Instructions: This exam is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero. Answer each question. **Show all work to receive full credit.** Unless the question specifies, you may provide either an exact answer or round to two decimal places. Write your answers on the test. You have **24 houra** to finish the exam. Answer all questions to the best of your ability. Unless otherwise specified, you are required to **SHOW ALL YOUR WORK** to receive full credit. The exam has 110 possible points. You will be graded out of 100 points.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.

I, ____________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this exam. I will show all my work to demonstrate my knowledge on the topic. If I do break this honor code, I will accept a 0 on this assignment.

Questions	Possible	Score		Possible	Score
Quiestion 1	20		Question 5	10	
Question 2	10		Question 6	20	
Question 3	10		Question 7	10	
Question 4	20		Question 8	10	
			Total		

True/False

1. Instructions: You must determine if the statement is true or false. If you say false, provide a counterexample. If you say true, explain your reasoning.

   a. Let A be a 3×3 matrix with $\det(A) = 1$. Then $\det(2A) = 2$.

   Solution 1. False. Take $A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Clearly $\det(2A) = 8$.

   b. For every 2×2 matrices A and B, we have that $\det(A + B) = \det(A) + \det(B)$.

   Solution 2. False. Let both A and B be the identity. Then $\det(A + B) = 4$ not 2.

   c. Performing row operations on a matrix does not change the eigenvalues.

   Solution 3. False. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues 0 and 2. Meanwhile $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 0 and 1.

   d. Every positive-definite matrix is invertible.

   Solution 4. True. All upper left determinants must be positive (nonzero included) including the determinant of the entire matrix.

   e. If S is a positive-definite matrix, then S^2 is also a positive-definite matrix.

   Solution 5. True. Remember that S^2 has eigenvalues that are the squares of the eigenvalues of S. This implies they are still positive. Also, S^2 is clearly symmetric since the dot product of rows and columns is either the length of one of them or a repeated dot product for the reflected position across the diagonal (a21 and a12).

Open-ended

2. Consider the predator vs prey model where the gazelle population shows fast growth (from 5g) but loss to lions (from -2ℓ), while the lion population always grows:

$$\frac{dg}{dt} = 5g - 2\ell \qquad \text{and} \qquad \frac{d\ell}{dt} = g + 2\ell.$$

Solve the differential equation

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} \mathbf{u}(t), \qquad \text{where } \mathbf{u}(t) = \begin{bmatrix} g(t) \\ \ell(t) \end{bmatrix}.$$

Leave two constants in your solution.

Solution 6. $\det(A - \lambda I) = (5-\lambda)(2-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3)$. Thus, $\lambda_1 = 3$ and $\lambda_2 = 4$. When $\lambda_1 = 3$,

$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \Rightarrow x = y \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. When $\lambda_2 = 4$,
$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \Rightarrow x = 2y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, $\mathbf{u}(t) = Ce^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} Ce^{3t} + 2De^{4t} \\ Ce^{3t} + De^{4t} \end{bmatrix}$

3. Use **Cramer's Rule** to solve for the unknown coefficients above when the initial populations are $g(0) = 50$ and $\ell(0) = 30$ for the above.

Solution 7. $\mathbf{u}(0) = \begin{bmatrix} C + 2D \\ C + D \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 50 \\ 30 \end{bmatrix}$. Now we apply Cramer's Rule to this linear system!

$B_1 = \begin{bmatrix} 50 & 2 \\ 30 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 50 \\ 1 & 30 \end{bmatrix}$. Thus, $\det(A) = -1$, $\det(B_1) = -10$ and $\det(B_2) = -20$.

Therefore, $C = -10/-1 = 10$ and $D = -20/-1 = 20$.

4. Consider the matrix $S = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

   a. Determine if S is positive-definite using one of the four tests.

   Solution 8. $3 > 0$ and $-9 - 16 < 0$. Since the determinant is negative, this is not positive definite.

b. Find an orthonormal matrix Q which diagonalizes S.

Solution 9. $\det(S - \lambda I) = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$. Thus, $\lambda_1 = 5$ and $\lambda_2 = -5$. For λ_1,

$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \Rightarrow 2y = x \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For λ_2,
$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \Rightarrow 2x = -y \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Since S is symmetric, the eigenvectors are orthogonal. Thus, we normalize them.
$\mathbf{q}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Therefore, the diagonalizing matrix Q is as follows:

$$Q = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \end{bmatrix}.$$

c. Using the diagonalization in (b), find a formula for S^k.

Solution 10. $S^k = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}^k \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & (-5)^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$
$\frac{1}{5} \begin{bmatrix} 2 \cdot 5^k & (-5)^k \\ 5^k & -2 \cdot (-5)^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 \cdot 5^k + (-5)^k & 2 \cdot 5^k - 2 \cdot (-5)^k \\ 2 \cdot 5^k - 2 \cdot (-5)^k & 5^k + 4 \cdot (-5)^k \end{bmatrix}$

d. Use the formula in (c) to calculate S^4.

Solution 11. $S^4 = \frac{1}{5} \begin{bmatrix} 4 \cdot 5^4 + (-5)^4 & 2 \cdot 5^4 - 2 \cdot (-5)^4 \\ 2 \cdot 5^4 - 2 \cdot (-5)^4 & 5^4 + 4 \cdot (-5)^4 \end{bmatrix} = \begin{bmatrix} 5^4 & 0 \\ 0 & 5^4 \end{bmatrix}$.

5. a. Given that the eigenvalues of a 2×2 matrix are 2 and 3, what is the determinant of that matrix?

Solution 12. $2 \cdot 3 = 6$

b. Suppose the trace of a 2×2 matrix is 7 and one of the eigenvalues is $\lambda_1 = -3$, what is the other eigenvalue λ_2?

Solution 13. $-3 + \lambda_2 = 7$. Thus, $\lambda_2 = 10$.

c. Suppose the trace of a 2×2 matrix is 4 and the determinant is 3, what are the eigenvalues of the matrix?

Solution 14. $\lambda_1 \lambda_2 = 3$ and $\lambda_1 + \lambda_2 = 4$. By substitution,
$4\lambda_2 - \lambda_2^2 = 3 \Rightarrow \lambda_2^2 - 4\lambda_2 + 3 = 0 \Rightarrow (\lambda_2 - 3)(\lambda_2 - 1) = 0$. Therefore, $\lambda_2 = 3$ or 1. We see that λ_1 is whichever value λ_2 is not by the original sum equation.

d. Suppose the trace of a 3×3 matrix is 6, the determinant is 6, and one of the eigenvalues is $\lambda_1 = 1$, what are the other two eigenvalues of the matrix?

Solution 15. $\lambda_2\lambda_3 = 6$ and $1 + \lambda_2 + \lambda_3 = 6 \Rightarrow \lambda_2 + \lambda_3 = 5$. By substitution, $5\lambda_3 - \lambda_3^2 = 6$. By arithmetic, $\lambda_3 = 3$ or 2. By the original sum equation, λ_2 is whichever of these values λ_3 is not.

6. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$ using the cofactor inverse formula. (Hint: Show OR EXPLAIN how you got the cofactors)

Solution 16. $C_{11} = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -4$

$C_{12} = -\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -C_{11} = 4$. $C_{13} = C_{11} = -4$. These determinants (when C_{ij} has that $i + j \neq 5$) within the cofactors are all identical except the antidiagonal.

$C_{14} = -\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = 4$. $C_{23} = 4, C_{32} = 4, C_{41} = 4$.

$C = \begin{bmatrix} -4 & -4 & -4 & 4 \\ -4 & -4 & 4 & -4 \\ -4 & 4 & -4 & -4 \\ 4 & -4 & -4 & -4 \end{bmatrix} = C^T$. Since A reduces down to the following: $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$, we get $\det(A) = 2\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 4\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -8 - 8 = -16.$

Therefore, $A^{-1} = \dfrac{C^T}{\det(A)} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$.

7. Find a 2×2 matrix with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$ and corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution 17. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

8. Let $\mathbf{u} = (1, 2, 1)$ and $\mathbf{v} = (3, 7, 1)$.

a. Find $\mathbf{u} \times \mathbf{v}$.

Solution 18. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 3 & 7 & 1 \end{vmatrix} = i \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = -5i + 2j + k = (-5, 2, 1)$

b. Let $\mathbf{w} = (5, -2, 3)$. Find the triple product of the three vectors.

Solution 19. $(-5, 2, 1) \cdot \mathbf{w} = -25 - 4 + 3 = -26$

c. Find the area of the tetrahedron (3-dimensional triangle) spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Solution 20. $Area = (1/2)|-26| = 13.$