Homework 1 Solutions

1.1

1) (a) True (b) False (c) False (d) False
   (e) True (f) False

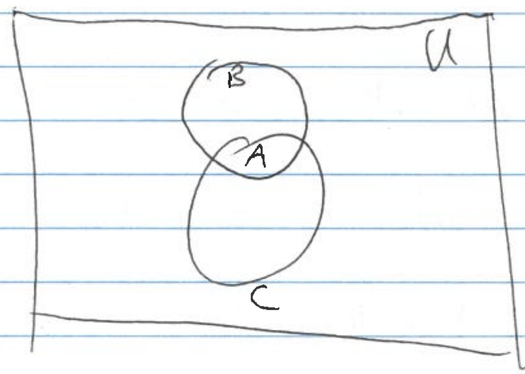
5) ~~{x | x is even, x ≤ 10}~~ Many Solutions
   (a) {x | x even positive integer less than 11}
   (b) {x | x is a vowel in the alphabet}
   (c) {x | x positive integer, $x^3 \le 125$}
   (d) {x | x integer, $x^2 \le 4$}

11) (a) True (b) False (c) False (d) True
    (e) True (f) True (g) True (h) True

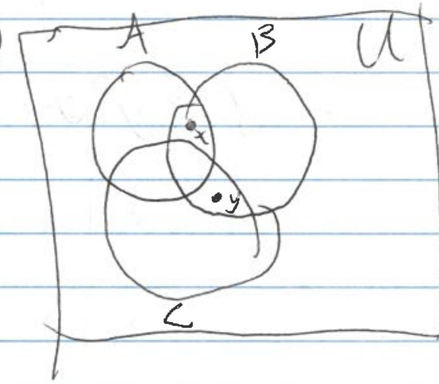
15) (a) False (b) True (c) False (d) True
    (e) True (f) False

18)

(a)



(b)



1.2

(3) (a) $\{a, b, c, d, e, f, g\}$ (b) $\emptyset$

(c) $B \cup C = \{a, c, d, e, f, g\} \Rightarrow A \cap (B \cup C) = \{a, c, g\}$

(d) $A \cup B = \{a, b, c, d, e, f, g\} \Rightarrow (A \cup B) \cap C = \{a, c, f\}$

(e) $\overline{A \cup B} = U \setminus (A \cup B) = \{h, k\}$ (f) $A \cap B = \{g\}$

$\overline{A \cap B} = U \setminus \{g\}$ or $\{a, b, c, d, e, f, h, k\}$

(5)

(a) $\{1,2,4,5,6,8,9\}$ (b) $C=\{1,2,3,4\} \Rightarrow A\cup C=\{1,2,3,4,6,8\}$

(c) $\{1,2,4,6,8,5\}$ (d) $\{1,2,3,4,5,9\}$ (e) $\{1,2,4\}$

(f) $\{8\}$ (g) $\{2,4\}$ (h) $\varnothing$ (i) $\{1,6,8\}$

(j) $\{5,9\}$ (k) C (l) $\{5,6,7,8,9\}$

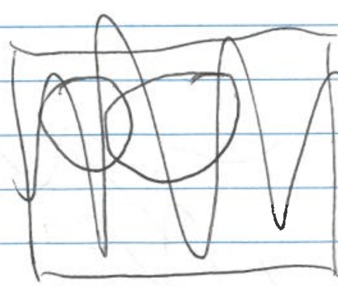
(m) $\{3,5,7,9\}$ (n) union of i and j $\{1,5,6,8,9\}$

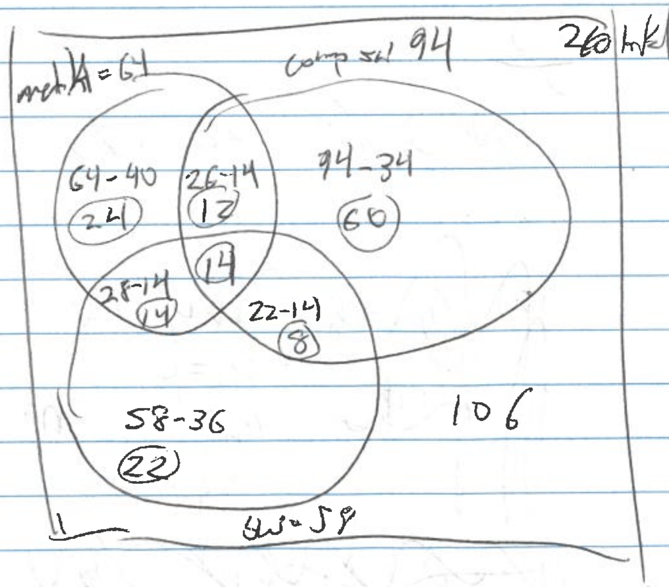
(o) Since $D\setminus C = 1)$, $\{1,2,3,4,7,8\}$ (p) $B\setminus C=\{5,9\} \Rightarrow \{1,3,5,9\}$
$C\setminus B=\{1,3\}$

(10) (a) False (b) True (c) False (d) True

(12) $C\cup(A\cap B)$ many answers.

(17)





(a) $260-(94+22+14+24)$
$= 260-(94+60)$
$= 260-154 = 106$

(b) 60

27) No take $A=\{1,2,3\}$ $B=\{1\}$, $C=\{2\}$

$A\cup B=A$ $A\cup C=A$ but $B\neq C$.

29) pf: let $x\in A\cup C$. Then $x\in A$ or $x\in C$.

If $x\in A$ then $x\in B$ since $A\subseteq B$. Thus $x\in B\cup D$.

If $x\in C$ then $x\in D$ since $C\subseteq D$. Thus, $x\in B\cup D$ □

pf: let $x\in A\cap C$. Then $x\in A$ and $x\in C$.

Since $x\in A$ and $A\subseteq B$, $x\in B$.

Since $x\in C$ and $C\subseteq D$, $x\in D$

Thus, $x\in B\cap D$ □

1.3

~~(7) 5, 25, 125, 625~~

~~$S_n = 5+25+125+\ldots 5^n$ geometric~~

~~$S_n = \frac{a(1-r^{n-1})}{1-r}$~~

~~(14)~~

~~(12) $a_n = a_{n-1}+\ldots$~~

~~$a_1 = 0$~~

$a_1 = 0$

$a_n = a_{n-1} + 2n+1$

recursive

~~(17) $a_n = 2+3(n-1)$, $n\geq 1$~~

1.3:

(7) 5, 25, 125, 625

12) $a_1 = 0$, $a_n = a_{n-1} + 2n - 1$ recursive

14) $a_n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$ explicit.

24) ~~$x \in (A \Delta B) \Delta C$. Then~~ $x \in (A \Delta B) \setminus C$ or $C \setminus (A \Delta B)$

~~If $x \in (A \Delta B) \setminus C$ then~~ $x \in A \Delta B$ ~~and~~ $x \notin C$.

~~$x \in A \Delta B \Rightarrow$ $x \in A \setminus B$ or $x \in B \setminus A$.~~

~~If $x \in A \setminus B$ $x \in A$ and $x \notin B$~~

~~If $x \in B \setminus A$, $x \in B$ and $x \notin A$~~

~~If $x \in C \setminus (A \Delta B)$ then~~

~~$x \in C$ and $x \notin A \Delta B \Rightarrow x \notin A \setminus B$ and $B \setminus A$.~~

1.4

(6)

$100 = 1\cdot 60 + 40$

$60 = 1\cdot 40 + \boxed{20}$

$40 = 2\cdot 20 + 0$

$100 = 50\cdot 2 = 2\cdot 2\cdot 25 = 5^2\cdot 2^2$

$60 = 30\cdot 2 = 2\cdot 2\cdot 15 = 2^2\cdot 3\cdot 5$

$\gcd(100, 60) = 2^2\cdot 5 = 20$

(8)

$58 = 1\cdot 34 + 24$

$34 = 1\cdot 24 + 10$

$24 = 2\cdot 10 + 4$

$10 = 2\cdot 4 + \boxed{2}$

$4 = 2\cdot 2 + 0$

$58 = 2\cdot 29$

$34 = 2\cdot 17$

$\gcd(58, 34) = 2$

(9)

$128 = 1\cdot 77 + 51$

$77 = 1\cdot 51 + 26$

$51 = 1\cdot 26 + 25$

$26 = 1\cdot 25 + \boxed{1}$

$25 = 1\cdot 25 + 0$

$77 = 7\cdot 11$

$128 = 2\cdot 64 = 2\cdot 2\cdot 32 = 2\cdot 2\cdot 2\cdot 16$

$= 2\cdot 2\cdot 2\cdot 2\cdot 8$

$= 2^5\cdot 4 = 2^6\cdot 2 = 2^7$

$\gcd(77, 128) = 1$

(12) $175 = 5\cdot 35 = 5^2\cdot 7$

$245 = 5\cdot 49 = 5\cdot 7^2$

$\mathrm{lcm}(175, 245) = 5^2\cdot 7^2$

(14) (a) ~~17 (mod 7)~~

$17 \equiv 3 \pmod 7$

(b) $48 \equiv 6 \pmod 7$

(c) $1207 \equiv 3 \pmod 7$

(d) $130 \equiv 4 \pmod 7$

(e) $93 \equiv 2 \pmod 7$

(f) $169 \equiv 1 \pmod 7$

extra 4

1.5

(14) $a+2b=4$ $\qquad$ $2c+d=4$

$2a-b=-2$ $\qquad$ $c-2d=-3$

$\Rightarrow a=4-2b$ $\qquad$ $\Rightarrow d=4-2c$

$\Rightarrow 2(4-2b)-b=-2$ $\qquad$ $\Rightarrow c-2(4-2c)=-3$

$8-4b-b=-2$ $\qquad$ $c-8+4c=-3$

$8-5b=-2$ $\qquad$ $5c-8=-3$

$b=2 \Rightarrow a=0$ $\qquad$ $c=1 \Rightarrow d=2$

(15) (a) $C+E=\begin{bmatrix} 4 & 0 & 2 \\ 9 & 6 & 2 \\ 3 & 2 & 4 \end{bmatrix}$

(b) $A=2\times3$

$B=3\times2$

$AB=\begin{bmatrix} 3(0)+1(1)+3(2) & 2(1)+1(2)+3(3) \\ 4(0)+1(1)+(-2)(2) & 4(1)+1(2)+(-2)(3) \end{bmatrix}=\begin{bmatrix} 7 & 13 \\ -3 & 0 \end{bmatrix}$

$BA=\begin{bmatrix} 0(2)+1(4) & 0(1)+1(1) & 0(3)+1(-2) \\ 1(2)+2(4) & 1(1)+2(1) & 1(3)+2(-2) \\ 2(2)+3(4) & 2(1)+3(1) & 2(3)+3(-2) \end{bmatrix}$

$=\begin{bmatrix} 4 & 1 & -2 \\ 10 & 3 & -1 \\ 16 & 5 & 0 \end{bmatrix}$

(c) $CB=$

$3\times3 \quad 3\times2$

$\begin{bmatrix} 1(0)+(-2)(1)+3(2) & 1(1)+(-2)(2)+3(3) \\ 4(0)+2(1)+5(2) & 4(1)+2(2)+5(3) \\ 3(0)+1(1)+2(2) & 3(1)+1(2)+2(3) \end{bmatrix}=\begin{bmatrix} 4 & 6 \\ 12 & 23 \\ 5 & 11 \end{bmatrix}$

$CB+F=\times$

(d) $AB = \begin{bmatrix} 7 & 13 \\ -3 & 0 \end{bmatrix}$

2×3 3×2

2×2

$DF = \begin{bmatrix} -3(-2)+2(4) & -3(3)+2(5) \\ 4(-2)+1(4) & 4(3)+1(5) \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ -4 & 17 \end{bmatrix}$

$AB + DF = \begin{bmatrix} 9 & 14 \\ -7 & 17 \end{bmatrix}$

(8) $A^T = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & -2 \end{bmatrix}$ $(A^T)^T = A$

(b) $C+E = \begin{bmatrix} 4 & 0 & 2 \\ 9 & 6 & 2 \\ 3 & 2 & 4 \end{bmatrix}$

$(C+E)^T = \begin{bmatrix} 4 & 9 & 3 \\ 0 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 2 & 1 \\ 3 & 8 & 2 \end{bmatrix}$ $E^T = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 4 & 1 \\ -1 & -3 & 2 \end{bmatrix}$

$C^T + E^T = \begin{bmatrix} 4 & 9 & 3 \\ 0 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} = (C+E)^T$!

(c) $AB = \begin{bmatrix} 7 & 13 \\ -3 & 0 \end{bmatrix}$

$\rightarrow (AB)^T = \begin{bmatrix} 7 & -3 \\ 13 & 0 \end{bmatrix}$

$B^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & -2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 0(2)+1(1)+2(3) & 0(4)+1(1)+2(-2) \\ 1(2)+2(1)+3(3) & 1(4)+2(1)+3(-2) \end{bmatrix}$

$= \begin{bmatrix} 7 & -3 \\ 13 & 0 \end{bmatrix} = (AB)^T$

$B^T C + A = \begin{bmatrix} 12 & 5 & 12 \\ 22 & 6 & 17 \end{bmatrix}$

(d) $B^T C =$

2×3 3×3

2×3

$= \begin{bmatrix} 0(1)+1(4)+2(3) & 0(-2)+1(2)+2(1) & 0(3)+1(5)+2(2) \\ 1(1)+2(4)+3(3) & 1(-2)+2(2)+3(1) & 1(3)+2(5)+3(2) \end{bmatrix} = \begin{bmatrix} 10 & 4 & 9 \\ 18 & 5 & 19 \end{bmatrix}$

(13) $A^2 = \begin{bmatrix} 3(3)+0(0)+0(0) & 3(0)+0(-2)+0(0) & 3(0)+0(0)+0(4) \\ 0(3)+(-2)(0)+0(0) & 0(0)+(-2)(-2)+0(0) & 0(0)+(-2)(0)+0(4) \\ 0(3)+0(0)+4(0) & 0(0)+0(-2)+4(0) & 0(0)+0(0)+4(4) \end{bmatrix}$

$= \begin{bmatrix} 9 & & 0 \\ & 4 & \\ 0 & & 16 \end{bmatrix} \Rightarrow A^3 = \begin{bmatrix} 27 & & 0 \\ & -8 & \\ 0 & & 64 \end{bmatrix}$

$A^k = \begin{bmatrix} 3^k & & 0 \\ & (-2)^k & \\ 0 & & 4^k \end{bmatrix}$

(18)

$AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

$a_{11}b_{11}+a_{12}b_{21} = 0$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1(0)+0(1) & 1(0)+0(0) \\ 0(0)+0(1) & 0(0)+0(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✓

1.6

(1) (a) closed $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

$ad-bc \neq 0$

$eh-fg \neq 0$

Is $(ae+bg)(cf+dh) - [(af+bh)(ce+dg)] \neq 0$?

identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

associative $(AB)C = A(BC)$ ✓

inverse: since $ad-bc \neq 0$

$\frac{1}{ad-bc}$ exists

Thus, inverse exists

$aecf + aedh + bgcf + bgdh$

$-(afce + dgaf + bhce + bhdg)$

$= aedh + bgcf - dgaf - bhce$

$= (ad-bc)eh + (bc-ad)gf$

$= (ad-bc)(eh-gf) \neq 0$ ✓

(2) inverse fails!