Lattice Theory Homework Problems

Problem 0.
Lattices can be defined as ordered sets of the form $\langle L, \leq \rangle$ or as algebras of the form $\langle L, \vee, \wedge \rangle$. Work out the details that show these two ways to define the notion of lattice are equivalent.

Problem 1.
Let $\mathbf{L}$ be a lattice and let $\theta, \varphi \in \operatorname{Con} \mathbf{L}$. Prove that $\theta \vee \varphi = \theta \cup \theta \circ \varphi \cup \theta \circ \varphi \circ \theta \cup \theta \circ \varphi \circ \theta \circ \varphi \cup \dots$.

Problem 2.
Consider the Galois connection established by a binary relation $R \subseteq A \times B$. Prove each of the following for all $X, X_0, X_1 \subseteq A$ and all $Y, Y_0, Y_1 \subseteq B$:

i. $X \subseteq X^{\rightarrow\leftarrow}$ and $Y \subseteq Y^{\leftarrow\rightarrow}$.
ii. $X_0 \subseteq X_1 \Rightarrow X_1^{\rightarrow} \subseteq X_0^{\rightarrow}$.
iii. $Y_0 \subseteq Y_1 \Rightarrow Y_1^{\rightarrow} \subseteq Y_0^{\rightarrow}$.
iv. $X^{\rightarrow\leftarrow\rightarrow} = X^{\rightarrow}$ and $Y^{\leftarrow\rightarrow\leftarrow} = Y^{\leftarrow}$.

Problem 3.
Let Γ be an algebraic closure operator and let $\mathbf{L}$ be the resulting lattice of closed sets. Prove that $\mathbf{L}$ is an algebraic lattice and that the compact elements of this lattice are exactly the closures of finite sets.

Problem 4.
Prove that any modular lattice has a type 2 representation.

Problem 5.
Let $\mathbf{L}$ be a lattice. Prove that the following are equivalent:

.a The equation $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$ holds in $\mathbf{L}$.
.b The equation $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ holds in $\mathbf{L}$.
.c The equation $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ holds in $\mathbf{L}$.
.d $\mathbf{L}$ has no sublattice isomorphic with $\mathbf{N}_5$ and no sublattice isomorphic with $\mathbf{M}_3$.