvee_{i\le n-1} \big(\Theta_{\boldsymbol{A}}(a_i,b_i)\cap\Theta_{\boldsymbol{A}}(e,f)\big) \vee \big((\Theta_{\boldsymbol{A}}(a_n,b_n)\cap\Theta_{\boldsymbol{A}}(e,f))\vee\alpha\big).$$

Finally, use this to get

$$\Theta_{\boldsymbol{A}}(c,d)\cap\Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\le n}\big(\Theta_{\boldsymbol{A}}(a_i,b_i)\cap\Theta_{\boldsymbol{A}}(e,f)\big).$$

3. (b) By Birkhoff's Subdirect Product Theorem, for every Σ-algebra $\boldsymbol{A}$ we have $A \cong ; \subseteq_{\mathrm{SD}} \prod_{i\in I}\boldsymbol{B}_i$ where the $\boldsymbol{B}_i$ are subdirectly irreducible (SDI). The algebras $\boldsymbol{B}_i$, $i \in I$, are called the SDI *factors* of $\boldsymbol{A}$. For any class K of Σ-algebras, let $\mathbf{F}_{\mathrm{SDI}}(\mathsf{K})$ be the class of SDI factors of all $\boldsymbol{A} \in \mathsf{K}$. Prove that, for any class K is Σ-algebras,

$$\mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K}) = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathbf{F}_{\mathrm{SDI}}(\mathsf{K}).$$

The following facts were all either established in the lectures at various places, or they are easy consequences of results that were established. You may use them in the proof of part (b) without justifying them.

$$\mathbf{F}_{\mathrm{SDI}}\big(\mathsf{Mod}_{\mathrm{prim}}(\Phi)\cap\mathsf{V}\big) \subseteq \mathsf{Mod}(\Phi)\cap\mathsf{V}$$
$$\mathbf{H}\,\mathbf{S}\big(\mathsf{Mod}(\Phi)\big) \subseteq \mathsf{Mod}(\Phi)$$
$$\mathbf{P}_{\mathrm{SD}}\big(\mathsf{Mod}(\Phi)\cap V\big) \subseteq \mathsf{Mod}_{\mathrm{prim}}(\Phi)\cap V.$$

Use the the above facts together with Jońsson's Lemma to show that $\mathbf{H}\,\mathbf{S}\,\mathbf{P}\big(\mathsf{Mod}_{\mathrm{prim}}(\Phi)\cap V\big) \subseteq \mathsf{Mod}_{\mathrm{prim}}(\Theta)\cap K$. Part (b) now follows as in the proof of Thm. 6.4.

Math 589, Spring 2002 **Problem Set 5** **Due May 8, 2002**

Corrected May 7, 2002

1. Prove the Ordered Second Isomorphism Theorem: Let $\boldsymbol{A}^{\leq}$ be be a poalgebra and $\alpha, \beta \in \operatorname{Qord}(\boldsymbol{A}^{\leq})$ such that $\beta \subseteq \alpha$. Define

$$\alpha/\beta = \alpha/(\beta \cap \breve{\beta}) = \left\{ \langle a/\beta \cap \breve{\beta},\ b/\beta \cap \breve{\beta} \rangle : a \,\alpha\, b \right\}.$$

Prove that α/β is a quasi-order of $\boldsymbol{A}^{\leq}/\beta$ and that $\boldsymbol{A}^{\leq}/\alpha \cong (\boldsymbol{A}^{\leq}/\beta)/(\alpha/\beta)$.

2. Let $h\colon \boldsymbol{A}^{\leq} \to \boldsymbol{B}^{\leq}$ be an order homomorphism of ordered algebras, and let α be the order-kernel of h Prove that there is an ordered subalgebra $\boldsymbol{C}^{\leq}$ of $\boldsymbol{B}^{\leq}$ such that $\boldsymbol{A}^{\leq}/\alpha \cong \boldsymbol{C}^{\leq}$.

[*Note:* This result was used at a couple places in the with the remark that it follows from the ordered homomorphism theorem; this it true, but the proof is not trivial.]

3. A *semiring* is an algebra $\langle A, +, . \rangle$ such that $\langle A, + \rangle$ is a commutative semigroup, $\langle A, . \rangle$ is a semigroup, and $\cdot$ distributives over $+$. A *partially ordered semiring* is a poalgebra $\langle \boldsymbol{A}, \leq \rangle$ such that $\boldsymbol{A}$ is a semiring. Prove that the class K of partially ordered semirings form an ordered variety, i.e., find a set E of inequations and prove that $\mathsf{K} = \mathsf{Mod}(E)$

An ordered variety V is *algebraizable* if there is a finite set of inequations $t_i(x, y) \preccurlyeq s_i(x, y)$, $i = 1, \ldots, n$, such that, for every $\boldsymbol{A}^{\leq} \in \mathsf{V}$ and for all $a, b \in A$,

$$a \leq^{\boldsymbol{A}} b \quad \text{iff} \quad \forall i \leq n \left(t_i^{\boldsymbol{A}}(a, b) \leq^{\boldsymbol{A}} s_i^{\boldsymbol{A}}(a, b) \text{ and } s_i^{\boldsymbol{A}}(a, b) \leq^{\boldsymbol{A}} t_i^{\boldsymbol{A}}(a, b) \right).$$

The ordered variety of lattices is algebraizable; take $n = 1$ and $t_1(x, y) \preccurlyeq s_1(x, y)$ to be $x \vee y \preccurlyeq y$.

4. Let V be an algebraizable ordered variety. Prove that, for every $\boldsymbol{A} \in \mathsf{V}$, the mapping $\alpha \mapsto \alpha \cap \breve{\alpha}$ from $\operatorname{Qord}(\boldsymbol{A}^{\leq})$ to $\operatorname{Co}(\boldsymbol{A})$ is injective. Use this to show that the ordered variety of semirings is not algebraizable.

Most of the definitions and theorems about poalgebras are exact analogues of those about unordered algebras, but there are some exceptions as the next problem shows.

5. Let V be an algebraizable ordered variety. Prove that, if $\boldsymbol{A}^{\leq}, \boldsymbol{B}^{\leq} \in \mathsf{V}$, then an order homomorphism $h\colon \boldsymbol{A}^{\leq} \to \boldsymbol{B}^{\leq}$ is an order isomorphism iff h is a bijection. Show by example that this may not be true if V is not algebraizable (take V to be the class of partially ordered semirings).