A Example Worked Several Ways

Consider the differential equation

$$y'' + y = \sin^2 t.$$

Here I will give several methods for finding the general solution to this differential equation.

1. The Method of Undetermined Coefficients

We know that the general solution has the form

$$y = y_p + y_h$$

where y_h is the general solution of

$$y'' + y = 0$$

and y_p can be any particular solution

$$y'' + y = \sin^2 t.$$

To find y_h we consider the characteristic equation

$$r^2 + r = 0$$

Using the quadratic formula we discover

$$r = \frac{0 \pm \sqrt{0^2 - 4}}{2} = \pm i.$$

In this way we are led to the two linearly independent solutions

$$y_1 = \sin t \quad \text{and} \quad y_2 = \cos t.$$

So $y_h = c_1 \sin t + c_2 \cos t$.

To find a particular solution it helps to use the trigonometric identity

$$\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t.$$

So we imagine that y_p has the form $A + B\sin 2t + C\cos 2t$. Then we calculate

$$y_p = A + B\sin 2t + C\cos 2t$$
$$y_p' = 0 - 2C\sin 2t + 2B\cos 2t$$
$$y_p'' = 0 - 4B\sin 2t - 4C\cos 2t$$

Adding the first line to the last line produces

$$\frac{1}{2} - \frac{1}{2}\cos 2t = A - 3B\sin 2t - 3C\cos 2t.$$

So we discover that $A = \frac{1}{2}, B = 0$, and $C = \frac{1}{6}$, This gives

$$y_p = \frac{1}{2} + \frac{1}{6}\cos 2t.$$

Just to be on the safe side, let us check this by taking derivatives (plugging this y_p in to our equation).

$$y_p = \frac{1}{2} + \frac{1}{6}\cos 2t$$
$$y_p' = 0 - \frac{1}{3}\sin 2t$$
$$y_p'' = 0 - \frac{2}{3}\cos 2t$$

So

$$y_p'' + y_p = \frac{1}{2} - \frac{1}{2}\cos 2t = \sin^2 t.$$

So our general solution is

$$\boxed{y = \tfrac{1}{2} + \tfrac{1}{6}\cos 2t + c_1 \sin t + c_2 \cos t.}$$

2. The Method of Variation of Parameters

Just as above, we find that $\sin t$ and $\cos t$ are two linearly independent solutions to the homogeneous equation. But here we imagine

$$y = u(t)\sin t + v(t)\cos t$$

for some functions $u(t)$ and $v(t)$ that we have to determine. In this method, we set

$$u'(t)\sin t + v'(t)\cos t = 0.$$

To plug y into our differntial equation we calculate

$$\begin{aligned} y &= u(t)\sin t + v(t)\cos t \\ y' &= u(t)\cos t - v(t)\sin t \\ y'' &= -u(t)\sin t - v(t)\cos t + u'(t)\cos t - v'(t)\sin t \end{aligned}$$

Then our differential equation produces

$$u'(t)\cos t - v'(t)\sin t = \sin^2 t.$$

In this way we get two equations

$$\begin{aligned} u'(t)\sin t + v'(t)\cos t &= 0 \\ u'(t)\cos t - v'(t)\sin t &= \sin^2 t \end{aligned}$$

in the unknown functions $u'(t)$ and $v'(t)$. Solving these equations á la Algebra II, we find

$$\begin{aligned} u'(t) &= \sin^2 t\cos t \\ v'(t) &= -\sin^3 t \end{aligned}$$

Now we figure out the antiderivatives.

$$\begin{aligned} u(t) &= \int \sin^2 t\cos t\ dt = \frac{1}{3}\sin^3 t + c_1 \\ v(t) &= \int -\sin^3 t\ dt = \int -(1-\cos^2 t)\sin t\ dt = \int -\sin t + \cos^2 t\sin t\ dt = \cos t - \frac{1}{3}\cos^3 t + c_2 \end{aligned}$$

So we get

$$y = \left[\frac{1}{3}\sin^3 t + c_1\right]\sin t + \left[\cos t - \frac{1}{3}\cos^3 t + c_2\right]\cos t.$$

Neatening this a bit we find our general solution

$$\boxed{y = \tfrac{1}{3}(\sin^4 t - \cos^4 t) + \cos^2 t + c_1 \sin t + c_2 \cos t.}$$

3. The Method of Laplace Transforms

First we apply the Laplace transform to our differential equation.

$$\mathcal{L}[y'' + y] = \mathcal{L}[\sin^2 t] = \mathcal{L}[\frac{1}{2} - \frac{1}{2}\cos 2t].$$

Using the linearity of the Laplace transform and how Laplace transforms behave for derivatives, we find

$$-y'(0) - sy(0) + s^2\mathcal{L}[y] + \mathcal{L}[y] = \frac{1}{2}\mathcal{L}[1] - \frac{1}{2}\mathcal{L}[\cos 2t].$$

Now with a bit of high school algebra we solve for $\mathcal{L}[y]$:

$$\begin{aligned}\mathcal{L}[y] &= \frac{1}{2}\mathcal{L}[1]\frac{1}{s^2+1} - \frac{1}{2}\mathcal{L}[\cos 2t]\frac{1}{s^2+1} + y'(0)\frac{1}{s^2+1} + y(0)\frac{s}{s^2+1} \\ &= \frac{1}{2}\mathcal{L}[1]\mathcal{L}[\sin t] - \frac{1}{2}\mathcal{L}[\cos 2t]\mathcal{L}[\sin t] + y'(0)\mathcal{L}[\sin t] + y(0)\mathcal{L}[\cos t]\end{aligned}$$

But we know how to use convolutions. This gives us

$$\mathcal{L}[y] = \frac{1}{2}\mathcal{L}[1 * \sin t] - \frac{1}{2}\mathcal{L}[\sin t * \cos 2t] + y'(0)\mathcal{L}[\sin t] + y(0)\mathcal{L}[\cos t]$$

So we discover that

$$y = \frac{1}{2}(1 * \sin t) - \frac{1}{2}(\sin t * \cos 2t) + y'(0)\sin t + y(0)\cos t.$$

To finish, we need to figure out those convolutions. Recall that $f(t) * g(t) = \int_0^t f(u)g(t-u)\ du$. So

$$1 * \sin t = \sin t * 1 = \int_0^t (\sin u) \cdot 1\ du = -\cos t\Big|_0^t = \cos 0 - \cos t = 1 - \cos t.$$

Likewise

$$\sin t * \cos 2t = \cos 2t * \sin t = \int_0^t \cos 2u \sin(t-u)\ du.$$

This integral we can do by parts.

$$\begin{aligned} w &= \sin(t-u) & dz &= \cos 2u\ du \\ dw &= -\cos(t-u)\ du & z &= \frac{1}{2}\sin 2u \end{aligned}$$

So

$$\begin{aligned}\int_0^t \cos 2u \sin(t-u)\ du &= \frac{1}{2}\sin 2u \sin(t-u)\Big|_0^t + \frac{1}{2}\int_0^t \sin 2u \cos(t-u)\ du \\ \int_0^t \cos 2u \sin(t-u)\ du &= \frac{1}{2}\int_0^t \sin 2u \cos(t-u)\ du\end{aligned}$$

We need to do integration by parts again. This time

$$\begin{aligned} w &= \cos(t-u) & dz &= \sin 2u\ du \\ dw &= \sin(t-u) & z &= -\frac{1}{2}\cos 2u \end{aligned}$$

So we find

$$\int_0^t \cos 2u \sin(t-u)\, du = \frac{1}{2}\left[-\frac{1}{2}\cos 2u \cos(t-u)\Big|_0^t + \frac{1}{2}\int_0^t \cos 2u \sin(t-u)\, du \right]$$
$$= -\frac{1}{4}(\cos 2t - \cos t) + \frac{1}{4}\int_0^t \cos 2u \sin(t-u)\, du.$$

Now just a bit of high school algebra gives

$$\sin t * \cos 2t = \int_0^t \cos 2u \sin(t-u)\, du = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t.$$

Putting things together, we find

$$\boxed{y = \tfrac{1}{2} - \tfrac{2}{3}\cos t + \tfrac{1}{6}\cos 2t + y'(0)\sin t + y(0)\cos t.}$$

4. We Got Three Different Solutions

We solved our differential equation $y'' + y = \sin^2 t$ three different ways and we got three different looking solutions:

$$y = \frac{1}{2} + \frac{1}{6}\cos 2t + c_1 \sin t + c_2 \cos t$$
$$y = \frac{1}{3}(\sin^4 t - \cos^4 t) + \cos^2 t + c_1 \sin t + c_2 \cos t$$
$$y = \frac{1}{2} - \frac{2}{3}\cos t + \frac{1}{6}\cos 2t + y'(0)\sin t + y(0)\cos t$$

Which is the right solution?

They are all the same!

$$\frac{1}{3}(\sin^4 t - \cos^4 t) = \frac{1}{3}(\sin^2 t + \cos^2 t)(\sin^2 t - \cos^2 t) = -\frac{1}{3}\cos 2t \quad \text{and} \quad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$$

give enough information to show that the first two solutions are the same.

The last solution rewrites as

$$y = \frac{1}{2} + \frac{1}{6}\cos 2t + y'(0)\sin t + \left(y(0) - \frac{2}{3} \right)\cos t$$

So $c_1 = y'(0)$ and $c_2 = y(0) - \frac{2}{3}$ gives the first solution. Moreover, using y in the first solution you can see that

$$y(0) = \frac{1}{2} + \frac{1}{6} + c_2$$

so that $c_2 = y(0) - \frac{2}{3}$, as we want.

Everything works out!

5. Which Method Is Better?

The method of undetermined coefficients seems the most straighforward. To pull it off, we had to use a trigonometric identity and solve a systems of three linear equations in three unknowns—in this example that was easy. On the other hand, we were lucky that the method applied. Maybe if $\sin^2 t$ was replaced by a more inconventient function we would be out of luck.

The method of variation of parameters required us to solve a system of two equations in two unknown functions and then to do some integration. In this example, the integration was facilitated by some trigonometric identities. Here things would work out if we replaced $\sin^2 t$ by some other function, so long as we could figure out the resulting integrals.

Both the method of undetermined coefficients and the method of variation of parameters required us the find two linearly independent solutions to the homogeneous equation.

The method of Laplace transforms seems to have taken a bit more work—it occupied 1.5 pages as opposed to 1 page for each of the other two. But actually, we found the solution, in some sense, after only half a page, although that solution involved the convolution operator $*$. To figure out the convolutions required some integration, which in this example worked by parts. It is interesting to note that the method of Laplace transforms did not require us to know two linearly independent solutions to the homogeneous equation. It may also be useful to remember that Laplace transforms can apply to functions that are only piecewise continuous.

In the end, all three methods worked on our example, and none of them depended on much beyond the methods of Calculus II. On the other hand, since differential equations come in many forms, it is good to have a variety of methods, since no one method can be conveniently applied to all differential equations.