Name: **Final Exam (Version Beta)**

Instructions: This exam is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Remove any smartwatches and non-religious head-wear. On your favorite question draw your favorite animal doing your favorite activity. Cheating of any kind will not be tolerated and will result in a grade of zero. Answer each question. **Show all work to receive full credit.** Write your answers on the test. You have **2 hours 30 minutes** to finish the exam. Answer all questions to the best of your ability. Unless otherwise specified, you are required to **SHOW ALL YOUR WORK** to receive full credit. The exam has 220 possible points. You will be graded out of 200 points.

Questions	Possible	Score		Possible	Score
Question 1	22		Question 8	12	
Question 2	14		Question 9	12	
Question 3	10		Question 10	12	
Question 4	14		Question 11	20	
Question 5	24		Question 12	24	
Question 6	18		Question 13	20	
Question 7	18		Total	220	

1. (1 pt each unless otherwise labeled) The below are short answer, fill-in the blank, or True-False questions. If you see a blank, then fill it in. If you see (T/F) at the beginning, fill the blank in with a T for true or F for false. If there is space left below the question, it is a short response question.

   a. (T/F) All sets have an even number of subsets. ______

   b. (2 pts) We know $|A \cup B| \leq |A| + |B|$. When are they equal?

   c. For a set A, $\overline{A}$ is called the __________ of A.

   d. (2 pts) If $\gcd(a, b) = 15$, $\text{lcm}(a, b) = 45$ and $a > b$, what are a and b?

   e. (T/F) $45 \equiv 21 \pmod 8$ ______

   f. Let A be the matrix of dimensions m by n and B be the matrix of dimension n by p. Then the dimensions of AB are __________.

   g. For arbitrary sets A and B, $A \Delta B =$ ______________.

   h. (2 pts) Determine a sequence, a_n, such that for any k, $a_k < a_{k+1}$.

   i. If p is a false statement, then $p \Rightarrow q$ is ______ for true or false q statement.

   j. (2 pts) What is the contrapositive of the statement "If John is swimming in the ocean and there is a shark in the water, then John will run out of the water screaming"?

   k. (T/F) $nC_k = \dfrac{nP_k}{k!}$ ______

   l. If S is the sample space and E is the event, $p(E) =$ __________.

   m. (T/F) If E and F are any events in the sample space S then $P(E \cup F) = P(E) + P(F)$. ______

   n. (2 pts) Give an example of a relation, other than '=', on $\mathbf{Z}$ to $\mathbf{Z}$ that is reflexive.

   o. (2 pts) Let $f : \{-1, 0, 1, 2\} \to \{-1, 0, 1\}$ be a function where $f(-1) = 1$, $f(2) = 0$, $f(0) = 0$, and $f(1) = -1$. Determine if f is onto, one-to-one, both, or neither.

   p. (T/F) The set $\{1, 2, 3\}$ is a closed under addition. ______

2. Use the Euclidean Algorithm to find the following:

   a. (5 pts) $\gcd(1180, 482)$

   b. (5 pts) Linear combination of 1180 and 482 equal to $\gcd(1180, 482)$.

   c. (2 pts) $\gcd(55, 54)$

   d. (2 pts) $\gcd(1111, 1112)$

3. (10 pts) Let a be a positive nonzero integer. Using (2c) and (2d) above, determine $\gcd(a, a+1)$ and prove your answer to using the Eulicdean Algorithm.

4. (14 pts) Using (3) prove the following: Prove that any subset of $\{1, 2, 3, \ldots, 2n\}$ of size $n+1$ contains 2 integers that are relatively prime.

5. (8 pts each) Prove or disprove the following statements. Please label which part you are proving below.

   a. $1 + 9 + \ldots + (2n+1)^2 = \dfrac{(n+1)(2n+1)(2n+3)}{3}$ for $n \geq 0$ (Notice that n starts at 0 so when $n = 0$ the right hand side is just the first number, 1).

   b. For any positive integer n, if n is prime, then $n^2 + 4$ is prime.

   c. Prove that for any $x \in \mathbf{Z}$ if $7x + 4$ is even then $3x - 11$ is odd.

6. Consider the following sequence of matrices.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}, \ldots$$

a. (4 pts) Write an explicit formula for this sequence of matrices.

b. (14 pts) Let a_n be the recursive form of the sequence of matrices such that

$$a_n = a_{n-1} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Use backtracking to verify your answer to (a).

7. a. (12 pts) Prove the following is a group under coordinate-wise addition:
$\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(a, b) |\ a, b \in \mathbf{Z}_2\}$.
(Example of coordinate-wise addition in general: $(a, b) + (c, d) = (a + c, b + d)$.) You may do this proof by group table or direct proof. If you choose to use the below table, you must then explain each condition for why this is a group.

$\mathbf{Z}_2 \times \mathbf{Z}_2$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)				
(1,0)				
(0,1)				
(1,1)				

b. (6 pts) Above you have shown that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is a group. Show $\mathbf{Z}_4$ is also a group. No explanation needed.

$\mathbf{Z}_4$	0	1	2	3
0				
1				
2				
3				

8. (12 pts) Let a and b be in an abelian group such that

$$a * b = e, \quad a \neq b,$$

$$a * a * a = e, \quad a * a \neq e,$$

$$b * b * b = e, \quad b * b \neq e.$$

If every element in the group is generated by a and b (formed by some combination of a and b with the operation) what is the maximum size of the group given the conditions above? Explain in detail. (Showing a single case of a and b does not do the job, however, writing out some combinations of a and b may clear things up.)

9. (12 pts) Use a truth table of p and q (MUST include q) to find a logically equivalent statement to $\sim p$.

10. (12 pts) Use a truth table to show that $((p \Rightarrow q) \Rightarrow q) \Rightarrow q \equiv p \Rightarrow q$.

11. (5 pts each) You have a standard deck of 52 cards. (ace is low and king is high)

    a. Find the probability of picking 2 black cards that are consecutive. (EX: 3 of spades, 4 of clubs or 2 of clubs)

    b. Find the probability you draw 3 of the same value (king of clubs and king of spades are the same value) where you put the card back in randomly before the next draw.

    c. Find the probability you either draw 2 face cards (not including aces) or 2 red cards that are consecutive.

    d. Find the probability that if you draw 3 cards, they alternate red, black, red and are less than 6.

12. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and R is a relation over such that xRy if and only if $x - y = 3n$ for some $n \in \mathbf{Z}$.

a. (5 pts) Find the matrix of the relation.

b. (6 pts) Use the matrix to find the in and out degree of every element in A. Explain how you were able to do this from the matrix without the digraph.

c. (5 pts) Draw the digraph of the relation.

d. (8 pts) Explain using either of the visuals above (or some combination) why R is an equivalence relation.

13. (20 pts) Write a problem that has to do with any topic from this class that you found the most interesting (Your problem should be similar to the difficulty of the problems on this exam besides problem 1). Then explain how you would solve/prove/answer this problem. Please be clever with this! This is a CLEAR indicator to me as to what you have learned in this class. I value this question the most of all the questions on this exam.