Midterm Exam Solutions

(1) (a) $A=\{a\}$ or any 1 element set

(b) $A=\{a\}$ or any set with odd # of elements

(c) $A=\mathbb{Z}$ or any set with infinite elements

(d) $A=\{1\}$ $B=\{2\}$ or any disjoint sets

(e) Any M with $N=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ or $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ other possible answers too

(f) $M=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ any symmetric matrix.

(g) $a, ar, \dots$ is infinite sum if $|r|\geq 1$.

(h) sum of finite things is finite.

(i) $a\equiv 2\ (n) \Rightarrow a=\cancel{4}k+2$ if $k=1$, $a=13$

(j) $30=2\cdot 3\cdot 5$ if $\gcd(30,b)=1$, b is a product of primes larger than 5.
$b=7^2$ or $b=7\cdot 11$, or ...

(2) (a) $\begin{pmatrix}a&0\\b&c\end{pmatrix}\begin{pmatrix}d&0\\e&f\end{pmatrix}=\begin{pmatrix}ad&0\\bd+ce&cf\end{pmatrix}$ this is a matrix in my set. ✓

(b) $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is in my set and is the matrix mult identity. ✓

(c) $\begin{pmatrix}a&0\\b&c\end{pmatrix}^{-1}=\frac{1}{ac}\begin{pmatrix}c&0\\-b&a\end{pmatrix}$ this is in the set except when $ac=0$ so NOT ALWAYS!

(d) closed and matrix mult is associative. ✓

(e) $\begin{pmatrix}d&0\\e&f\end{pmatrix}\begin{pmatrix}a&0\\b&c\end{pmatrix}=\begin{pmatrix}ad&0\\ae+fb&fc\end{pmatrix}$ I need $ae+fb=bd+ce$ which is not always true!
comes from product in (a).

(3) $\{1,2\}$ with addition
$1+2=3$ $3\notin\{1,2\}$

(4) (a)

$550 = 1\cdot 510 + 40$ Thus, $\gcd(550, 510) = 10$.

$510 = 12\cdot 40 + 30$

$40 = 1\cdot 30 + 10$

$30 = 3\cdot 10 + 0$

(b) $10 = 40 - 1\cdot 30$ $\Rightarrow$ $10 = 40 - 1(510 - 12\cdot 40)$

$30 = 510 - 12\cdot 40$ $= 40 - 1\cdot 510 + 12\cdot 40 = 13\cdot 40 - 1\cdot 510$

$40 = 550 - 1\cdot 510$ $= 13(550 - 1\cdot 510) - 1\cdot 510$

$= 13\cdot 550 - 13\cdot 510 - 1\cdot 510 = \underline{13}\cdot 550 - \underline{14}\cdot 510$.

(c) $\gcd(a,b) \mid a$ $a \mid \mathrm{lcm}(a,b)$ $\Rightarrow \gcd(a,b) \mid \mathrm{lcm}(a,b)$.

(5) $\gcd(m,n) = 2\cdot 5$ means $2\cdot 5$ $2\cdot 5$

$\mathrm{lcm}(m,n) = 2^2\cdot 3\cdot 5$ means either 1 gets a 2 and a 3

or one gets 2 the other gets 3.

If 1 gets 2 and 3, then we have $2^2\cdot 3\cdot 5$ and $2\cdot 5$ but $2\cdot 5 \mid 2^2\cdot 3\cdot 5$. This violates one of the conditions

Thus, the 2 and the 3 go to different ones and give $\underset{30}{\underset{\shortparallel}{2\cdot 3\cdot 5}}$ and $\underset{20}{\underset{\shortparallel}{2^2\cdot 5}}$.

(6) (a)

p	q	$\sim p$	$\sim p \vee q$	$\sim q$	$(\sim p \vee q) \Rightarrow \sim q$
T	T	F	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

(7) $p \wedge \sim p$ and $p \vee \sim p$

contradiction — can never both be true

tautology — one or the other is true.

(b)

p	q	r	$\sim p$	$q \Rightarrow \sim p$	$(q \Rightarrow \sim p) \wedge r$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F

(8) (a) (contrapositive)

Pf: Suppose $\frac{1}{x} = \frac{m}{n}$ $m,n \in \mathbb{Z}$ $n \neq 0$ $\gcd(m,n)=1$ (rational). We show x is rational to prove the contrapositive. $\frac{1}{x} = \frac{m}{n} \Rightarrow x = \frac{n}{m}$ $m \neq 0$ since $\frac{1}{x} \neq 0$. Thus, x is rational ☐

(b) Pf: (contradiction) Suppose for a contradiction the sum is rational. Then if x is rational and y is irrational, we have

$\frac{m}{n} + y = \frac{a}{b}$ where $x = \frac{m}{n}$ $m,n,a,b \in \mathbb{Z}$ $n,b \neq 0$ and $\gcd(m,n) = \gcd(a,b) = 1$.

Then $y = \frac{a}{b} - \frac{m}{n} = \frac{an - mb}{bn}$ since $n,b \neq 0$ and $an - mb \in \mathbb{Z}$, y is rational!

Thus, contradiction since y is irrational ☐

(c) Pf: To show $A \subseteq B$, we find k s.t. when $x = 4l+1$ it is also $4k-3$.

$4l+1 = 4k-3$

$\Rightarrow 4l+4 = 4k \Rightarrow l+1 = k$. Thus $k = l+1$. Therefore $A \subseteq B$.

By similar argument, $B \subseteq A$ for $k = l-1$.

Thus, $A = B$. ☐

(9) Pf: (by induction).

~~Base case~~: $n=5$. $1+5^2 = 26 < 32 = 2^5$ ✓

Induction step: Suppose true for $n \leq k$. That is when $n=k$, $1+k^2 < 2^k$.

We show true for $n=k+1$. That is, $1+(k+1)^2 < 2^{k+1}$.

$\underline{k^2+2k+2}$

$1+k^2 < 2^k \Rightarrow 2+2k^2 < 2^{k+1}$

$\Rightarrow \underline{2+k^2}+k^2 < 2^{k+1}$ $\quad k^2 \geq 2k$ when $k \geq 3$. Thus.

$k^2+2k+2 < 2+k^2+k^2 = 2+2k^2 < 2^{k+1}$. By Induction, we are done. ☐

(10) (a) Suppose the vowels are only in spot 1, 3, and 5. Then we have

$\underset{v}{\underline{3}}\cdot\underset{c}{\underline{4}}\cdot\underset{v}{\underline{2}}\cdot\underset{c}{\underline{3}}\cdot\underset{v}{\underline{1}}\cdot\underset{c}{\underline{2}}\cdot\underset{c}{\underline{1}} = 4!\,3!$ choices. Now we multiply this by the number of ways the vowels can be in the 4 odd spots, ${}_4C_3 = 4$

Thus, there are a total of $4 \cdot 4! \cdot 3!$ "words" possible.

(b) The number must end in 0 to be divisible by 10. Thus, 0 is used. 5 others left.

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{1} = 5 \cdot 4 \cdot 3 = 60$

must be 0.

(c) order matters! Thus, $\underline{8} \cdot \underline{7} \cdot \underline{6} = 8 \cdot 7 \cdot 6$

(d) Many ways: Assume each die different so that order matters. Then

6^3 possible outcomes

5^3 possibly with no 2.

Thus, to have a 2 is $6^3 - 5^3$.