Instructions: This quiz is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Answer each question. **Show all work to receive full credit.** Unless the question specifies, you should provide an exact answer. If you get stuck, please attempt to explain what you want to do. This may give more partial credit.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.

I, ____________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this quiz. I will show all my work to demonstrate my knowledge on the topic.

1. For each of the below determine whether the statement is true or false. If the statement is false, explain why.

   a. The length of the sum of two vectors is the same as the sum of their lengths.

   Solution 1. False. less than or equal to.

   b. If we have two vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbf{R}^3$, then $\mathbf{v}_1 \cdot \mathbf{v}_2 \in \mathbf{R}^3$.

   Solution 2. False. $\mathbf{v}_1 \cdot \mathbf{v}_2 \in \mathbf{R}$

   c. $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 1/5 \end{bmatrix}$ is a unit vector.

   Solution 3. False. $||\mathbf{v}|| = \sqrt{9 + 16 + (1/25)} \neq 0$

   d. $k\mathbf{v} + l\mathbf{u} + e\mathbf{w}$ will always cover all of $\mathbf{R}^3$ for any fixed vectors $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ with variable $k, l, e \in \mathbf{R}$.

   Solution 4. False. One could be a scalar multiple of another.

   e. A square matrix will always have a unique inverse.

   Solution 5. False. May not have an inverse.

2. Let $\mathbf{v} = (x, x, y, y)$ have length $2\sqrt{2}$ and $\mathbf{w} = [x, x, x, x]$ have length 4. Determine all possible vectors, $\mathbf{v}$, for which this is true.

Solution 6. $||\mathbf{w}|| = \sqrt{4x^2} = 4$
$4x^2 = 16 \; x^2 = 4 \Rightarrow x = \pm 2$
$||\mathbf{v}|| = \sqrt{2x^2 + 2y^2} = 2\sqrt{2}$
$2x^2 + 2y^2 = 8 \Rightarrow x^2 + y^2 = 4 \Rightarrow 4 + y^2 = 4$
$y = 0$. Thus, $\mathbf{v} = (-2, -2, 0, 0)$ or $\mathbf{v} = (2, 2, 0, 0)$.

3. Consider the following vectors:

$$\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix}, \text{ and } \mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$$

with $k = -1$, $l = 2$, $e = -4$. Answer/find the following:

a. Determine if any of the vectors are perpendicular/orthogonal.

Solution 7. $\mathbf{v} \cdot \mathbf{w} = 10 - 6 - 4 = 0$. Thus, perpendicular.
$\mathbf{v} \cdot \mathbf{u} = -2 - 9 + 7 \neq 0$. Thus, not perpendicular.
$\mathbf{w} \cdot \mathbf{u} = -5 + 6 - 28 \neq 0$. Thus, not perpendicular.

b. $(\mathbf{v} \cdot \mathbf{u})\mathbf{v} + l\mathbf{w}$

Solution 8. $= (-4)\mathbf{v} + 2\mathbf{w} = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \\ -12 \end{bmatrix}$

c. Find unit vectors in the same direction as each vector.

Solution 9. $||\mathbf{v}|| = \sqrt{14}$, $||\mathbf{w}|| = \sqrt{45}$, and $||\mathbf{u}|| = \sqrt{59}$.
$\mathbf{v}_1 = \begin{bmatrix} (2/\sqrt{14}) \\ (-3/\sqrt{14}) \\ (1/\sqrt{14}) \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} (5/\sqrt{45}) \\ (2/\sqrt{45}) \\ (-4/\sqrt{45}) \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} (-1/\sqrt{59}) \\ (3/\sqrt{59}) \\ (7/\sqrt{59}) \end{bmatrix}$.

d. $e\mathbf{v} + k\mathbf{w} + l\mathbf{u}$

Solution 10. $= \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix} + \begin{bmatrix} -5 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} -15 \\ 16 \\ 14 \end{bmatrix}$.

4. Given the following system with arbitrary $\mathbf{b}$, find $\mathbf{x}$ with entries in terms of the entries in $\mathbf{b}$.

$$A\mathbf{x} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{b}$$

Solution 11. $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & -2 & 6 & b \\ 0 & -15 & -9 & c \end{bmatrix}$ $R_3/-3 \to R_3$ $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & -2 & 6 & b \\ 0 & 5 & 3 & -c/3 \end{bmatrix}$

$2R_3 - R_2 \to R_2$ $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & 12 & 0 & (-2c/3) - b \\ 0 & 5 & 3 & -c/3 \end{bmatrix}$ $R_2/12 \to R_2$ $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 0 & (-c/18) - (b/12) \\ 0 & 5 & 3 & -c/3 \end{bmatrix}$

$R_3 - 5R_2 \to R_3$ $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 0 & (-c/18) - (b/12) \\ 0 & 0 & 3 & (-c/3) + (5c/18) + (5b/12) \end{bmatrix}$ $R_3/3 \to R_3$ $\begin{bmatrix} 1 & -3 & 1 & a \\ 0 & 1 & 0 & (-c/18) - (b/12) \\ 0 & 0 & 1 & (-c/54) + (5b/36) \end{bmatrix}$

$3R_2 + R_1 \to R_1$ $\begin{bmatrix} 1 & 0 & 1 & a - (c/6) - (b/4) \\ 0 & 1 & 0 & (-c/18) - (b/12) \\ 0 & 0 & 1 & (-c/54) + (5b/36) \end{bmatrix}$ $R_1 - R_3 \to R_1$ $\begin{bmatrix} 1 & 0 & 0 & a - (8c/54) - (14b/36) \\ 0 & 1 & 0 & (-c/18) - (b/12) \\ 0 & 0 & 1 & (-c/54) + (5b/36) \end{bmatrix}$

$x_1 = a - (4c/27) - (7b/18)$, $x_2 = (-c/18) - (b/12)$, and $x_3 = (-c/54) + (5b/36)$.

5. Given your answer above, if $a = 1$, $b = 2$, and $c = -3$ what is $\mathbf{x}$?

Solution 12. $x_1 = 1 - (4(-3)/27) - (7(2)/18) = 2/3$, $x_2 = (-(-3)/18) - ((2)/12) = 0$, and $x_3 = (-(-3)/54) + (5(2)/36) = 1/3$.

6. Below are two matrices A and B, find the following (if not possible, say so and explain):

a. A^{-1}

Solution 13. $\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -2 & 4 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{bmatrix}$ $R_3 - R_2 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -2 & 4 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

$R_1 + R_2 \to R_1$ $\begin{bmatrix} 0 & 0 & 7 & 1 & 1 & 0 \\ -1 & -2 & 4 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ $3R_2 + R_3 \to R_2$ $\begin{bmatrix} 0 & 0 & 7 & 1 & 1 & 0 \\ 0 & -6 & 12 & 0 & 2 & 1 \\ 3 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

$R_1/7, R_2/6, R_3/3$ $\begin{bmatrix} 0 & 0 & 1 & 1/7 & 1/7 & 0 \\ 0 & -1 & 2 & 0 & 1/3 & 1/6 \\ 1 & 0 & 0 & 0 & -1/3 & 1/3 \end{bmatrix}$ $R_2 - 2R_1 \to R_2$ $\begin{bmatrix} 0 & 0 & 1 & 1/7 & 1/7 & 0 \\ 0 & -1 & 0 & -2/7 & 1/21 & 1/6 \\ 1 & 0 & 0 & 0 & -1/3 & 1/3 \end{bmatrix}$

$-R_2$ and swap R_1 and R_3 $\begin{bmatrix} 1 & 0 & 0 & 0 & -1/3 & 1/3 \\ 0 & 1 & 0 & 2/7 & -1/21 & -1/6 \\ 0 & 0 & 1 & 1/7 & 1/7 & 0 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 0 & -1/3 & 1/3 \\ 2/7 & -1/21 & -1/6 \\ 1/7 & 1/7 & 0 \end{bmatrix}$$

b. B^{-1}

Solution 14. There does not exist an inverse.
$$\begin{bmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{bmatrix} R_3 - R_2 \to R_3 \begin{bmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & -1 & 1 \end{bmatrix}$$
$$2R_2 + R_1 \to R_2 \begin{bmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 1 & 2 & 0 \\ 0 & 3 & -3 & 0 & -1 & 1 \end{bmatrix}$$
Row 3 is -Row 2. Thus, not 3 pivots and no inverse exists.

c. $(AB)^{-1}$

Solution 15. $(AB)^{-1} = B^{-1}A^{-1}$. However, B does not have an inverse. Thus, AB has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \\ 2 & -2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

7. Give an example of a square matrix which does not have an inverse. Then show that no inverse exists using elimination. Explain why this shows there is no inverse.

Solution 16. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Determinant of the matrix is 0. Thus, not invertible.
$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} R_1 - R_2 \to R_2 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ No second pivot exists. Thus, not invertible.