# The Logic of Equations: MATH 768

MW 3:55–5:10
LeConte 303B

Instructor: George McNulty
Office Hours: Monday–Thursday 2:30–4:30

MATH 701 will provide a background sufficient for this course.
The course will use the manuscript of a forthcoming book (no charge).

The concepts that can be expressed by means of equations and the kinds of proofs that may be devised using equations are central concerns of equational logic. The concept of a ring is usually presented by saying that a ring is a system $\langle R, +, -, \cdot, 0, 1\rangle$ in which the following equations are true:

$$
\begin{array}{ccc}
x + (y + z) \approx (x + y) + z & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z & x \cdot (y + z) \approx x \cdot y + x \cdot z \\
x + y \approx y + x & x \cdot 1 \approx x & (x + y) \cdot z \approx x \cdot z + y \cdot z \\
-x + x \approx 0 & 1 \cdot x \approx x & \\
x + 0 \approx x & &
\end{array}
$$

A ring is an *algebra*—meaning here a nonempty set endowed with a system of finitary operations. *Equations*, on the other hand, are certain strings of formal symbols. The concept of truth establishes a binary relation between equations and algebras: an equation $s \approx t$ *is true* in an algebra $\mathbf{A}$. This relationship underlies virtually all work in equational logic. By way of this relation each *equational theory*—that is, each set of equations closed under logical consequence—is associated with a *variety of algebras*: the class of all algebras in which each equation of the theory is true. Through this connection syntactical and computational tools developed for dealing with equations can be brought to bear on algebraic questions about varieties. Conversely, algebraic techniques and concepts from the theory of varieties can be employed on the syntactical side.

It turns our that a variety is a class of similar algebras closed with respect to forming homomorphic images, subalgebras, and arbitrary direct products. The classification of algebras into varieties is compatible with most commonly encountered algebraic constructions. It allows us to gather algebras into classes that can be easily comprehended and manipulated. In many cases, it allows us to distinguish algebras that strike our intuitions as genuinely different, as well as to group together algebras which seem to belong together.

This course will focus on topics like

(1) **Finite Axiomatizability:** Which varieties can be specified by finitely many equations?
(2) **Decision Problems:** For which varieties is it possible to have a computer algorithm that determines which equations are true in the variety? Is there a computer algorithm which would determine whether a finite set of equations specifies exactly the class of all groups?
(3) **The Lattice of Equational Theories:** Set-inclusion imposes a partial order on the set of equational theories. The structure of this ordered set reflects the comparative strength of the equational theories.

### What You Can Expect to Get Out of This Course

Your hard work in this course should lead

(1) to increased mathematical sophistication,
(2) a stronger ability to see underlying mathematical principles,
(3) an increased ability to discover proofs and to pursue their details with a diligent craftmanship, and
(4) an increased ability to express your mathematics verbally and in writing.

### How Grades Will Be Determined

Grades will be determined by performance on problem sets.