Homework Problems
Some Even with Solutions

Problem 0
Let A be a nonempty set and let Q be a finitary operation on A. Prove that the rank of Q is unique.

To say that n is the rank of Q is to assert that $Q : A^n \to A$. So suppose that n and m are natural numbers and Q has rank m as well as rank n. We need to prove that $n = m$.

It follows that $A^n = A^m$ since both A^n and A^m turn out to be the domain of Q. Now in general A^X is the set of all functions from X into A. Recalling that $n = \{0, 1, \ldots, n-1\}$ and $m = \{0, 1, \ldots, m-1\}$ we can understand more clearly what A^n and A^m are.

Now, using that A is not empty, pick $a \in A$. The function with domain n having a as its constant value is just $\{(0, a), (1, a), \ldots, (n-1, a)\} \in A^n$. Since $A^n = A^m$ we see that for some $b_0, b_1, \ldots, b_{m-1} \in A$ we have

$$\{(0, a), (1, a), \ldots, (n-1, a)\} = \{(0, b_0), (1, b_1), \ldots, (n-1, b_{m-1})\}.$$

But the set on the left has exactly n distinct elements while the set on the right has exactly m distinct elements. Therefore $n = m$, as desired.

Problem 1
Construct a semigroup that cannot be expanded to a monoid.

To say that a semigroup cannot be expanded to a monoid is the same as saying that it has no element that can play the role of the identity. For example the even integers equipped with multiplication constitutes a semigroup that cannot be expanded to a monoid.

Problem 2
Construct a semigroup that is not the multiplicative semigroup of any ring.

This is a bit harder. In our understanding of the notion of ring we require that our ring have both a zero and a one. So a cheap answer is to use the solution to Problem 1. A little bit more sophisticated example would be to take the positive integers under multiplication. Then any attempt to add some weird "addition" (it need not be the one you know and love) must lead to zero element, say ζ with the property that $\zeta n = n\zeta = \zeta$ for all positive integers n. There is no such positive integer ζ so there is no weird addition that would make this semigroup into a ring.

But what about an example of a semigroup that has both a 0 and a 1 but which cannot be expanded to a ring? Here is one such.

On the five element set $\{1, 2, 3, 4, 5\}$ impose a product $*$ that gives the cyclic group of order 5 whose identity element is 1. So this gives a nice semigroup with a unit. Still it has no zero. So throw in 0 and extend the operation $*$ by declaring

$$0 * x = x * 0 = 0$$

for all $x \in \{0, 1, 2, 3, 4, 5\}$. You should check that this extended binary operation $*$ is still associative. Now we have in fact a six element commutative semigroup with a zero and a one. Moreover, every nonzero element has an inverse. If we could throw on a wierd addition and produce a ring then we would have in fact a field with six elements. But the size on any finite field must be the power of a prime. Six is of course the smallest positive natural number that is not the power of some prime.

We could argue more directly. Imagine we a found a wierd addition $\oplus$ that gives us a ring. Then the additive reduct of this ring would be an Abelian group with six elements. Up to isomorphism, there is only one such group: the cyclic group of order six. It follows that $1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 = 0$. But also

$1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 = (1 \oplus 1) * (1 \oplus 1 \oplus 1)$. Since we have an integeral domain, it follows that either $1 \oplus 1 = 0$ or $1 \oplus 1 \oplus 1 = 0$. That is 1 has order either 2 or 3 in the additive group of our ring. But then, since $a * (1 \oplus 1) = a \oplus a$ and $a * (1 \oplus 1 \oplus 1) = a \oplus a \oplus a$ for all elements a. Then every nonzero element must have order either 2 or 3. But remember, our additive group is the cyclic group of order 6. Therefore, there must be an element of order 6. This is a contradiction.

Problem 3

Let A be a set and denote by $\mathbf{Eqv}\,A$ the set of all equivalence realtions on A. For $R, S \in \mathbf{Eqv}\,A$ define

$$R \wedge S = R \cap S$$
$$R \vee S = R \cup R \circ S \cup R \circ S \circ R \cup R \circ S \circ R \circ S \cup \dots$$

where $\circ$ stands for the relational product (that is $a(R \circ S)b$ means that there is some $c \in A$ sucn that both aRc and cSb). Prove that $\langle \mathbf{Eqv}\,A, \wedge, \vee \rangle$ is a lattice.

There are several ways to proceed. We could demonstrate all the equations which axiomatize lattices. The only ones that present any challenge are the associative law for $\vee$ and the two absorption laws. Another approach is to show that $\subseteq$ is a lattice order of $\mathbf{Eqv}\,A$ and that $\wedge$ and $\vee$ are the greatest lower bound and the least upper bound with respect to this order. This is the approach taken here.

CLAIM 1. *If $R, S \in \mathbf{Eqv}\,A$, then $R \wedge S \in \mathbf{Eqv}\,A$.*

Proof. We need to show that $R \cap S$ is reflexive, symmetric, and transitive. Let $a \in A$. Since R and S are reflexive, we know $\langle a, a \rangle \in R$ and $\langle a, a \rangle \in S$. So $\langle a, a \rangle \in R \cap S$ and $R \cap S$ is reflexive. For symmetry, suppose $\langle a, b \rangle \in R \cap S$. Then $\langle a, b \rangle \in R$ and $\langle a, b \rangle \in S$. Since R and S are symmetric, we see $\langle b, a \rangle \in R$ and $\langle b, a \rangle \in S$. Therefore $\langle b, a \rangle \in R \cap S$ and so $R \cap S$ is symmetric. Finally, to establish transitivity observe

$$\begin{aligned} \langle a.b \rangle \in R \cap S \text{ and } \langle b, c \rangle \in R \cap S &\implies \langle a, b \rangle, \langle b, c \rangle \in R \text{ and } \langle a, b \rangle, \langle b, c \rangle \in S \\ &\implies \langle a, c \rangle \in R \text{ and } \langle a, c \rangle \in S \\ &\implies \langle a, c \rangle \in R \cap S \end{aligned}$$

■

Since $R \cap S$ is the largest *set* included in both R and S, it follows from the claim above that it must be the largest member of $\mathbf{Eqv}\,A$ included in both R and S.

Dealing with the join is more trouble.

CLAIM 2. *If $R, S, T \in \mathbf{Eqv}\,A$ and $R \subseteq T$ and $S \subseteq T$, then $R \vee S \subseteq T$.*

Proof. First observe that if P and Q are any binary relations on A and $P \cup Q \subseteq T$, then $P \circ Q \subseteq T$. This is a simple consequence of the transitivity of T. Then by a straightforward induction we have $R \circ S \circ R \circ S \circ \dots \circ R \circ S \subseteq T$ and also the similar inclusion ending in R. Therefore $R \vee S \subseteq T$ as desired. ■

In view of this last claim, we would be finished once we show that $R, S \subseteq R \vee S$ and $R \vee S$ is an equivalence relation on A. Of course $R \subseteq R \vee S$ by the definition of the join. But observe that $S \subseteq R \circ S$, since R is reflexive. Thus we only need the next claim.

CLAIM 3. *If $R, S \in \mathbf{Eqv}\,A$, then $R \vee S \in \mathbf{Eqv}\,A$.*

Proof. $R \vee S$ is reflexive since R is reflexive and $R \subseteq R \vee S$. To see symmetry it helps to have a little notation. For any binary relation P let

$$P^{\smile} = \{\langle b, a\rangle \mid \langle a, b\rangle \in P\}.$$

The relation $P^{\smile}$ is called the **converse** of the relation P. The statement that P is symmetric is just that $P = P^{\smile}$. Now if P and Q are binary relations, it is not hard to see that $(P \circ Q)^{\smile} = Q^{\smile} \circ P^{\smile}$. Now observe that

$$\begin{aligned}(R \circ S \circ R \circ \cdots \circ R \circ S)^{\smile} &= S^{\smile} \circ R^{\smile} \circ \cdots \circ R^{\smile} \circ S^{\smile} \circ R^{\smile}\\ &= S \circ R \circ \cdots \circ R \circ S \circ R\\ &\subseteq R \circ S \circ R \circ \cdots \circ R \circ S \circ R \subseteq R \vee S\end{aligned}$$

The converses drop out at the next-to-last line since R and S are symmetric. The first inclusion on the last line is a consequence of the reflexivity of R. Since the converse of each piece in the union that defines $R \vee S$ is included in $R \vee S$, it follows that $R \vee S$ is symmetric.

Finally we have to establish the transitivity of $R \vee S$. It is an immediate consequence of reflexivity and transitivity that $R \circ R = R$ and $S \circ S = S$. From the associativity of composition we see

$$\begin{aligned}(R \circ S \circ \cdots \circ S \circ R) \circ (R \circ S \circ \cdots \circ S \circ R \circ S) &= R \circ S \circ \cdots \circ S \circ R \circ R \circ S \circ \cdots \circ S \circ R \circ S\\ &= R \circ S \circ \cdots \circ S \circ R \circ S \circ \cdots \circ S \circ R \circ S \subseteq R \vee S\end{aligned}$$

The transitivity of $R \vee S$ follows from considering $\langle a, b\rangle, \langle b, c\rangle \in R \vee S$. Both ordered pairs must belong to (perhaps different) pieces that make up $R \vee S$ as a union. The display above shows why $\langle a, c\rangle \in R \vee S$. ■

PROBLEM 4
Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. Prove

$$\hom(\mathbf{A}, \mathbf{B}) = (\operatorname{Sub} \mathbf{A} \times \mathbf{B}) \cap \{h \mid h \text{ is a function from } A \text{ into } B\}$$

$$
\begin{aligned}
g \in \hom(\mathbf{A},\mathbf{B}) \Leftrightarrow & g \in \{h \mid h \text{ is a function from } A \text{ into } B\} \text{ and} \\
& g(Q^{\mathbf{A}}(a_0,\dots,a_{r-1}) = Q^{\mathbf{B}}(g(a_0),\dots,g(a_{r-1})) \\
& \text{for all operation symbols } Q \text{ and} \\
& \text{all } a_0,\dots,a_{r-1} \in A \text{ where } r \text{ is the rank of } Q \\
\Leftrightarrow & g \in \{h \mid h \text{ is a function from } A \text{ into } B\} \text{ and} \\
& \langle Q^{\mathbf{A}}(a_0,\dots,a_{r-1}), Q^{\mathbf{B}}(g(a_0),\dots,g(a_{r-1}))\rangle \in g \\
& \text{for all operation symbols } Q \text{ and} \\
& \text{all } a_0,\dots,a_{r-1} \in A \text{ where } r \text{ is the rank of } Q \\
\Leftrightarrow & g \in \{h \mid h \text{ is a function from } A \text{ into } B\} \text{ and} \\
& Q^{\mathbf{A}\times\mathbf{B}}(\langle a_0, g(a_0)\rangle,\dots,\langle a_{r-1}, g(a_{r-1})\rangle) \in g \\
& \text{for all operation symbols } Q \text{ and} \\
& \text{all } a_0,\dots,a_{r-1} \in A \text{ where } r \text{ is the rank of } Q \\
\Leftrightarrow & g \in \{h \mid h \text{ is a function from } A \text{ into } B\} \text{ and} \\
& Q^{\mathbf{A}\times\mathbf{B}}(\langle a_0, b_0\rangle,\dots,\langle a_{r-1}, b_{r-1}\rangle) \in g \\
& \text{for all operation symbols } Q \text{ and} \\
& \text{all } a_0,\dots,a_{r-1} \in A \text{ and all } b_0,\dots,b_{r-1} \in B \\
& \text{such that } \langle a_0, b_0\rangle,\dots,\langle a_{r-}, b_{r-1}\rangle \in g, \\
& \text{where } r \text{ is the rank of } Q \\
\Leftrightarrow & g \in \{h \mid h \text{ is a function from } A \text{ into } B\} \text{ and} \\
& g \text{ is a subuniverse of } \mathbf{A}\times\mathbf{B} \\
\Leftrightarrow & g \in (\operatorname{Sub}\mathbf{A}\times\mathbf{B}) \cap \{h \mid h \text{ is a function from } A \text{ into } B\}
\end{aligned}
$$

PROBLEM 5
Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I\rangle$ be a system of similar algebras. Prove that each projection function on $\prod \mathbf{A}$ is a homomorphism.

Recall that the projection function p_j is defined so that

$$p_j(\langle a_i \mid i \in I\rangle) = a_j$$

for all $\langle a_i \mid i \in I\rangle \in \prod A$. We need to show that p_j preserves the operations. So let Q be an operation symbol and, wlog, suppose that Q has rank 3. Then let $a = \langle a_i \mid i \in I\rangle, b = \langle b_i \mid i \in I\rangle, c = \langle c_i \mid i \in I\rangle$. Just observe

$$
\begin{aligned}
p_j(Q^{\prod \mathbf{A}}(a,b,c) &= p_j(\langle Q^{\mathbf{A}_i}(a_i,b_i,c_i) \mid i \in I\rangle \\
&= Q^{\mathbf{A}_j}(a_j,b_j,c_j) \\
&= Q^{\mathbf{A}_j}(p_j(a),p_j(b),p_j(c))
\end{aligned}
$$

PROBLEM 6
Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I\rangle$ be a system of similar algebras. Further, assume $\mathbf{B}$ is an algebra of the same signature and that $B = \prod A$. Prove that if each projection function on B is a homomorphism, then $\mathbf{B} = \prod \mathbf{A}$.

Since $\mathbf{B}$ and $\prod \mathbf{A}$ have the same universe, it only remains to show that they also have the same operations. So let Q be an operation symbols. Let's suppose that its rank is 3. Then let $a = \langle a_i \mid i \in I\rangle, b = \langle b_i \mid i \in I\rangle, c = \langle c_i \mid i \in I\rangle$. Now notice that for all $j \in I$

$$p_j(Q^{\mathbf{B}}(a,b,c)) = Q^{\mathbf{A}_j}(a_j, b_j, c_j) = p_j(Q^{\prod \mathbf{A}}(a,b,c).$$

Thus, $Q^{\mathbf{B}}(a,b,c)$ and $Q^{\prod \mathbf{A}}(a,b,c)$ agree at each coordinate. Therefore they are identical. Consequently, $\mathbf{B} = \prod \mathbf{A}$.

Problem 7

Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I\rangle$ be a system of similar algebras. Let $\mathbf{B}$ be an algebra of the same signature and let h_i be a homomorphism from $\mathbf{B}$ into $\mathbf{A}_i$, for each $i \in I$. Prove that there is a homomorphism g from $\mathbf{B}$ into $\prod \mathbf{A}$ such that $h_i = p_i \circ g$ for all $i \in I$. (Here p_i denotes the i^{th} projection function.

The definition of g is forced on us by the wish that $h_i = p_i \circ g$. That is $g(b) = \langle h_i(b) \mid i \in I\rangle$ for all $b \in B$. We need only show that this g is a homomorphism. So let Q be an operation symbol and suppose it has rank, say 3. Let $a, b, c \in B$. Then

$$\begin{aligned} g(Q^{\mathbf{B}}(a,b,c)) &= \langle h_i(Q^{\mathbf{B}}(a,b,c)) \mid i \in I\rangle \\ &= \langle Q^{\mathbf{A}_i}(h_i(a), h_i(b), h_i(b)) \mid i \in I\rangle \\ &= Q^{\prod \mathbf{A}}(\langle h_i(a) \mid i \in I\rangle, \langle h_i(b) \mid i \in I\rangle, \langle h_i(c) \mid i \in I\rangle) \\ &= Q^{\prod \mathbf{A}}(g(a), g(b), g(c)) \end{aligned}$$

Problem 8

Let $\mathbf{A}$ be an algebra. Prove

$$\text{Con } \mathbf{A} = (\text{Sub}\, \mathbf{A} \times \mathbf{A}) \cap \{\theta \mid \theta \text{ is an equivalence relation on } A\}.$$

This is similar to Problem 4. To simplify matters this time, suppose that $\theta \in \mathbf{Eqv}\, A$. What we need to show is that θ is a congruence if and only if θ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. This amounts to checking somethings about every basic operation. So suppose Q is an operation symbol, with rank let us say 3 this time. Let $a, a', b, b', c, c' \in A$. Now observe that

$$\langle a, a'\rangle, \langle b, b'\rangle . \langle c, c'\rangle \in \theta \implies \langle Q^{\mathbf{A}}(a,b,c), Q^{\mathbf{A}}(a,b,c)\rangle \in \theta$$

is what needs to be checked to see that θ is a congruence. But this implication is also exactly what needs to be checked to see that θ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. Thus these to notions are the same.

Problem 9

Let $\mathbf{A}$ be an algebra and let h be an endomorphism of $\mathbf{A}$. Prove that $h \circ h^{-1}$ is a congruence of $\mathbf{A}$. Observe that $h^{-1} = \{(b,a) \mid h(a) = b \text{ and } a \in A\}$.

First notice that $a h \circ h^{-1} b \Leftrightarrow h(a) = h(b)$. This means that $h \circ h{-}1 = \ker h$. Since h is an endomorphism, we know that its kernel must be a congruence.

Problem 10

Let $\mathbf{A}$ be an algebra and let θ be a congruence of $\mathbf{A}$. Prove that $\theta = \bigcup\{\text{Cg}^{\mathbf{A}}(a,a') \mid a\theta a'\}$.

If $a\theta a'$ then $\text{Cg}^{\mathbf{A}}(a,a') \subseteq \theta$ since $\text{Cg}^{\mathbf{A}}(a,a')$ is the intersection of all congruences containing the pair $\langle a, a'\rangle$. Therefore $\bigcup\{\text{Cg}^{\mathbf{A}}(a,a') \mid a\theta a'\} \subseteq \theta$.

For the reverse inclusion, notice

$$\theta = \bigcup\{\{\langle a.a'\rangle\} \mid \langle a, a'\rangle \in \theta\} \subseteq \bigcup\{\text{Cg}^{\mathbf{A}}(a,a') \mid a\theta a'\}$$

Problem 11

Let $\mathbf{A}$ be an algebra and let $X \subseteq A$ such that $\mathrm{Sg}^{\mathbf{A}} X = A$. Suppose that $\mathbf{B}$ is an algebra with the same signature and let h and g be homomorphisms from A into B such that $h(x) = g(x)$ for all $x \in X$. Prove that $h = g$.

Let $U = \{u \mid u \in A$ and $h(u) = g(u)\}$. Plainly, $X \subseteq U$. We are going to show that U is a subuniverse of $\mathbf{A}$. From this it follows that $\mathrm{Sg}^{\mathbf{A}} X \subseteq U$. Since $\mathrm{Sg}^{\mathbf{A}} X = A$ we see that $U = A$ and so h and g agree everywhere on their domain.

To see that U is a subuniverse we only need to show it is closed with respect to all the operations. So let Q be an operation symbol and let us say for variety that it has rank 3. Let $a, b, c \in U$. Now

$$Q^{\mathbf{A}}(a,b,c) \in U \Leftrightarrow g(Q^{\mathbf{A}}(a,b,c)) = h(Q^{\mathbf{A}}(a,b,c)).$$

But notice

$$\begin{aligned} g(Q^{\mathbf{A}}(a,b,c)) &= Q^{\mathbf{B}}(g(a), g(b), g(c)) \\ &= Q^{\mathbf{B}}(h(a), h(b), h(c)) \\ &= h(Q^{\mathbf{A}}(a,b,c)) \end{aligned}$$

where the link to the middle line happens because $a, b, c \in U$ gives us $g(a) = h(a), g(b) = h(b)$ and $g(c) = h(c)$.

Consequently, from $a, b, c \in U$ we draw the conclusion that $Q^{\mathbf{A}}(a,b,c) \in U$. Hence, U is a subuniverse just as we wanted.

Problem 12

Prove that every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

Suppose it were otherwise. Let $\mathbf{A}$ be a finite algebra with the least number of elements that is not decomposable has a direct product of directly indecomposable algebras. Now $\mathbf{A}$ must have at least two elements, since one-element algebras can be decomposed as a product of an empty system of algebras. Also $\mathbf{A}$ is not itself indecomposable (else we have the decomposition before us). This means that $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ where both $\mathbf{B}$ and $\mathbf{C}$ have fewer elements than $\mathbf{A}$. By the minimality of $\mathbf{A}$, we can decompose $\mathbf{B}$ and $\mathbf{C}$ into direct products of directly indecomposable algebras. Putting these decompositions together we get a decomposition of $\mathbf{A}$, which isn't supposed to have one.

Problem 13

Find two algebras $\mathbf{A}$ and $\mathbf{B}$ so that neither $\mathbf{A}$ nor $\mathbf{B}$ can be embedded into $\mathbf{A} \times \mathbf{B}$.

Let $\mathbf{A} = \langle \{0,1\}, f \rangle$ and $\mathbf{B} = \langle \{0,1,2\}, g \rangle$ where $f(0) = 1, f(1) = 0, g(0) = 1, g(1) = 2, g(2) = 0$. So f is a 2-cycle and g is a 3-cycle. Now show that $\mathbf{A} \times \mathbf{B}$ has a basic operation which is a 6-cylce. In particular, $\mathbf{A} \times \mathbf{B}$ has no proper subalgebras at all.

Problem 14

Prove that $\mathbf{A}$ has factorable congruences if and only if $\beta = (\beta \vee \phi) \wedge (\beta \vee \phi^*)$ for every pair ϕ, ϕ^* of complementary factor congruences of $\mathbf{A}$ and every $\beta \in \mathrm{Con}\ A$.

Problem 15

Prove that if $\mathbf{Con}\ A$ is a distributive lattice, then $\mathbf{A}$ has factorable congruences.

This follows immediately from Problem 14, since $\phi \vee \phi^* = 1_{\mathbf{B}}$, making the equality from Problem 14 into an instance of the distributive law.

PROBLEM 16
Suppose that $\beta, \eta_0, \eta_1 \in \mathrm{Con}\ \mathbf{B}$ and $\eta_0 \wedge \eta_1 \leq \beta$. Prove that $\mathbf{B}/\beta \in \boldsymbol{HP_s}(\mathbf{B}/\eta_0, \mathbf{B}/\eta_1)$.

Define $h_0 : B/\eta_0 \wedge \eta_1 \to B/\eta_0$ by putting $h_0(b/\eta_0 \wedge \eta_1) = b/\eta_0$ for all $b \in B$. It is routine to show that this definition is definite and that h_0 is a homomorphism. Define h_1 similarly. Then $\langle h_0, h_1 \rangle$ separates the points of $B/\eta_0 \wedge \eta_1$ and the image of h_i is B/η_i. This means that $\mathbf{B}/\eta_0 \wedge \eta_1 \in \mathsf{P_s}(\mathbf{B}/\eta_0, \mathbf{B}/\eta_1)$. Now $\mathbf{B}/\beta$ is a homomorphic image of $\mathbf{B}/\eta_0 \wedge \eta_1$ by the Second Isomorphism Theorem, because $\eta_0 \wedge \eta_1 \leq \beta$.

PROBLEM 17
Show that $\boldsymbol{SH} \neq \boldsymbol{HS}, \boldsymbol{PS} \neq \boldsymbol{SP}$, and $\boldsymbol{PH} \neq \boldsymbol{HP}$.

PROBLEM 18
Show that restricted to classes of commutative semigroups, the operators $\boldsymbol{SPHS}, \boldsymbol{SHPS}$, and $\boldsymbol{HSP}$ are distinct. In fact, if $\mathcal{K}$ is the class of finite cyclic groups considered as semigroups (i.e., multiplication groups), then the three operators applied to $\mathcal{K}$ give different classes.

PROBLEM 19
Verify that if $\mathcal{K}$ is a class of Abelian groups then $\boldsymbol{HS}\mathcal{K} = \boldsymbol{SH}\mathcal{K}$. Formulate a property of varieties involving the behavior of congruences suc that if $\mathcal{V}$ has the property and $\mathcal{K} \subseteq \mathcal{V}$, then $\boldsymbol{HS}\mathcal{K} = \boldsymbol{SH}\mathcal{K}$.

PROBLEM 20
(See the book for notation) Prove that if w is a term and if w' is a proper initial segment of w, then w' is not a term.

PROBLEM 21
Prove that the equality $\boldsymbol{V} = \boldsymbol{HP_s}$ holds for class operators. (Hint: If $\mathcal{V} = \boldsymbol{V}(\mathbf{A})$ and X is sufficiently large, the $\mathbf{F}_{\mathcal{V}}(X)$ is a subdirect power of $\mathbf{A}$.)

PROBLEM 22
If $\mathcal{V}_0 \subseteq \mathcal{V}_1$ and $X \subseteq Y$, then $\mathbf{F}_{\mathcal{V}_0}(X)$ is a homomorphic image of $\mathbf{F}_{\mathcal{V}_1}(Y)$ in a natural way, and $\mathbf{F}_{\mathcal{V}_0}(X)$ is isomorphic to the subalgebra of $\mathbf{F}_{\mathcal{V}_1}(Y)$ generated by X, in a natural way.

PROBLEM 23
Prove that if $\mathcal{V}$ is a nontrivial variety, then in a free algebra $\mathbf{F}_{\mathcal{V}}(X)$, the set X is a minimal generating set.

PROBLEM 24
Prove that an algebra $\mathbf{A}$ is arithmetical if and only if this version of the Chinese Remainder Theorem hold is $\mathbf{A}$:

> For every finite sequence $a_0, a_1, \ldots, a_n, \psi_0, \ldots, \psi_n$ of elements and congruences of $\mathbf{A}$, if $\langle a_i, a_j \rangle \in \psi_i \vee \psi_j$ for all $i, j \leq n$, then there exists an element $x \in A$ such that $\langle a_i, x \rangle \in \psi_i$ for all $i \leq n$.

PROBLEM 25
Prove that a variety $\mathcal{V}$ is arithmetical if and only if $\mathcal{V}$ has a Mal'cev term $p(x, y, z)$ and a term $q(x, y, z)$ so that $\mathcal{V} \models q(x, x, y) \approx q(x, y, x) \approx q(y, x, x) \approx x$.

PROBLEM 26
Prove that for any integer $n > 1$ the variety of rings obeying the law $x^n \approx x$ is arithmetical.