Instructions: This homework is an individual effort. Answer each question. This is due on **Monday, May 18th**. **Show all work to receive full credit.**

1 Chapter 1

1. Find the angle between the pairs of vectors.

   a. $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   Solution 1. $\cos\theta = \dfrac{\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{||\mathbf{v}||||\mathbf{u}||} = \dfrac{1}{2(1)} = \dfrac{1}{2}$
   $\theta = \arccos(1/2) = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$. Since both our vectors are in quadrant 1, $\theta = \dfrac{\pi}{3}$.

   b. $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

   Solution 2. $\mathbf{v} \cdot \mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 4 - 2 - 2 = 0$. Thus, the two vectors are perpendicular and have $\theta = \dfrac{\pi}{2}$.

2. Let $\mathbf{v} = (1, 1, ..., 1)$ be in $\mathbb{R}^3$. Find a unit vector in the same direction. Then find a unit vector perpendicular to $\mathbf{v}$.

   Solution 3. $\mathbf{v} = (1, 1, 1) \in \mathbb{R}^3$. $||\mathbf{v}|| = \sqrt{3}$. Thus, $\mathbf{u}_v$ is the unit vector $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ in the same direction as $\mathbf{v}$.
   We first find a vector perpendicular to $\mathbf{v}$. Say that is $\mathbf{w} = (1, 0, -1)$. Now we unitize this to get $\mathbf{u}_w = \left(\dfrac{1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}\right)$.

3. Pick any numbers x, y, z such that $x + y + z = 0$. Let $\mathbf{v}$ be the vector (x, y, z) and $\mathbf{w}$ be the vector (z, x, y). Find the angle between the two vectors. Find $\cos\theta$ for the two vectors. Then take different x, y, z and do the same. Explain the phenomenon.

Solution 4. This is the trickiest one! Take $\mathbf{v} = (1, -1, 0)$. Then $\mathbf{w} = (0, 1, -1)$. $\mathbf{v} \cdot \mathbf{w} = -1$, $||\mathbf{v}|| = \sqrt{2}$, and $||\mathbf{w}|| = \sqrt{2}$. Thus, $\cos\theta = \frac{-1}{2}$. Repeat this as the question states and you will always get $\frac{-1}{2}$. Now we explain why.

$$\cos\theta = \frac{xz + yx + zy}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2 + z^2}} = \frac{xz + yx + zy}{x^2 + y^2 + z^2} = \frac{xz + yx + zy}{(x + y + z)(x + y + z) - 2xy - 2xz - 2yz}$$

$x + y + z = 0$ by the statement of the problem. Thus,

$$\cos\theta = \frac{xz + yx + zy}{-2(xy + xz + yz)} = \frac{-1}{2}.$$

4. Find a linear combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3 = 0$ with $x_1 = 1$,

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solution 5. We can set up the following system and perform row operations on the augmented matrix!

$$\begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} 2R_1 - R_2 \to R_2 \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & 3 & 6 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} 3R_1 - R_3 \to R_3 \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 12 & 0 \end{bmatrix}$$

$$R_2/3, R_3/6 \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} R_2 - R_3 \to R_3 \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 4R_2 \to R_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0 \quad (1)$$
$$x_2 + 2x_3 = 0 \quad (2)$$

Since $x_1 = 1$, (1) tells me $x_3 = 1$ and (2) tells me $x_2 = -2$. Thus, $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

2 Chapter 2

1. Make the following system upper triangular with only 2 row operations:

$$\begin{aligned} 2x+3y+z &= 8 \\ 4x+7y+5z &= 20 \\ -2y+2z &= 0. \end{aligned}$$

Solution 6. We will eliminate 4 and then -2.
$$\begin{bmatrix} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{bmatrix} R_2 - 2R_1 \to R_2 \begin{bmatrix} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{bmatrix} 2R_2 + R_3 \to R_3 \begin{bmatrix} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$
Thus, we now have the system

$$\begin{aligned} 2x+3y+z &= 8 \\ y+3z &= 4 \\ 8z &= 8. \end{aligned}$$

2. Use elimination to solve the following system:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

Solution 7. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{bmatrix} 2R_1 - R_2 \to R_2 \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 5 & 7 & 6 \end{bmatrix} 3R_1 - R_3 \to R_3 \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -3 \end{bmatrix}$

$R_3 - R_2 \to R_3 \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ Last row tells us that this system has no solutions since $0 \neq -3$.

3. Choose the numbers a, b, c, d such that the augmented matrix has:

   a. No solution

   Solution 8. Like above if $d = 0$ and c is anything but 0, this has no solutions regardless of a and b.

   b. Infinite solutions

   Solution 9. Opposite of above, if $d = 0$ and $c = 0$, this has infinite solutions regardless of a and b since we have a free variable x_3 that can be anything.

   c. one solution

   Solution 10. If $d = 1$ then this has a solution no matter what because x_3 is fixed to the value of c.

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

4. Find A^2, A^3, and A^4.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution 11. $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

5. Write two 3 x 3 matrices with nonzero entries. Call them A and B. Find A^{-1}, B^{-1}, AB, BA, and $(AB)^{-1}$.

Solution 12. You may choose any but for simplicity I will choose the following:
$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$.
Neither is invertible since the rows are all the same. This also means the product is not invertible. To invert, you should be using the elimination method from the lectures.

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} = BA$$