Homework 5 solutions

4.2

(2) $D(R) = \{a, b, c, d\}$

$Range(R) = \{1, 2\}$

D: graph not possible

Matrix

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & 1 & 1 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 1 & 0 & 0 \end{array}$$

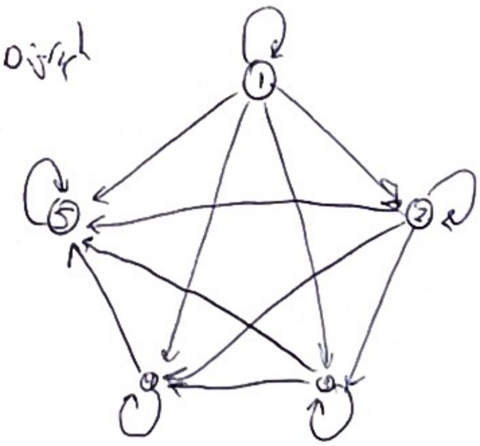
(8) $D(R) = \{1, 2, 3, 4, 5\}$

$Range(R) = \{1, 2, 3, 4, 5\}$

matrix

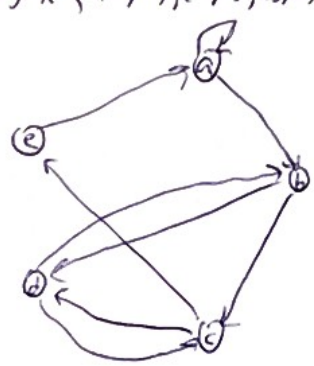
$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Digraph



D.

20) $R = \{(a,a), (a,b), (b,c), (b,d), (c,d), (c,e), (d,b), (d,c), (e,a)\}$



22) $d(1) =$ 1 in, 3 out

$d(2) =$ 2 in, 2 out

$d(3) =$ 0 in, 2 out

$d(4) =$ 2 in, 3 out

$d(5) =$ 1 in, 0 out

4.4

(2) ref: no $(1,1)\notin R$
irref: yes
sym: $(1,2)\in R$ but $(2,1)\notin R$ no
asym: yes
antisym: yes since asym
trans: yes

(6) ref: no $(1,1)\notin R$
irref: no $(2,2)\in R$
sym: yes
asym: no
antisym: no
trans: no $(1,3)\in R$ $(3,1)\in R$ $(1,1)\notin R$

(10) ref: $(1,1)\notin R$
irref: yes
sym: $(1,2)\in R$ $(2,1)\notin R$
asym: yes
antisym yes by asym
trans: yes since no chains of 2. paths

(12) ref: yes
irref: no
sym: yes
asym: no
antisym: no $(1,2)$ and $(2,1)$ $1\neq 2$
trans: yes

(16) ref: yes odd+odd is even and even+even is even
irref: no
sym: yes addition is commutative
asym: no
antisym: no 2 evens do not need to be even
trans: yes only adding odds or all evens.

(20) ref: yes
irref: no
sym: yes equality is commutative
asym: no
antisym: no $b\neq d$.
trans: yes equality

4.5

(2) ref: diagonal yes
sym: no $(1,3)$ $(3,1)$
trans: yes NOT equiv

(4) ref: yes all to selves
sym: no $1\to 2$ $2\not\to 1$ No equiv
trans: no $3\to 1$ $1\to 2$ $3\not\to 2$

(6) ref: yes
sym: yes equiv.
trans: yes

(10) ref: yes
sym: yes equiv.
trans: yes

(18) $R_1=\{(a_1,b_1)\ldots\}$
$R_2=\{(c_1,d_1)\ldots\}$

ref: For R_1 and R_2 to be equiv relations both must have $(a_1,a_1);\ldots(a_n,a_n)$ where all of A is satisfied. Thus in the intersection. This gives reflexive for intersection.

sym: Both R_1 and R_2 have sets of 2 that form a symmetric relation. If they share 1 of these then they must have the other of the pair to remain symmetric. Thus the intersection is reflexive.

trans: If they have the element (a_1,a_2) then there are 3 cases, $a_1=a_2$ which then all done.
If $a_1\neq a_2$ but it pairs with (a_2,a_1), we know by reflexive that $(a_1,a_1)\in R_1\cap R_2$ thus done.
If $a_1\neq a_2$ and it pairs with (a_2,a_3) where $a_3\neq a_1$ then they both must have (a_1,a_3) to remain transitive. Thus, intersection transitive.

5.1

(8) (a) $f(-2) = -1$

Then $g(-1) = g(f(-2)) = 3$

(b) $g(-2) = 6$

$f(6) = f(g(-2)) = 7$

(c) $f(x) = x-1$

$g(x-1) = g(f(x)) = (x-1)^2+2$

(d) $g(x) = x^2+2$

$f(x^2+2) = x^2+3 = f(g(x))$

(e) $f(y) = y+1$

$f(y+1) = y+2 = f(f(y))$

(f) $g(y) = y^2+2$

$g(y^2+2) = (y^2+2)^2+2 = g(g(y))$

(10) ~~[illegible]~~

(a) ~~[illegible]~~ Not one-to-one $|-1| = |1|$

yes onto since all outputs are positive reals or 0.

(b) not one to one since $f(1,2) = f(1,3)$

yes onto since $a \in \mathbb{R}$.

(c) not one to one since $f(1,b) = f(2,b) = (1,a)$

onto not $(2,a)$ does not exist.

(d) no ~~[illegible]~~

$f((a,b)) = f((c,d))$

$(a+b, a-b) = (c+d, c-d)$

$a+b = c+d$ and $a-b = c-d$

if $c \neq a$ and $d \neq b$ then ↑ is impossible so

Thus one-to-one

onto yes pick a pair and solve for a and b.

(e) not one to one $(1)^2 = (-1)^2$

yes onto since all positive #'s can be squarerooted and still be real.

(12)

(a) $x = \sqrt{y} + 1$

$(x-1)^2 = f^{-1}(x)$

(b) $x = y^3 + 1$

$\sqrt[3]{(x-1)} = f^{-1}(x)$

(c) $x = \frac{2y-1}{3}$

$\frac{3x+2}{2} = f^{-1}(x)$

(d) $f^{-1} = \{(3,1), (2,2), (4,3), (5,4), (1,5)\}$

(16) Counting argument:

n^n since each element has n choices to go to and there are n of them.

(20) $f: A \to B$ onto $\quad g: B \to C$ onto

$g \circ f: A \to C$

f being onto maps to everything in B. g being onto maps to everything in C. Thus, g has access to all of B to map into C the way it needs with being onto. Thus, all of C is mapped to and $g \circ f$ is onto.

9.4

(2) $(\mathbb{Z}, -)$

yes!

closed $a-b\in\mathbb{Z}$ ✓

inverse $a-(a)=0$ ✓ $-a\in\mathbb{Z}$

identity $a-0=a$ ✓ $0\in\mathbb{Z}$

associative $a-(b-c)=a+(-b+c)$ and addition is associative

not abelian! $1-2\neq 2-1$

(4) $(\mathbb{Q}, \cdot)$

no!

0 does not have any unique inverse.

(6) $a+b+2\in\mathbb{R}$ closed

$a+b+2=a \Rightarrow b=-2$ identity $\in\mathbb{R}$

$a+b+2=-2$

$a+b=-4$

$b=-4-a\in\mathbb{R}$ inverse

addition is associative

$a+b+2=b+a+2$ ✓ abelian

(8) $a+b+ab\in\mathbb{R}$ closed

$a+b+ab=a$

$b+ab=0$

$b(1+a)=0$ $\quad a\neq -1$

$b=0$ is identity $\in\mathbb{R}\setminus\{-1\}$

$a+b+ab=0$

$b(1+a)=-a$

$b=\frac{-a}{1+a}\in\mathbb{R}$ is it -1?

inverse $-\frac{a}{1+a}=-1 \Rightarrow -1-a=-a$

$-1=0$ ✗ good!

associative $(a+b+ab)*c \qquad a*(b+c+bc)$

$=(a+b+ab)+c+(a+b+ab)c \qquad a+(b+c+bc)+a(b+c+bc)$

$=a+b+ab+c+ac+bc+abc = a+b+c+bc+ab+ac+abc$

(12)

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_4	f_3	f_2	f_6	f_5
f_4	f_4	f_3	f_6	f_5	f_1	f_2
f_5	f_5	f_6	f_2	f_1	f_2	f_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1

$f_1(f_6)$

closed by this, inverse by this, identity is f_1, and associative by table but not abelian.

$f_6(f_2) = \frac{1-x}{1-x-1} = \frac{x-1}{x} = f_5$

$f_4(f_5) = \frac{1}{1-(\frac{x-1}{x})} = \frac{1}{\frac{1}{x}} = x$

$f_6(f_3) = \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{\frac{1}{x}}{\frac{1-x}{x}} = \frac{1}{1-x} = f_4$

$f_3(f_5) = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = \frac{-\frac{1}{x}}{\frac{x-1}{x}} = 1-x$

$f_6(f_5) = \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{\frac{x-1}{x}}{-\frac{1}{x}} = 1-x$

$f_4(f_3) = \frac{1}{1-\frac{1}{x}} = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1} = f_6$

$f_4(f_4) = \frac{1}{1-\frac{1}{1-x}} = \frac{1}{\frac{-x}{1-x}} = \frac{x-1}{x} = f_5$

$f_4(f_6) = \frac{1}{1-\frac{x}{x-1}} = \frac{1}{\frac{-1}{x-1}} = 1-x$

$f_6(f_4) = \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{\frac{1}{1-x}}{\frac{x}{1-x}} = \frac{1}{x}$

$f_5(f_6) = \frac{\frac{x}{x-1}-1}{\frac{x}{x-1}} = \frac{\frac{1}{x-1}}{\frac{x}{x-1}} = \frac{1}{x}$

$f_6(f_6) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{\frac{x}{x-1}}{\frac{1}{x-1}} = x$

abelian yes by definition

con

9.4

(20) (a) yes

even + even = even
thus closed.

inverses are also even
and 0 is even!

(b) no since odd + odd = even
so not closed.

(22) $3^n \cdot 3^k = 3^{n+k}$ closed! ✓

inverse is 3^{-n} ~~[illegible]~~ ✓

identity 3^0 ✓

note $\mathbb{Z}$ is not a group under multiplication though!!

Thus, is not actually a subgroup.