Name: Solutions **Quiz 5**

Instructions: This quiz is an individual effort. Answer each question to the best of your ability. You are not permitted to use any resource besides your own knowledge to complete this quiz. The total points is 20. You may do work on a seperate piece of paper, but all answers must be written on the quiz and you must turn in your work. This means your work must be organized for the instructor to follow your thought process.

If you are unable to complete a question, whether it be due to time or not, write out what you would do or what you know relates to the question. This shows me you at least have knowledge on the subject matter and some knowledge on how to approach the problem, but are stuck on the execution.

1. a. Prove that $\mathbb{Z}_6$ is an abelian group under addition.

$\{0,1,2,3,4,5\}$

$6k+r+6l+r_2 = 6(k+l)+r+r_2$ if $r+r_2 \geq 6$ then we know $r+r_2 = 6q+r_3$ $0 \leq r_3 < 6$

Thus $= 6(k+l+q)+r_3$ $r_3 \in \mathbb{Z}_6$. Thus closed.

inverse: ~~6k...~~ $6k+r+6l+r_2 = 6m$ when $r_2 = 6-r \in \mathbb{Z}_6$.

identity: 0 or 6k.
remainder

associative addition is associative and closedness shows how to reduce remainder sum to find in group.

Thus group.

b. Is $\mathbb{Z}_3$ a subgroup of the above? Prove that it is or explain why it is not.

yes!

closed by same args above, inverse by same arg above, same identity,

and $\mathbb{Z}_3 \subseteq \mathbb{Z}_6$.

c. (Challenge problem) Explain or show why $\mathbb{Z}_6$ is not a group under multiplication.

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2. Let $A = \{x \in \mathbb{Z} \mid 0 \leq x \leq 9\}$ and define the relation $a \sim b$ as the following:

$$a \sim b \Leftrightarrow a \equiv b \pmod{4} \text{ for } A \text{ to } A, \text{ or}$$

a and b have the same remainder when divided by 4.

a. Find the domain and range of the relation.

$D(R) = A$

$Range(R) = A$. ← reflexivity gives this

b. Find the matrix of the relation.

(M)

$$\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\ 1 & & 1 & & & & 1 & & & & 1 \\ 2 & & & 1 & & & & 1 & & & \\ 3 & & & & 1 & & & & 1 & & \\ 4 & 1 & & & & 1 & & & & 1 & \\ 5 & & 1 & & & & 1 & & & & 1 \\ 6 & & & 1 & & & & 1 & & & \\ 7 & & & & 1 & & & & 1 & & \\ 8 & 1 & & & & 1 & & & & 1 & \\ 9 & & 1 & & & & 1 & & & & 1 \end{array}$$

c. Find the digraph of the relation.

$0 \to 0$, $\to 4$, $\to 8$ $\quad$ $1 \to 1$, $\to 5$, $\to 9$ $\quad$ $2 \to 2$, $\to 6$ $\quad$ $3 \to 3$, $\to 7$ $\quad$ $4 \to 0$, $\to 4$, $\to 8$ $\quad$ $5 \to 1$, $\to 5$, $\to 9$ $\quad$ $6 \to 2$, $\to 6$ $\quad$ $7 \to 3$, $\to 7$ $\quad$ $8 \to 0$, $\to 4$, $\to 8$ $\quad$ $9 \to 1$, $\to 5$, $\to 9$

Draw that!

d. Determine if this is an equivalence relation through proof or contradiction. (The above may even be enough to say this or not say this, but you must explain that.)

yes it is

reflexive by all mapping to themselves

symmetric since if they are equivalent one way they are the other as well.

transitive only chains of 3 when all 3 equivalent so yes.

e. Is this relation antisymmetric? Show with proof or contradiction.

NO since $3 \equiv 7$ and $7 \equiv 3$ but $3 \neq 7$.

3. Let f, g and h be the following possible functions:

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } f(x) = \frac{3x+1}{1-2x}$$

$$g : \mathbb{R} \to \mathbb{R} \text{ such that } g(x) = x^4$$

$$h : \mathbb{R} \to \{0, 1\} \text{ such that } h(x) = \chi_{\mathbb{Z}}(x) = \left\{ \begin{array}{ll} 0 & x \in \mathbb{Z} \\ 1 & x \notin \mathbb{Z} \end{array} \right\}$$

a. Are these all functions? Explain.

yes all have 1 and only 1 output.

b. Determine which functions are onto, one-to-one, both or neither. Some proof or explanation is required.

f: is neither since it does not map $\frac{1}{2}$ to anything and it does have an inverse, thus cannot be either.

g: not onto since no negative #'s hit.
~~not~~ not one to one since $1^4=(-1)^4$ and same that goes on 1.

h: onto yes! exist #s that are ≠0 and some that $=0$.
not one to one since 2 and 3 both $\chi_{\mathbb{Z}}=0$.

c. Determine which functions have an inverse. If they have an inverse, find it.

f: it does $x=\frac{3y+1}{1-2y}$ ⇒ $x-2yx=3y+1$ ⇒ $3y+2yx=x-1$ ⇒ $y=\frac{x-1}{3+2x}$ $f^{-1}(x)$

only $\frac{1}{2}$. does not exist otherwise

$\frac{1}{2}=\frac{x-1}{3+2x}$

$3+2x=2x-2$!

$1\neq 0$ issue!

g: No unless restricted
$g^{-1}(x)=\sqrt[4]{x}$

h: not a function!
0 → any integer.

d. Find $f \circ g$ and $g \circ f$.

$f\circ g=\frac{3x^4+1}{1-2x^4}$

$g\circ f=\left(\frac{3x+1}{1-2x}\right)^4$