Instructions: This quiz is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Answer each question. **Show all work to receive full credit.** Unless the question specifies, you should provide an exact answer. If you get stuck, please attempt to explain what you want to do. This may give more partial credit.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.

I, ______________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this quiz. I will show all my work to demonstrate my knowledge on the topic.

1. Use the matrix, A, and vector, b, to answer the below questions.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

a. Find the matrices U and L such that U is upper triangular, L is lower triangular, and $A = LU$ If you are stuck here, email me and you will lose full points on this part, but I will give you the solution to move forward).

Solution 1. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ $R_2 - R_1 \to R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{bmatrix}$ $R_3 - R_1 \to R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

$R_3 - 2R_2 \to R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$

$$L = \left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Can check by multiplying L and U.

b. Solve the system $L\mathbf{c} = \mathbf{b}$ for $\mathbf{c}$.

Solution 2. $[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & 0 & 7 \\ 1 & 2 & 1 & 11 \end{bmatrix}$ $R_2 - R_1 \to R_2$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 11 \end{bmatrix}$

$R_3 - R_1 \to R_3$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 6 \end{bmatrix}$ $R_3 - 2R_2 \to R_3$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Thus, $\mathbf{c} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$.

c. Solve the system $U\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$.

Solution 3. $[U\ \mathbf{c}] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $R_1 - R_2 \to R_1$ $\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $R_1 + R_3 \to R_1$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $R_2 - 2R_3 \to R_2$ $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Thus, $\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

d. Show this is correct by solving the system $A\mathbf{x} = \mathbf{b}$ by elimination (You can do full row reduction or stop at the upper triangle and solve).

Solution 4. By part (a), we get to the following reduced augmented matrix: $[A\ \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. By the same exact row reduction as above, we get $\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

2. Write all possible 3 x 3 permutation matrices.

Solution 5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

3. Which of the matrices you found above is P in the below.

$$PA = P \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution 6. $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

4. Let A and B be the following matrices:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, B = \begin{bmatrix} d & e \\ e & f \end{bmatrix}.$$

Show that if AB is symmetric then $AB = BA$.

Solution 7. $AB = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ e & f \end{bmatrix} = \begin{bmatrix} ad + be & ae + bf \\ bd + ce & be + cf \end{bmatrix}$. For this to be symmetric, $ae + bf = bd + ce$.

$$BA = \begin{bmatrix} d & e \\ e & f \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} da + eb & db + ec \\ ea + fb & eb + fc \end{bmatrix}.$$

For this to be AB we need $ea + fb = bd + ce$ and $db + ec = ae + bf$ which we have above!

5. Suppose you are given the following:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

a. Write the following system as a matrix equation, $A\mathbf{x} = \mathbf{b}$.

Solution 8. $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

b. Determine the rank of the matrix A.

Solution 9. $rank(A) = dim(C(A)) = 2$ since column 1 and 2 are scalars of each other.

c. Determine the generator(s) (spanning vectors) of the column space of the matrix A.

Solution 10. Column 1 and 3 generate the column space.

d. Determine the generator(s) (spanning vectors) and dimension of the null space of the matrix A.

Solution 11. $[A\ \mathbf{b}] = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 6 & 9 & 0 \\ -1 & -3 & 3 & 0 \end{bmatrix}$ $R_1 + R_3 \to R_3$ $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 6 & 9 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$

$R_2 - 2R_1 \to R_2$ $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$ $R_3 - 2_2 \to R_3$ $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 - R_2 \to R_1$ $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2/3$ $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, column 2 is free and $z = 0$. If $y = 1$, $x = -3$. Therefore, the vector $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ generates the null space of A and the dimension of the null space is 1.

e. Give the complete solution of the system $A\mathbf{x} = \mathbf{b}$ as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$.

Solution 12. By the same row reduction as above, we get the reduced augmented matrix $\begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, $z = 1$ automatically. Since y is free, we give it value 0 for the particular solution. Therefore, $x = -2$. Our complete solution given part (c) is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$