Ph.D. Comprehensive Examination in Algebras, Lattices, Varieties
August 1993

Problem 0.
Describe a class $\mathcal{K}$ of similar algebras such that $\mathbf{HS}\mathcal{K} \neq \mathbf{SH}\mathcal{K}$.

Problem 1.
An algebra $\mathbf{A}$ is said to have *the Chinese Remainder Property* provided for any finite sequence $\theta_0, \theta_1, \ldots, \theta_{n-1}$ of congruences of $\mathbf{A}$, and any $a_0, a_1, \ldots, a_{n-1} \in A$, if every pair $x\theta_i a_i$ and $x\theta_j a_j$ has a solution, then there is a simultaneous solution $x \in A$ such that $x\theta_i a_i$ for all $i < n$. Prove that an algebra $\mathbf{A}$ has the Chinese Remainder Property if and only if $\theta \cap (\phi \circ \psi) \subseteq (\theta \cap \phi) \circ (\theta \cap \psi)$ for all congruences $\phi, \psi, \theta \in \mathbf{Con\ A}$.

Problem 2.
Let $\mathcal{B}$ be the variety of Boolean algebras, where the fundamental operations are join, meet, and complementation. Find all the binary terms $t(x, y)$, such that $t(x, y)$ generates the clone of all term operations of $\mathcal{B}$. Another way to frame this problem is to find all the ways in which $\mathcal{B}$ can be term equivalent to a variety of groupoids.

Problem 3.
Construct an algebra $\mathbf{A}$ with just one basic operation such that the basic operation is essentially binary (it depends on both variables) but so that no term operation of $\mathbf{A}$ depends on more than two variables.

Problem 4.
Let $\mathbf{A}$ be an algebra and let $\theta \in \operatorname{Con}\mathbf{A}$. Recall that θ is said to be a *factor congruence* provided there is $\phi \in \operatorname{Con}\mathbf{A}$ so that θ and ϕ permute, $\theta \cap \phi = O_{\mathbf{A}}$, and $\theta \vee \phi = 1_{\mathbf{A}}$. Now suppose that $\mathbf{A}$ is congruence distributive. Prove that the factor congruences of $\mathbf{A}$ permute with all the congruences of $\mathbf{A}$, and that the factor congruences constitute a sublattice of the lattice of all congruences of $\mathbf{A}$.

Problem 5.
Call a lattice *sectionally complemented* provided it has a least element 0 and for every $a > 0$ the interval $[0, a]$ is complemented. Let $\mathbf{L}$ be a finite distributive lattice. Prove that $\operatorname{Con}\mathbf{L}$ is sectionally complemented. Hint: First consider the case for principal congruences.

Problem 6.
Let $\mathbf{L}$ be a complemented, atomic modular lattice. Prove that $\mathbf{L}$ is subdirectly irreducible if and only if for each pair p_0 and p_1 of distinct atoms, there is a third atom p_2 distinct from p_0 and p_1 such that $p_0 \vee p_1 \geq p_2$. Hint: First verify that $\mathbf{L}$ is subdirectly irreducible iff any two coverings in $\mathbf{L}$ are projective. It may also be helpful to prove that the binary relation on the atoms imposed above between p_0 and p_1 is an equivalence relation.