Homework 2 solutions

2.1

(5) (a) True (b) True (c) True (d) False

(7) 2 is odd and -3 is positive or 0 (d) ✓
not negative

(8) (a) $p \wedge \sim r$ (b) ~~[illegible]~~ $q \vee p$

(c) ~~[illegible]~~ $\sim p \wedge \sim q$ (d) $r \wedge q$

(11) (a) For all x there exist y such that x+y is even
such that

(b) There exists x for all y ~~such that~~ x+y is even

(19)

p	q	p ↓ q	r
T	T	F	T
T	F	F	T
F	T	F	F
F	F	T	F

p	q	r
T	T	T
T	F	T
T	F	F
F	T	T
F	F	T
F	T	F
F	F	F

p q r

(18)

p	q	r	$p \downarrow q$	$(p \downarrow q) \downarrow r$
T	T	T	F	F
T	F	F	F	T
T	F	T	F	F
T	T	F	F	T
F	T	T	F	F
F	F	T	T	F
F	T	F	F	T
F	F	F	T	F

(19)

p	q	r	$p \downarrow q$	$p \downarrow r$	$(p \downarrow q) \wedge (p \downarrow r)$
T	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	F	F
T	T	F	F	F	F
F	T	T	F	F	F
F	F	T	T	F	F
F	T	F	F	T	F
F	F	F	T	T	T

(20)

p	q	r	$p \downarrow q$	$p \downarrow r$	$(p \downarrow q) \downarrow (p \downarrow r)$
T	T	T	F	F	T
T	F	F	F	F	T
T	F	T	F	F	T
T	T	F	F	F	T
F	T	T	F	F	T
F	F	T	T	F	F
F	T	F	F	T	F
F	F	F	T	T	F

2.2

(2) (a) $\sim p \Leftrightarrow r$ (b) $r \Rightarrow (p \wedge q)$

(c) $(p \wedge \sim r) \Rightarrow \sim q$ (d) $(\sim p \wedge r) \Rightarrow q$

(4) (a) If I am the Queen of England then $2+2 \neq 4$.

(b) If I do not walk to work then I am President

(c) If I did take the train to work then I am not late

(d) If I didn't go to the store then I don't have time

or I was too tired.

(e) If I didn't bought a car or I didn't bought a house then I didn't

have enough money

(9) If $p \Rightarrow q$ is false p is true and q is false

Thus $p \wedge q$ is false and $\sim(p \wedge q)$ is true

and $(\sim(p \wedge q)) \Rightarrow q$ is false

(11)

p	q	$p \downarrow p$	$q \downarrow q$	$(p \downarrow p) \downarrow (q \downarrow q)$	$p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

$\underbrace{\hphantom{(p \downarrow p) \downarrow (q \downarrow q) \quad p \wedge q}}_{\text{equivalent!}}$

2.3

(1) T

contrapositive of 1st statement

(3) F nothing about not bringing to work is said

(7) T contrapositives

(11) ~~(⇒)~~

Pf: (⇒) Suppose $A=B$. Then $A \subseteq B$ and $B \subseteq A$ ✓

(⇐) Suppose $A \subseteq B$ and $B \subseteq A$. Let $x \in A$.

Then $x \in B$ by $A \subseteq B$. Let $x \in B$. Then $x \in A$ by $B \subseteq A$

Thus any element in one is always in the other $\Rightarrow A = B$. ☺

(15) $n + (n+1) + (n+2) + (n+3) + (n+4)$ is divisible by 5.

Suppose $n \in \mathbb{Z}$

Pf Sum: $n + n+1 + n+2 + n+3 + n+4 = 5n + 10 = 5(n+2)$

$\Rightarrow 5 \mid \text{Sum}$ ☺

(add 1) Sum of 3 integers odd when?

all odd, or 1 is odd

Pf: $a = 2k+1$ $b = 2l+1$ $c = 2m+1$

~~$a+b+c = 2(k+l+m) + 3 = 2(k+l+m+1) + 1$~~

~~$= 2(k+l+m+1) + 1$~~

~~since $k+l+m+1$~~

(add 2) " " even when?

all even or 1 even

Suppose a, b, c are all odd such that

Pf $a=2k+1 \quad b=2l+1 \quad c=2m+1 \quad k, l, m \in \mathbb{Z}^+$

$a+b+c = 2k+2l+2m+3 = 2k+2l+2m+2+1$
$= 2(k+l+m+1)+1$

Since $k+l+m+1 \in \mathbb{Z}^+$, $a+b+c$ is odd.

Suppose a is odd and b and c are even such that

$a=2k+1, \ b=2l \quad c=2m$, $k, l, m \in \mathbb{Z}^+$. Then

$a+b+c = 2k+1+2l+2m = 2(k+l+m)+1$

Since $k+l+m \in \mathbb{Z}^+$, $a+b+c$ is odd.

Suppose ~~a is even~~ a and b are odd and c is even such that

$a=2k+1 \quad b=2l+1 \quad c=2m \quad k, l, m \in \mathbb{Z}^+$.

Then $a+b+c = 2k+1+2l+1+2m = 2k+2l+2m+2 = 2(k+l+m+1)$.

Since $k+l+m+1 \in \mathbb{Z}^+$, $a+b+c$ is even.

Suppose a b and c are all even such that

$a=2k \quad b=2l \quad c=2m \quad k, l, m \in \mathbb{Z}^+$

$a+b+c = 2k+2l+2m = 2(k+l+m)$

Since $k+l+m \in \mathbb{Z}^+$, $a+b+c$ is even.

2.4

(i) Pf: (by Induction)

Since $n \geq 1$, we first show this is true for $n=1$.

Base Case: $1 = 1(2(1)-1) = 1$ ✓

Inductive step: Now suppose this is true for $n \leq k$. We show this is true for $n = k+1$.

Goal: $(k+1)(2(k+1)-1) = (k+1)(2k+1)$

$1 + 5 + 9 + \cdots + (4k-3) + (4(k+1)-3)$

$= k(2k-1) + (4(k+1)-3) = k(2k-1) + (4k+1)$

$= 2k^2 - k + 4k + 1 = 2k^2 + 3k + 1 = (2k+1)(k+1)$

By Induction we are done □

(ii) $1 + 2^n < 3^n \quad \forall n \geq 2$.

Pf: (by Induction)

We first show true for $n=2$.

Base Case: $1 + 2^2 = 5 < 3^2 = 9$ ✓

Inductive step: Suppose this is true for $n \leq k$. We show true for $n = k+1$.

$1 + 2^k < 3^k \Rightarrow \cancel{3+3} \quad \cancel{4+2\cdot 3^k} \quad 2 + 2^{k+1} < 2 \cdot 3^k$

$\cancel{2\cdot 3^k}$

$\Rightarrow 1 + 2^{k+1} < 2 + 2^{k+1} < 2 \cdot 3^k < 3 \cdot 3^k = 3^{k+1}$

$\Rightarrow 1 + 2^{k+1} < 3^{k+1}$. By Induction we are done.

(12)

Pf: Sup. (By induction) We start at $n=1$.

Base case: $1 < \frac{(2(1)+1)^2}{8} = 9/8$ ✓

Induction step: Suppose true for $n \leq k$. We show true for $k+1$.

This means $1+2+\cdots+k+(k+1) < \frac{(2(k+1)+1)^2}{8} = \frac{(2k+3)^2}{8} = \frac{4k^2+12k+9}{8}$

$1+2+\cdots+k+(k+1) < \frac{(2k+1)^2}{8} + k+1 = \frac{(2k+1)^2+8k+8}{8}$

$= \frac{4k^2+4k+1+8k+8}{8} = \frac{4k^2+12k+9}{8}$

Thus by Induction we are done □

(13) Pf: (By Induction)

Base case: $n=0 \Rightarrow A=\emptyset$ $P(\emptyset) = 1 = 2^0$ (itself)
$n=1 \Rightarrow A=\{a\}$ $P(A) = 2 = 2^1$

Induction step: Suppose true for $n \leq k$. We show true for $n=k+1$.

That is $P(A) = 2^{k+1}$ when $|A| = k+1$

Consider $A = \{a_1, \ldots, a_k, a_{k+1}\}$. ~~Then~~ Consider the subset of A

$\{a_1, \ldots, a_k\}$ of k elements. By I.H. this has 2^k subsets. All of

these subsets are in A as well. The only subsets of A not in $\{a_1, \ldots, a_k\}$ are

ones that contain a_{k+1}.

Each of the subsets of $\{a_1, \dots, a_k\}$ can form a new subset with the injection a_{k+1}. That is now $2 \cdot 2^k$.

This is all of them. ~~...~~ Thus by Induction we are done. $\square$

(15)

(21)

Pf (by induction) We show for $n=1$ this is true

$A^2 A^1 = A^{2+1} = A^3$

$A^{2+1} = A^3$ not much proof needed here

Ind. Step: Suppose true for $n \leq k$. We show true for $n = k+1$.

That is $A^2 A^{k+1} = A^{2+(k+1)}$

$A^2 A^{k+1} = A^2 A^k \cdot A = A^{2+k} \cdot A = A^{2+k+1}$ $\square$ by Induction we are done $\square$

(15) ~~Base~~

Pf: Base cases: $n=1$ clearly true

$n=2$ $x \in \overline{A_1 \cap A_2} \Rightarrow$ ~~...~~

Thus $x \notin A_1 \cap A_2$, ~~...~~ Thus $x \in \bar{A_1}$ or $x \in \bar{A_2}$ thus $\overline{A_1 \cap A_2} \subset \bar{A_1} \cup \bar{A_2}$

Similar argument the other way (De Morgan's law.)

Ind. Step. Suppose true for $n \leq k$. Then we show true for $n = k+1$.

$\overline{\bigcap_{i=1}^{k+1} A_i} = \overline{\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}}$

$\bigcup_{i=1}^{k+1} \bar{A_i} = \bigcup_{i=1}^{k} \bar{A_i} \cup \bar{A}_{k+1} = \overline{\bigcap_{i=1}^{k} A_i} \cup \bar{A}_{k+1} = \overline{\bigcap_{i=1}^{k+1} A_i}$ by De Morgan's law $\square$. ~~Done~~ by Induction $\square$