Instructions: This quiz is closed book, closed note, and an individual effort. **Show all work to receive full credit.** Unless the question specifies, you should provide an exact answer. If you get stuck, please attempt to explain what you want to do. This may give more partial credit.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.
I, ____________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this quiz. I will show all my work to demonstrate my knowledge on the topic.

1. If A has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = (1, 0)$ and $\lambda_2 = 5$ with eigenvector $\mathbf{x}_2 = (1, 1)$, find A.

   Solution 1. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$

2. The block B has eigenvalues 1 and 2. The block C has eigenvalues 3 and 4. The block D has eigenvalues 5 and 7. Find the eigenvalues of A if

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

   Before rolling your sleeves up and doing this out, try to reason what may happen.

   Solution 2. A will have eigenvalues 1, 2, 5 and 7 since C when taking the determinant of A the values in C will always mutliply the bottom left 2 by 2 matrix of all zeros. Meanwhile, B and D are not!

3. Diagonalize A and compute $X\Lambda X^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$$

   Solution 3. $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$
   Thus, $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $X^{-1} = (1/2) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Thus, $A = (1/2) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$A^k = (1/2) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right)^k \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = (1/2) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$= (1/2) \begin{bmatrix} 1 & -3^k \\ 1 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = (1/2) \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

4. Find two $\lambda's$ and $\mathbf{x}'s$ so that $\mathbf{u} = e^{\lambda t}\mathbf{x}$ solves

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}.$$

What combination $\mathbf{u} = \mathbf{c}_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$ starts from $\mathbf{u}(0) = (5, -2)$?

Solution 4. $\det(A - \lambda I) = (4-\lambda)(1-\lambda) = (\lambda - 4)(\lambda - 1)$. Thus, $\lambda_1 = 1$ and $\lambda_2 = 4$.

$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow 3x + 3y = 0 \Rightarrow x = -y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, $\mathbf{u} = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 e^t + c_2 e^{4t} \\ c_1 e^t \end{bmatrix}$. Since $\mathbf{u}(0) = (5, -2)$, we have $-c_1 + c_2 = 5$ and $c_1 = -2$. Thus, $c_2 = 3$. Therefore, $\mathbf{u} = \begin{bmatrix} 2e^t + 3e^{4t} \\ 2e^t \end{bmatrix}$.

5. Find an orthonormal matrix Q that diagonalizes $S = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

Solution 5. $\det(S - \lambda I) = (-2-\lambda)(7-\lambda) - 36 = \lambda^2 - 5\lambda - 50 = (\lambda - 10)(\lambda + 5)$. Thus, $\lambda_1 = 10$ and $\lambda_2 = -5$.

$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \Rightarrow 6x - 3y = 0 \Rightarrow 2x = y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \Rightarrow 3x + 6y = 0 \Rightarrow x = -2y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. These are orthogonal since S is symmetric. Thus, we normalize.

$\mathbf{q}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Therefore, $Q = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$.

6. For which numbers b and c are ALL these matrices positive definite?

$$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}$$

Solution 6. 1) $9 - b^2$ must be positive. This is true for $b^2 < 9$. Thus, $-3 < b < 3$.
2) $2c - 16$ must be positive. Thus, $c > 8$. This also takes care of the condition for the upper left c being positive.
3) $c^2 - b^2$ must be positive. This is true for $c^2 > b^2$. This is true given the above restrictions on c and b.
Thus, the conditions composed in (1) and (2) are enough for all 3 matrices to be positive definite.