Instructions: This homework is an individual effort. Answer each question. This is due on **Monday, June 15th**. **Show all work to receive full credit.**

1 Chapter 6

1. Find the eigenvalues and eigenvectors of these four matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Solution 1. $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)-8 = \lambda^2-4\lambda-5 = (\lambda-5)(\lambda+1)$.

Thus, $\lambda = 5, -1$. When $\lambda = 5$,

$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x - 2y = 0 \Rightarrow x = y$. Thus, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. By similar process, the next eigenvector is $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For the rest of the matrices, I will not show the process since it is identical to this.

For B, $\lambda = 0, 6$ with eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For C, $\lambda = 1$ twice with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For D, $\lambda = 1$ twice with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A-\lambda I) = (\lambda-4)(\lambda-5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

Solution 2. $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix}$.

3. Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

Solution 3. First we find the eigenvalues. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = (\lambda-1)(\lambda-3)$. Thus, $\lambda_1 = 1$ and $\lambda_2 = 3$. When $\lambda_1 = 1$,

$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. When $\lambda_2 = 3$,
$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x + 2y = 0 \Rightarrow x = y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we have the following:

$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. By similar process we get the following for the second matrix:

$\lambda_1 = 0$ and $\lambda_2 = 4$. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Thus,
$X = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$, and $X^{-1} = (1/4)\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

4. Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (Hint: Use $X\Lambda X^{-1}$)

Solution 4. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}(-1/2)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = (-1/2)\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$
$= (-1/2)\begin{bmatrix} -\lambda_1 - \lambda_2 & -\lambda_1 + \lambda_2 \\ -\lambda_1 + \lambda_2 & -\lambda_1 - \lambda_2 \end{bmatrix}$.

5. The rabbit population shows fast growth (from 6r) but loss to wolves (from -2w). The wolf population always grows in this model:

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

Solution 5. Let $\mathbf{u} = \begin{bmatrix} r \\ w \end{bmatrix}$. $\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}$ is the system we are looking at. Let us find the eigenvalues and eigenvectors of the matrix.

$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda)+4 = \lambda^2 - 7\lambda + 10 = (\lambda-5)(\lambda-2)$. Thus, $\lambda_1 = 2$ and $\lambda_2 = 5$. When $\lambda_1 = 2$,

$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x - y = 0 \Rightarrow 2x = y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. When $\lambda_2 = 5$,
$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x - 2y = 0 \Rightarrow x = 2y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Now we have the following solution:

$\mathbf{u} = Ce^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + De^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} Ce^{2t} + 2De^{5t} \\ 2Ce^{2t} + De^{5t} \end{bmatrix}$. When $\mathbf{u}(0) = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$. Thus, $C + 2D = 30$ and $2C + D = 30$. By substitution, $C + 60 - 4C = 30 \Rightarrow C = 10, D = 10$. Thus, $\mathbf{u} = \begin{bmatrix} 10e^{2t} + 20e^{5t} \\ 20e^{2t} + 10e^{5t} \end{bmatrix}$.
Ratio$= \lim_{t\to\infty} \dfrac{r(t)}{w(t)} = \lim_{t\to\infty} \dfrac{10e^{2t} + 20e^{5t}}{20e^{2t} + 10e^{5t}} = \lim_{t\to\infty} \dfrac{10e^{2t}(1+2e^{3t})}{10e^{2t}(2+e^{3t})} = \lim_{t\to\infty} \dfrac{1+2e^{3t}}{2+e^{3t}} = \lim_{t\to\infty} \dfrac{2e^{3t}}{e^{3t}} = 2$.

6. Find the eigenvalues and the unit eigenvectors of $S = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.(Hint: Use the fact that S is symmetric.)

Solution 6. $\det(S-\lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (2-\lambda)\lambda^2 + 4\lambda + 4\lambda = -\lambda^3 + 2\lambda^2 + 8\lambda = -\lambda(\lambda^2 - 2\lambda - 8) = \lambda(\lambda-4)(\lambda+2)$. Thus, $\lambda_1 = 0$, $\lambda_2 = 4$, and $\lambda_3 = -2$. When $\lambda_1 = 0$,

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. When $\lambda_2 = 4$,
$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ We reduce to get $\begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix}$. Thus, $-2y + 2z = 0$, $2x - 4y = 0$ and $2x - 4z = 0$. Therefore, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. When $\lambda_3 = -2$,

$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. We reduce to get $\begin{bmatrix} 0 & -2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$. Thus, $-2y + 2z = 0$, $2x + 2y = 0$, and $2x + 2z = 0$. Therefore, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since S is symmetric, we are already orthogonal.

Now we unitize! $\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

7. Find an orthonormal matrix Q that diagonalizes $S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$.

Solution 7. By similar process to the above we get $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = -3$ with $\mathbf{x}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$. These vectors normalized are $\mathbf{q}_1 = \frac{1}{3} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{q}_2 = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{q}_3 = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$. Thus,

$$Q = \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & -2/3 \end{bmatrix}.$$

Check it!

8. Which of S_1, S_2, S_3, S_4 has positive eigenvalues? Use a test, don't compute the $\lambda' s$.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$$

Solution 8. Not S_1 since the determinant is negative.
Not S_2 since the upper left value is negative.
Not S_3 since the determinant is 0.
S_4 is since the upper left is positive and so is its determinant.

9. Compute the upper left determinants of S to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

Solution 9. The uppermost determinant is 2. The 2 by 2 determinant is $10 - 4 = 6$. The full matrix has the determinant $2(40-9)-2(16-0)=62-32=30$. Thus, S is positive definite.

Reduced S is $\begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

We see that $6/2=3$ and $30/6=5$ which are the second and third pivots.