Instructions: This quiz is closed book, closed note, and an individual effort. **Show all work to receive full credit.** Unless the question specifies, you should provide an exact answer. If you get stuck, please attempt to explain what you want to do. This may give more partial credit.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.

I, ____________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this quiz. I will show all my work to demonstrate my knowledge on the topic.

1. Construct a matrix with the required properties or explain why it is not possible.

   a. Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.

   Solution 1. $\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}$ since row 1 and 2 generate all of $\mathbf{R}^2$ which certainly contains the two vectors required.

   b. Column space has basis $\begin{bmatrix}1\\1\\3\end{bmatrix}$, nullspace has basis $\begin{bmatrix}3\\1\\1\end{bmatrix}$.

   Solution 2. Not possible since $\dim(C(A)) = r = 1$ and $\dim(N(A)) = n - r = 1$. Thus, $n = 2$. However, $N(A) \in \mathbf{R}^m$. Therefore, the basis element cannot be 3-dimensional. Also seen since to be in the nullspace this must be true: $A\begin{bmatrix}3\\1\\1\end{bmatrix} = \mathbf{0}$. However, A has dimensions 3 x 2 and the vector is 3 x 1.

2. Determine of the following pairs are orthonormal, orthogonal, independent or any combination. Show work testing each!

   a. $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$

   Solution 3. Not orthonormal since the second vector has length $\sqrt{2}$. Thus, we test independence and orthogonality.
   $\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$ $R_1 + R_2 \to R_1$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. Thus, for these two vectors to be linear combination equal to $\mathbf{0}$, they must be multiplied by scalars of 0.

b. $\begin{bmatrix} .6 \\ .8 \end{bmatrix}$ and $\begin{bmatrix} .4 \\ -.3 \end{bmatrix}$

Solution 4. $\begin{bmatrix} .6 & .4 \\ .8 & -.3 \end{bmatrix}$ $10R_1, 10R_2$ $\begin{bmatrix} 6 & 4 \\ 8 & -3 \end{bmatrix}$ $R_1 + R_2 \to R_2$ $\begin{bmatrix} 6 & 4 \\ 14 & 1 \end{bmatrix}$
$4R_2 - R_1 \to R_1$ $\begin{bmatrix} 50 & 0 \\ 14 & 1 \end{bmatrix}$ $R_1/50$ $\begin{bmatrix} 1 & 0 \\ 14 & 1 \end{bmatrix}$ $R_2 - 14R_1 \to R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Thus, independent.

Now we test the length. $\sqrt{.36 + .64} = 1$ and $\sqrt{.16 + .09} \neq 1$. Thus, not orthonormal. For orthogonality, their dot product must be 0. We get $.24 - .24 = 0$. Thus, orthogonal.

3. Find $A^T A$ if the columns are unit vectors, all mutually perpendicular (all perpendicular to each other).

Solution 5. $A = \begin{bmatrix} col(1) & . & . & . & col(n) \end{bmatrix}$ where $col(i)$ are vectors. Then

$$A^T A = \begin{bmatrix} col(1) \\ . \\ . \\ . \\ col(n) \end{bmatrix} \begin{bmatrix} col(1) & . & . & . & col(n) \end{bmatrix} = \begin{bmatrix} col(1) \cdot col(1) & ... & col(1) \cdot col(n) \\ col(2) \cdot col(1) & & \\ . & & \\ . & & \\ col(n) \cdot col(1) & ... & col(n) \cdot col(n) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ . & & & \\ . & & & \\ 0 & 0 & ... & 1 \end{bmatrix} = I_n$$

4. Given the following set of points, find the below: $(0,0),(1,8),(3,8),(4,20)$.

a. Find the line of best fit for the points.

Solution 6.

$$\begin{aligned} 0 &= 0m + b \\ 8 &= 1m + b \\ 8 &= 3m + b \\ 20 &= 4m + b \end{aligned}$$

These must be true for an exact solution. Let's turn it into a matrix problem:

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. It is clear no line can go through all of these points (also seen since the matrix will have free variables). Thus, we use a projection to get the closest! Multiply both sides by A^T.

$\begin{bmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 26 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 112 \\ 36 \end{bmatrix}$. Now we can solve.

$\begin{bmatrix} 26 & 8 & 112 \\ 8 & 4 & 36 \end{bmatrix} R_1 - 2R_2 \to R_1 \begin{bmatrix} 10 & 0 & 40 \\ 8 & 4 & 36 \end{bmatrix} R_1/10, R_2/4 \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 9 \end{bmatrix}$
$R_2 - 2R_1 \to R_2 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$. Thus, $m = 4$ and $b = 1$ gives the line of best fit: $y = 4x + 1$.

b. Find the closest parabola to the points (You may use a calculator to do some of the arithmetic).

Solution 7.

$$\begin{aligned} 0 &= a(0)^2 + b(0) + c \\ 8 &= a(1)^2 + b(1) + c \\ 8 &= a(3)^2 + b(3) + c \\ 20 &= a(4)^2 + b(4) + c \end{aligned}$$

These must be true for an exact solution. Let's turn it into a matrix problem:

$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. It is clear no parabola can go through all of these points (also seen since the matrix will have free variables). Thus, we use a projection to get the closest! Multiply both sides by A^T.

$$\begin{bmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 400 \\ 112 \\ 36 \end{bmatrix}.$$

Now we can solve.

$$\begin{bmatrix} 338 & 92 & 26 & 400 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{bmatrix} R_1/2, R_2/2, R_3/2 \begin{bmatrix} 169 & 46 & 13 & 200 \\ 46 & 13 & 4 & 56 \\ 13 & 4 & 2 & 18 \end{bmatrix} R_2 - 2R_3 \to R_2 \begin{bmatrix} 169 & 46 & 13 & 200 \\ 20 & 5 & 0 & 20 \\ 13 & 4 & 2 & 18 \end{bmatrix}$$

$$R_1 - 13R_3 \to R_3 \begin{bmatrix} 169 & 46 & 13 & 200 \\ 20 & 5 & 0 & 20 \\ 0 & -6 & -13 & -34 \end{bmatrix} R_1 + R_3 \to R_1 \begin{bmatrix} 169 & 40 & 0 & 166 \\ 20 & 5 & 0 & 20 \\ 0 & -6 & -13 & -34 \end{bmatrix}$$

$$R_1 - 8R_2 \to R_1 \begin{bmatrix} 9 & 0 & 0 & 6 \\ 20 & 5 & 0 & 20 \\ 0 & -6 & -13 & -34 \end{bmatrix} R_1/9, R_2/5, -R_3 \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 4 & 1 & 0 & 4 \\ 0 & 6 & 13 & 34 \end{bmatrix} R_2 - 4R_1 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 6 & 13 & 34 \end{bmatrix} R_3 - 6R_2 \to R_3 \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 13 & 26 \end{bmatrix} R_3/13 \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, we get the best fitting parabola, $y = (2/3)x^2 + (4/3)x + 2$.

c. Find the closest cubic to the points (You may use a calculator to do some of the arithmetic).

Solution 8.

$$\begin{aligned} 0 &= a(0)^3 + b(0)^2 + c(0) + d \\ 8 &= a(1)^3 + b(1)^2 + c(1) + d \\ 8 &= a(3)^3 + b(3)^2 + c(3) + d \\ 20 &= a(4)^3 + b(4)^2 + c(4) + d \end{aligned}$$

These must be true for an exact solution. Let's turn it into a matrix problem:

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. The matrix is square! Thus, there is an exact solution and we can solve it by elimination!

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 27 & 9 & 3 & 1 & 8 \\ 64 & 16 & 4 & 1 & 20 \end{bmatrix} R_3{-}R_2 \to R_3 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 26 & 8 & 2 & 0 & 0 \\ 64 & 16 & 4 & 1 & 20 \end{bmatrix} R_4{-}2R_3 \to R_4 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 26 & 8 & 2 & 0 & 0 \\ 12 & 0 & 0 & 1 & 20 \end{bmatrix}$$

$$\text{Swap } R_4 \text{ and } R_1 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 1 & 1 & 1 & 1 & 8 \\ 26 & 8 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_3/2 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 1 & 1 & 1 & 1 & 8 \\ 13 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_3{-}R_2 \to R_2 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 12 & 3 & 0 & -1 & -8 \\ 13 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_2 - R_1 \to R_2 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 0 & 3 & 0 & -2 & -28 \\ 13 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_2 + 2R_4 \to R_2 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 0 & 3 & 0 & 0 & -28 \\ 13 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_2/3 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & 0 & -28/3 \\ 13 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_3 - 4R_2 \to R_3 \begin{bmatrix} 12 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & 0 & -28/3 \\ 13 & 0 & 1 & 0 & 112/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_1 - R_4 \to R_1$$

$$\begin{bmatrix} 12 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & -28/3 \\ 13 & 0 & 1 & 0 & 112/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_1/12 \begin{bmatrix} 1 & 0 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 0 & -28/3 \\ 13 & 0 & 1 & 0 & 112/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} R_3{-}13R_1 \to R_3 \begin{bmatrix} 1 & 0 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 0 & -28/3 \\ 0 & 0 & 1 & 0 & 47/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, the cubic containing all the points is $y = (5/3)x^3 - (28/3)x^2 + 47/3x$.

5. Let **a**, **b**, and **c** be the following:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

a. Show the vectors are linearly independent (i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{x} = \mathbf{0}$ if $\mathbf{x} = \mathbf{0}$) (This is a hint for problem 1 as well!).

Solution 9. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ is the matrix we will reduce down.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix} R_1 + R_2 \to R_1 \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix} R_3 - R_1 \to R_3 \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} R_3/3 \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - R_3 \to R_1 \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1/2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 - R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, the vectors are independent.

b. Find orthonormal vectors $\mathbf{q}_a$, $\mathbf{q}_b$, and $\mathbf{q}_c$.

Solution 10. $\mathbf{u}_a = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}{\sqrt{1+1+4}} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\mathbf{b}' = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{c}' = \mathbf{c} - \frac{\mathbf{u}_a^T \mathbf{c}}{\mathbf{u}_a^T \mathbf{u}_a} \mathbf{u}_a - \frac{\mathbf{u}_b^T \mathbf{c}}{\mathbf{u}_b^T \mathbf{u}_b} \mathbf{u}_b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_c = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Check it!