**Instructions:** This homework is an individual effort. Answer each question. This is due on **Monday, May 25th**. **Show all work to receive full credit.**

# 1 Rest of Chapter 2

1. Solve the system $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$.

   a. $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$.

   **Solution 1.** $[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 11 \end{bmatrix} R_2 - 4R_1 \to R_2 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Thus, $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

   $[U\ \mathbf{c}] = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix} R_1/2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} R_1 - 2R_2 \to R_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$. Thus, $\mathbf{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

   b. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

   **Solution 2.** $[L\ \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 5 \\ 1 & 1 & 1 & 6 \end{bmatrix} R_3 - R_2 \to R_3 \begin{bmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_2 - R_1 \to R_2 \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

   Thus, $\mathbf{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$.

   $[U\ \mathbf{c}] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_1 - R_2 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_2 - R_3 \to R_2 \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

   Thus, $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

2. Let $P$ be the matrix with 1's on the antidiagonal (i.e. from the top right position to the bottom left position). Describe $PAP$.

   **Solution 3.** We will show for a general 2 x 2 and infer for larger matrices.
   $PAP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix} = A$

   We say that $PAP$ swaps the first and third row and then the first and third column.

3. Describe in words what the 5 x 5 permutation matrices would do to the matrix $A$.

a. $P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Solution 4.** $\begin{bmatrix} R_4 \\ R_5 \\ R_2 \\ R_3 \\ R_1 \end{bmatrix}$

b. $P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

**Solution 5.** $\begin{bmatrix} R_1 \\ R_5 \\ R_2 \\ R_4 \\ R_3 \end{bmatrix}$

c. $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

**Solution 6.** $\begin{bmatrix} R_3 \\ R_2 \\ R_4 \\ R_1 \\ R_5 \end{bmatrix}$

4. If $A = A^T$ and $B = B^T$, which are symmetric? Show work as to why or why not.

a. $A^2 - B^2$

**Solution 7.** $(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A^T)^2 - (B^T)^2 = A^2 - B^2$ Thus, symmetric.

b. $(A + B)(A - B)$

**Solution 8.** $((A + B)(A - B))^T = (A - B)^T(A + B)^T = (A^T - B^T)(A^T + B^T) = (A - B)(A + B)$ Thus, not symmetric! Matrix multiplication is not communative.

c. $ABAB$

**Solution 9.** $(ABAB)^T = (AB)^T(AB)^T = B^TA^TB^TA^T = BABA$ Thus, not symmetric.

# 2 Chapter 3

1. Construct a 3 x 3 matrix whose column space contains (1,1,0) and (1,0,1) but not (1,1,1). Then construct a 3 x 3 matrix whose column space is a line in $\mathbb{R}^3$.

   **Solution 10.** $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

2. If $S = C(A)$ and $T = C(B)$, then describe the matrix whose column space is $S + T$.

   **Solution 11.** The temptation is to say $A+B$. However, there is the potential to sum leads to columns that are scalar multiples of each other. Thus, $S+T$ would not describe the column space. So, we say that it is $A+B$ unless the sum of the columns forms scalar multiples.

3. Construct a matrix whose column space contains (1,1,1) and whose nullspace is the multiples of (1,1,1,1).

   **Solution 12.** $\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to be in the null space.

   We also need $\dim(N(A)) = 1$. Since $\dim(N(A)) = n - r$, $\dim(C(A)) = 3$. Thus, the fourth column must be a multiple of one of the other 3. Consider the following:

   $$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

   With these columns, we get the desired. Please feel free to check.

4. What are the special solutions to $A\mathbf{x} = \mathbf{0}$ for the following matrices:

   $$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   **Solution 13.** The first matrix has 2 pivots. Thus, there should be 2 free columns/variables ($x_3$ and $x_4$). If $x_3 = 1$ and $x_4 = 0$, then $x_1 = -2$ and $x_2 = -4$. Therefore, the special solutions are $\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$.

The second matrix has 1 pivot. Thus, there are 2 free columns/variables ($x_1$ and $x_3$). If $x_1 = 1$ and $x_3 = 0$, then $x_2 = 0$. If $x_1 = 0$ and $x_3 = 1$, then $x_2 = -2$. Therefore, the special solutions are $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-2\\1\end{bmatrix}$.

5. Find the complete solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix}1 & 2 & 1 & 0\\2 & 4 & 4 & 8\\4 & 8 & 6 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4\\2\\10\end{bmatrix}$$

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**Solution 14.** $\begin{bmatrix}1 & 2 & 1 & 0 & 4\\2 & 4 & 4 & 8 & 2\\4 & 8 & 6 & 8 & 10\end{bmatrix}$ $R_3 - R_2 \to R_3$ $\begin{bmatrix}1 & 2 & 1 & 0 & 4\\2 & 4 & 4 & 8 & 2\\2 & 4 & 2 & 0 & 8\end{bmatrix}$ $R_3/2, R_2/2$ $\begin{bmatrix}1 & 2 & 1 & 0 & 4\\1 & 2 & 2 & 4 & 1\\1 & 2 & 1 & 0 & 4\end{bmatrix}$ $R_1 - R_3 \to R_3$ $\begin{bmatrix}1 & 2 & 1 & 0 & 4\\1 & 2 & 2 & 4 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $R_2 - R_1 \to R_2$ $\begin{bmatrix}1 & 2 & 1 & 0 & 4\\0 & 0 & 1 & 4 & -3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ $R_1 - R_2 \to R_1$ $\begin{bmatrix}1 & 2 & 0 & -4 & 7\\0 & 0 & 1 & 4 & -3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ Pivots in column 1 and column 3. Thus, $x_2$ and $x_4$ are free. For the particular solution, we let both of these be 0. Therefore, $\mathbf{x}_p = \begin{bmatrix}7\\0\\-3\\0\end{bmatrix}$. Now we assume the last column of our augmented matrix is all 0's to get the special solution(s). If $x_2 = 1$ and $x_4 = 0$, then $x_1 = -2$ and $x_3 = 0$. If $x_4 = 1$ and $x_2 = 0$, then $x_1 = 4$ and $x_3 = -4$. Thus, $\mathbf{x}_n = x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}4\\0\\-4\\1\end{bmatrix}$. We then get the complete solution $\mathbf{x} = \begin{bmatrix}7\\0\\-3\\0\end{bmatrix} + x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}4\\0\\-4\\1\end{bmatrix}$.

6. Show that $(b_1, b_2, b_3)$ is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix}1 & 3 & 1\\3 & 8 & 2\\2 & 4 & 0\end{bmatrix}$$

**Solution 15.** This vector is in the column space if there is a solution, $\mathbf{x}$.

$$\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} R_2{-}2R_1 \to R_2 \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 1 & 2 & 0 & b_2 - 2b_1 \\ 2 & 4 & 0 & b_3 \end{bmatrix} R_3{-}2R_2 \to R_3 \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 1 & 2 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + 4b_1 \end{bmatrix}$$

Thus, for this vector to be in the column space, $b_3 - 2b_2 + 4b_1 = 0$.

7. Find the rank of the $A$ and $A^T$ depending on $q$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix}$$

**Solution 16.** $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} R_3 - R_2 \to R_3 \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & q-2 \end{bmatrix} R_2 - R_1 \to R_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

If $q = 2$, then $rank(A) = 2$. Otherwise, $rank(A) = 3$. By similar process, the same conditions hold for the transpose.

8. Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

**Solution 17.** $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} R_1{+}R_2 \to R_1 \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$

$R_3 + R_4 \to R_4 \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} R_1 + R_4 \to R_4 \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 + R_3 \to R_1 \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, there are 3 pivots possible when reducing this matrix which is equivalent to 3 being the maximum number of independent vectors.

9. Find 3 different bases for the column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

**Solution 18.** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ form a basis.