Instructions: This worksheet is for practice and is NOT to be turned in for a grade. Answer each question.

1 Chapter 8

1. Which of these transformations is not linear? The input is $\mathbf{v} = (v_1, v_2)$:

   (a) $T(\mathbf{v}) = (v_2, v_1)$

   Solution 1. We check that $T(k\mathbf{v} + \ell\mathbf{u}) = kT(\mathbf{v}) + \ell T(\mathbf{u})$.
   $T(k\mathbf{v} + \ell\mathbf{u}) = T((kv_1 + \ell u_1, kv_2 + \ell u_2)) = (kv_2 + \ell u_2, kv_1 + \ell u_1) = k(v_2, v_1) + \ell(u_2, u_1)$
   $= kT(\mathbf{v}) + \ell T(\mathbf{u})$. Thus, this is a linear transformation.

   (b) $T(\mathbf{v}) = (v_1, v_1)$

   Solution 2. We check that $T(k\mathbf{v} + \ell\mathbf{u}) = kT(\mathbf{v}) + \ell T(\mathbf{u})$.
   $T(k\mathbf{v} + \ell\mathbf{u}) = T((kv_1 + \ell u_1, kv_2 + \ell u_2)) = (kv_1 + \ell u_1, kv_1 + \ell u_1) = k(v_1, v_1) + \ell(u_1, u_1) =$
   $kT(\mathbf{v}) + \ell T(\mathbf{u})$. Thus, this is a linear transformation.

   (c) $T(\mathbf{v}) = (0, v_1)$

   Solution 3. This is linear by similar argument above.

   (d) $T(\mathbf{v}) = (0, 1)$

   Solution 4. This is not linear since the identity does not exist in this map!

   (e) $T(\mathbf{v}) = v_1 - v_2$

   Solution 5. We check that $T(k\mathbf{v} + \ell\mathbf{u}) = kT(\mathbf{v}) + \ell T(\mathbf{u})$.
   $T(k\mathbf{v}+\ell\mathbf{u}) = T((kv_1+\ell u_1, kv_2+\ell u_2)) = kv_1+\ell u_1-kv_2-\ell u_2 = k(v_1-v_2)+\ell(u_1-u_2) =$
   $kT(\mathbf{v}) + \ell T(\mathbf{u})$. Thus, this is a linear transformation.

   (f) $T(\mathbf{v}) = v_1 v_2$

   Solution 6. We check that $T(k\mathbf{v} + \ell\mathbf{u}) = kT(\mathbf{v}) + \ell T(\mathbf{u})$.
   $T(k\mathbf{v} + \ell\mathbf{u}) = T((kv_1 + \ell u_1, kv_2 + \ell u_2)) = (kv_1 + \ell u_1)(kv_2 + \ell u_2) = k^2 v_1 v_2 + \ell k u_1 v_2 +$
   $\ell k u_2 v_1 + \ell^2 u_1 u_2 \neq kT(\mathbf{v}) + \ell T(\mathbf{u})$. Thus, this is not a linear transformation! Try finding a counterexample!

2. Suppose a linear transformation T satisfies $T(1,1) = (2,2)$ and $T(2,0) = (0,0)$. Find $T(\mathbf{v})$ when:

   (a) $\mathbf{v} = (2, 2)$

   Solution 7. $T(2,2) = T(2(1,1)) = 2T(1,1) = 2(2,2) = (4,4)$

   (b) $\mathbf{v} = (3, 1)$

   Solution 8. $T(3,1) = T((1,1) + (2,0)) = T(1,1) + T(2,0) = (2,2) + (0,0) = (2,2)$

   (c) $\mathbf{v} = (-1, 1)$

Solution 9. $T(-1,1) = T((1,1)-(2,0)) = T(1,1) - T(2,0) = (2,2) - (0,0) = (2,2)$

(d) $\mathbf{v} = (a,b)$

Solution 10. $T(a,b) = T\left(\frac{a-b}{2}(2,0) + b(1,1)\right) = (0,0) + (2b,2b) = (2b,2b)$

3. Suppose T is a linear transformation from V to W. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for V and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be a basis for W. Suppose $T(\mathbf{v}_1) = \mathbf{w}_2$ and $T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{w}_1 + \mathbf{w}_3$. Find the matrix A that corresponds to T. Use A to find $T(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$.

Solution 11. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ since the columns represent the original basis and the rows represent the new basis. Thus, we take the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ since this is the coefficients of the desired linear combination and we do as follows:

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3.$$

4. Let T be linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. Take the standard basis for $\mathbb{R}^3$ and let $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ be the basis for $\mathbb{R}^2$. Find the matrix representation A under these bases for the linear transformation T where $T(1,0,0) = (1,0)$ and $T(0,1,0) = (2,2)$ and $T(0,0,1) = (-1,1)$. Use the matrix A to compute $T(1,2,1)$.

Solution 12. $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$. Thus, $T(1,2,1) = T((1,0,0) + 2(0,1,0) + (0,0,1))$ and $A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Therefore, $T(1,2,1) = 5(1,1) - (1,0) = (4,5)$.

5. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ so that $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 5v_1 + v_2 \\ -2v_1 + 2v_2 \end{bmatrix}$. Find bases for $\mathbb{R}^2$ so that the matrix representation of T is diagonal.

Solution 13. Since we have the transformations for the basis elements, we get $A = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$. Let us find the eigenvalues and eigenvectors.

$det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$. Thus, $\lambda_1 = 3$ and $\lambda_2 = 4$. For λ_1,

$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow 2x + y = 0 \Rightarrow 2x = -y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. For $\lambda_2 = 4$,

$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow x + y = 0 \Rightarrow x = -y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, with this basis of eigenvectors, $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \Lambda$.