Instructions: This exam is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero. Answer each question. **Show all work to receive full credit.** Write your answers on the test. You have **1 hour 40 minutes** to finish the exam. Answer all questions to the best of your ability. Unless otherwise specified, you are required to **SHOW ALL YOUR WORK** to receive full credit. The exam has 110 possible points. You will be graded out of 100 points.

Questions	Possible	Score		Possible	Score
Question 1	20		Question 6	14	
Question 2	12		Question 7	4	
Question 3	4		Question 8	12	
Question 4	16		Question 9	4	
Question 5	8		Question 10	16	
			Total	110	

1. (each is worth 2 pts) Give an example for each part that satisfies the condition and show why it satisfies the condition. If there is no example, please say so and explain or prove why. Let A and B be sets. Let M and N be 2 x 2 matrices where $M \neq N$.

   a. $|A| = 1$.

   b. $|A|$ is odd.

   c. $|A| = \infty$.

   d. $|A \cup B| = |A| + |B|$.

   e. $MN = NM$.

   f. $M^T = M$.

   g. An infinite geometric sequence whose sum is infinite.

   h. A finite sequence whose sum is infinite.

   i. a such that $a \equiv 2 \pmod{11}$.

   j. b such that b is coprime to 30 and b is not prime.

2. (12 pts) Let A be the set of all lower triangular 2 by 2 matrices. What this means is that all elements above the main diagonal are 0. For example,

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}.$$

Below is a list of possible properties. Consider the structure $(A, \cdot)$. Determine if the properties are true for the structure with either proof or example for why it is false.

a. Closed

b. Existence of an identity

c. Existence of an inverse for all in the set

d. Associative

e. Commutative

3. (4 points) Come up with your own structure that is not closed and show it is not closed. (Think simple!)

4. (16 pts) Let $a = 550$ and $b = 510$.

   a. Use the Euclidean Algorithm to find $d = \gcd(a, b)$.

   b. Use the Extended Euclidean Algorithm to find s and t (integers) such that $d = a \cdot s + b \cdot t$.

   c. Determine and explain if the following is true or false: $\gcd(a, b) |\ \mathrm{lcm}(a, b)$ for any $a, b \in \mathbf{Z}$.

5. (8 pts) Let m and n be positive integers such that $\gcd(m, n) = 2 \cdot 5$, $\mathrm{lcm}(m, n) = 2^2 \cdot 3 \cdot 5$, $m > n$, and $n \nmid m$. Only one pair of (m, n) is possible given the above information. Determine this pair. Show all work for how you got your pair. Guess and check will give you 0 CREDIT!

6. (14 pts) Use truth tables to determine all outcomes of the following:

   a. $(\sim p \vee q) \Rightarrow \sim q$

   b. $(q \Rightarrow \sim p) \wedge r$

7. (4 pts) Give an example of a tautology using the arbitrary statements p, q, and r (you do not need to use all of them). Then give an example of a contradiction (all false). Explain both of your answers either with a truth table or words.

8. (12 pts) Prove or disprove the following statements. (proof types: Contradiction, Direct, Contrapositive, and counterexample)

   a. If x is an irrational number, then so is $1/x$.

   b. The sum of an irrational number and a rational number is irrational.

   c. Let $A = \{x | x = 4k + 1 \text{ for some } k \in \mathbf{Z}\}$ and $B = \{x | x = 4k - 3 \text{ for some } k \in \mathbf{Z}\}$. Prove $A = B$. (Hint: what do you need to do to x to make it look like an element in the other set?)

9. (4 pts) Use a proof by induction to show the following:

$$1 + n^2 < 2^n$$

for $n \geq 5$. (Hint: multiply by 2)

10. (4 pts each)

    a. Find the number of permutations of the letters of the word REMAINS such that the vowels always occur in odd places. (Hint: consider the case of vowel placement first)

    b. How many 4 digit numbers that are divisible by 10 can be formed from the numbers 0, 3, 5, 7, 8, and 9 such that no number repeats?

    c. How many ways can we award a 1st, 2nd and 3rd place prize among eight contestants?

    d. There are three dice each of them having faces with a number from 1 to 6. These dices are rolled. Find the number of possible outcomes such that at least one of the dice shows the number 2. NO WRITING OUT ALL POSSIBILITIES. (Hint: If you consider a roll of 1-1-2 and 1-2-1 to be the same, how can you handle the placement of a 2?)