Instructions: This homework is an individual effort. Answer each question. This is due on **Monday, June 8th**. **Show all work to receive full credit.**

1 Chapter 5

1. If a 4×4 matrix has determinant $\det(A) = \frac{1}{2}$, find $\det(2A), \det(-A), \det(A^2)$ and $\det(A^{-1})$.

   Solution 1. $\det(2A) = 2^4 \det(A) = 8, \det(-A) = (-1)^4 \det(A) = 1/2, \det(A^2) = \det(A)\det(A) = 1/4$ and $\det(A^{-1}) = 1/\det(A) = 2$.

2. True or false, with reason if true or a counterexample if false:

   a. The determinant of $I + A$ is $1 + \det(A)$.

      Solution 2. $\det(I + I) \neq 2$

   b. The determinant of ABC is $|A||B||C|$.

      Solution 3. True. $\det(ABC) = \det(AB)\det(C) = \det(A)\det(B)\det(C)$.

   c. The determinant of $4A$ is $4|A|$.

      Solution 4. False. $\det(4A) = 4^n \det(A)$ and the 4 in $4|A|$ only distributes to a row.

3. Reduce A to an upper triangular matrix and find $\det(A)$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Solution 5. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $R_2 - R_1 \to R_2$ and $R_3 - R_1 \to R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $R_3 - R_2 \to R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since it is upper triangular, $\det(A) = 1$.

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $R_2 - 2R_1 \to R_2$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 3 & 3 & 3 \end{bmatrix}$ $R_3 - 3R_1 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix}$ $R_3 - (3/2)R_2 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -(3/2) \end{bmatrix}$. Since it is upper triangular, $\det(A) = 3$.

4. Compute the determinants of A and B. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution 6. $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2$. Thus, independent.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $R_2 - 4R_1 \to R_2$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$ $R_3 - 7R_1 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ $R_3 - 2R_2 \to R_3$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is a zero row or the diagonal has a 0 pivot, $\det(B) = 0$. This means the columns are not independent.

5. The n x n determinant C_n has $1's$ above and below the main diagonal: $C_1 = |0|$, $C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$,

$C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$, $C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

a. What are these determinants C_1, C_2, C_3, C_4? $C_1 = 0$, $C_2 = -1$, $C_3 = -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$, and

$$C_4 = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -C_2 = 1$$

b. By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

Solution 7. $C_n = -C_{n-2}$. Thus, $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

6. Solve these linear equations by Cramer's Rule:

a.

$$\begin{aligned} 2x + 5y &= 1 \\ x + 4y &= 2 \end{aligned}$$

Solution 8. $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$B_1 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$\det(A) = 3$, $\det(B_1) = -6$, and $\det(B_2) = 3$. Thus, $x = -6/3 = -2$ and $y = 3/3 = 1$.

b.

$$\begin{aligned} 2x + y &= 1 \\ x + 2y + z &= 0 \\ y + 2z &= 0 \end{aligned}$$

Solution 9. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $B_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, and $B_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$\det(A) = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 6 - 2 = 4$, $\det(B_1) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $\det(B_2) = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$, and $\det(B_3) = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$. Thus, $x = 3/4$, $y = -1/2$, and $z = 1/4$.

7. Find A^{-1} form the cofactor formula $C^T / \det(A)$.

a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution 10. We will just list the cofactors to save space.

$C_{11} = 3, C_{12} = 0, C_{13} = 0, C_{21} = -2, C_{22} = 1, C_{23} = -7, C_{31} = 0, C_{32} = 0, C_{33} = 3$.
Thus, $C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}$. $\det(A) = \begin{vmatrix} 3 & 0 \\ 7 & 1 \end{vmatrix} = 3$. Thus, $A^{-1} =$

$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -7/3 & 1 \end{bmatrix}.$$

Now for b.

$C_{11} = 3, C_{12} = 2, C_{13} = 1, C_{21} = 2, C_{22} = 4, C_{23} = 2, C_{31} = 1, C_{32} = 2, C_{33} = 3$. Thus, $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. $\det(A) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 2 = 4$. Thus,

$$A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}.$$

8. Find the cofactors of A and multiply AC^T to find $\det(A)$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

If you change that 4 to 100, why is $\det(A)$ unchanged?

Solution 11. $C_{21} = 3, C_{22} = 1, C_{23} = -1, C_{31} = -6, C_{32} = 2, C_{33} = 1$. Thus, $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$.

$AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \det(A)I \Rightarrow \det(A) = 3$. The determinant remains unchanged nomatter what is in the 4 spot because the cofactor is always 0!

9. a. Find the area of the parallelogram with edges $\mathbf{v} = (3, 2)$ and $\mathbf{w} = (1, 4)$.

   Solution 12. We show a generalization by triangles! If we assume one end point of the parallelogram is (0,0), then the vectors are the coordinates of two other points. Since the line connecting the two end points of the vector bisects the area of the parallelogram we get that $Area(parallelogram) = \det([v_1, v_2])$. Thus, $Area = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$.

   b. Find the area of the triangle with sides $\mathbf{v}, \mathbf{w}$, and $\mathbf{v} + \mathbf{w}$. Draw it.

   Solution 13. $Area = (1/2)\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 5$. Drawing is hard to type, but please ask me if this is a question.

10. A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$ and $(1, 3, 1)$ and $(1, 1, 3)$. Find its volume.

Solution 14. $Volume = ((3,1,1)\times(1,3,1))\cdot(1,1,3) = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} =$ $-2-2+24 = 20$.

11. Find $\mathbf{u}\times\mathbf{v}$ when $\mathbf{u} = (1,1,1)$ and $\mathbf{v} = (5,-1,7)$.

Solution 15. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 5 & -1 & 7 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ -1 & 7 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} = 8i - 2j - 6k$. Thus, the cross product is the vector $(8,-2,-6)$.