Instructions: This homework is an individual effort. Answer each question. This is due on **Monday, June 1st**. **Show all work to receive full credit.**

1 Rest of Chapter 3

1. Find a basis and the dimension for each of the four subspaces associated with the following matrices.

   a. $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

   Solution 1. First we find column space! $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ $R_2 - R_1 \to R_1$ $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ $R_1 - R_3 \to R_3$ $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 - 4R_1 \to R_2$ $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Thus the column space is generated by 2 vectors and a basis is column 2 and column 4. Row space is 2 also seen from the reduction. A basis is row 1 and 2. Now we work on the null spaces. Given the reduction, there are 3 free variables; x_1, x_3, and x_5. Thus, we get the special solutions to be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. These generate the null space and give dimension 3. The left null space we reduce the transpose of the original matrix.

   $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix}$ By obvious manipulation, $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 - 3R_1 \to R_2$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & 6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 - 4R_1 \to R_3$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ By obvious manipulation, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Thus, 2 pivots and a free variable. The free variable is y_2. This gives the generating vector $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$

Solution 2. Let's reduce! $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$ $R_2 - 2R_1 \rightarrow R_2$ $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ $R_1 - 2R_2 \rightarrow R_1$ $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$.Thus, we have a column space with dimension 2 and generated by the first 2 columns. The row space is going to be 2 since no rows turned to 0's. Thus, generated by the only 2 rows. We have one free variable, x_3, which gives us the null space generator $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$. The left null space will have only the 0 vector since the row space is full.

2. Let $V = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right)$.

a. Find a matrix A that has V as its row space .

Solution 3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

b. Find a matrix B that has V as its nullspace.

Solution 4. $B = \begin{bmatrix} -1 & 2 & -1 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ This was created since the following had to be true: $a_{11} + a_{12} + a_{13} = 0$ and $2a_{11} + a_{12} = 0$.

c. Find AB.

Solution 5. $AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ -2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ -4 & 8 & -4 \end{bmatrix}$.

2 Chapter 4

1. If $\mathbf{P}$ is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\mathbf{P}^\perp$ (The orthogonal complement of P). Construct a matrix with $\mathbf{P}$ as its nullspace.

Solution 6. A basis for $\mathbf{P}^\perp$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Any matrix with the correct dimensions and 1's in every position has this null space.

2. Suppose A is the 4 x 4 identity matrix without its last column. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What is the projection matrix, P?

Solution 7. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. We now find $\widehat{\mathbf{x}}$. $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Since the identity has inverse identity, $\widehat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus, $\mathbf{p} = A\widehat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$. $P = A(A^T A)^{-1}A^T = AA^T = I_4$ with a zero in the last diagonal spot..

3. What is the orthogonal complement of $S = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$?

Solution 8. This implies we want all $\mathbf{x}$ such that $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By row reducing, we get $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, x_2 is free and we get special solution $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The space generated or spanned by this vector is the orthogonal complement.

4. Project $\mathbf{b}$ onto the line throught $\mathbf{a}$. Check that $\mathbf{e}$ (the error) is perpendicular to $\mathbf{a}$.

   a. $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   Solution 9. $\mathbf{p} = \widehat{\mathbf{x}}\mathbf{a} = \dfrac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a} = \dfrac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \dfrac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.
   $\mathbf{e} \cdot \mathbf{a} = \dfrac{-10}{9} + \dfrac{5}{9} + \dfrac{5}{9} = 0$

   b. $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

   Solution 10. $\mathbf{p} = \widehat{\mathbf{x}}\mathbf{a} = \dfrac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. $\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \mathbf{0}$. $\mathbf{e} \cdot \mathbf{a} = \dfrac{-10}{9} + \dfrac{5}{9} + \dfrac{5}{9} = 0$.
   Clearly $\mathbf{e}$ is perpendicular to $\mathbf{a}$.

5. In both of the above, find the projection matrix P and find the project $\mathbf{p}$.

Solution 11. $P_1 = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$P_2 = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$.

6. Project $\mathbf{b}$ onto the column space of A by solving $A^TA\widehat{\mathbf{x}} = A^T\mathbf{b}$ and $\mathbf{p} = A\widehat{\mathbf{x}}$.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

Solution 12. $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$.

$A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 14 \end{bmatrix} R_2 - R_1 \to R_2 \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 10 \end{bmatrix} R_2/2 \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix} R_1 - R_2 \to R_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$. Thus,

$\widehat{\mathbf{x}} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$. $\mathbf{p} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}$.

7. What linear combination of (1,2,-1) and (1,0,1) is closest to $\mathbf{b} = (2, 1, 1)$?

Solution 13. $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Transpose both sides: $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Now simplify and solve. $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. $\begin{bmatrix} 4 & 0 & 3 \\ 0 & 2 & 3 \end{bmatrix}$ $R_1/4, R_2/2$ gives the solution, $\mathbf{x} = \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix}$.

8. Find the line of best fit for the points (0,1), (1,5), (3,13), (4,17). Do any of these points lie on the line?

Solution 14.

$$1 = m(0) + b$$
$$5 = m(1) + b$$
$$13 = m(3) + b$$
$$17 = m(4) + b$$

Matrix form: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$. Transpose both sides since no line goes through all of the points.

$\begin{bmatrix} 26 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 112 \\ 36 \end{bmatrix}$

$\begin{bmatrix} 26 & 8 & 112 \\ 8 & 4 & 36 \end{bmatrix} R_1{-}2R_2 \to R_1 \begin{bmatrix} 10 & 0 & 40 \\ 8 & 4 & 36 \end{bmatrix} R_1/10, R_2/4 \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 9 \end{bmatrix} R_2{-}2R_1 \to R_2 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$.

Thus, the best fit line is $y = 4x + 1$.

9. Find the closest parabola to the points (0,1), (1,5), (3,13), (4,17). Do any of these points lie on the parabola?

Solution 15.

$$\begin{aligned} 1 &= a(0)^2 + b(0) + c \\ 5 &= a(1)^2 + b(1) + c \\ 13 &= a(3)^2 + b(3) + c \\ 17 &= a(4)^2 + b(4) + c \end{aligned}$$

Matrix form: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$. Transpose both sides since the matrix is not square.

$\begin{bmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 394 \\ 112 \\ 36 \end{bmatrix}$ Now we solve.

$\begin{bmatrix} 338 & 92 & 26 & 394 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{bmatrix} R_2{-}2R_3 \to R_2 \begin{bmatrix} 338 & 92 & 26 & 394 \\ 40 & 10 & 0 & 40 \\ 26 & 8 & 4 & 36 \end{bmatrix} R_2/10 R_3/2, R_1/2 \begin{bmatrix} 169 & 46 & 13 & 197 \\ 4 & 1 & 0 & 4 \\ 13 & 4 & 2 & 18 \end{bmatrix}$

$R_3 - 4R_2 \to R_3 \begin{bmatrix} 169 & 46 & 13 & 197 \\ 4 & 1 & 0 & 4 \\ -3 & 0 & 2 & 2 \end{bmatrix} R_1 - 46R_2 \to R_1 \begin{bmatrix} -15 & 0 & 13 & 13 \\ 4 & 1 & 0 & 4 \\ -3 & 0 & 2 & 2 \end{bmatrix}$

$R_1{-}(13/2)R_3 \to R_1 \begin{bmatrix} 9/2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 4 \\ -3 & 0 & 2 & 2 \end{bmatrix} 2/9R_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 4 \\ -3 & 0 & 2 & 2 \end{bmatrix} R_2{-}4R_1 \to R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ -3 & 0 & 2 & 2 \end{bmatrix}$

$R_3 + 3R_1 \to R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} R_3/2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Thus, the parabola is actually the line

$y = 4x + 1$.

10. Find the closest cubic to the points (0,1), (1,5), (3,13), (4,17). Do any of these points lie on the cubic?

Solution 16. Matrix form: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$. We can get an exact solution this time since the dimensions match!

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 \\ 27 & 9 & 3 & 1 & 13 \\ 64 & 16 & 4 & 1 & 17 \end{bmatrix} R_2 - R_1 \to R_2 \text{ Rep for all rows } \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 4 \\ 27 & 9 & 3 & 0 & 12 \\ 64 & 16 & 4 & 0 & 16 \end{bmatrix}$$

$$R_4 - 4R_2 \to R_4 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 4 \\ 27 & 9 & 3 & 0 & 12 \\ 60 & 12 & 0 & 0 & 0 \end{bmatrix} R_3 - 9R_2 \to R_3 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 4 \\ 18 & 0 & -6 & 0 & -24 \\ 60 & 12 & 0 & 0 & 0 \end{bmatrix} R_4/12.R_3/6$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 4 \\ 3 & 0 & -1 & 0 & -4 \\ 5 & 1 & 0 & 0 & 0 \end{bmatrix} R_2{+}R_3 \to R_2 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & -4 \\ 5 & 1 & 0 & 0 & 0 \end{bmatrix} R_4{-}R_2 \to R_4 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 - 3R_4 \to R_3 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 - 4R_4 \to R_2 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{Swap } R_1 \text{ and } R_4$$

and $-R_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Thus, $y = 4x + 1$.

11. Find an orthonormal basis for the column space of A given:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Solution 17. $\mathbf{q}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\mathbf{v}_2' = \mathbf{v}_2 - \frac{\mathbf{q}_1^T \mathbf{v}_2}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -1/2 \\ 1/2 \\ 5/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{1}{2\sqrt{52}} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}.$$ Check it!

12. Find the projection of **b** onto the column space above.

Solution 18. We find $\widehat{\mathbf{x}}$ first. $A^T A \widehat{\mathbf{x}} = A^T \mathbf{b}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \widehat{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 4 & 2 & -4 \\ 2 & 14 & 11 \end{bmatrix} R_1/2 \begin{bmatrix} 2 & 1 & -2 \\ 2 & 14 & 11 \end{bmatrix} R_2 - R_1 \to R_2 \begin{bmatrix} 2 & 1 & -2 \\ 0 & 13 & 13 \end{bmatrix} R_2/13 \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} R_1 - R_2 \to$$
$$R_1 \begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} R_1/2 \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1 \end{bmatrix}.$$ Thus, $\widehat{\mathbf{x}} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$. Now we find $\mathbf{p}$.

$$\mathbf{p} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7/2 \\ -3/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$