Instructions: This exam is closed book, closed note, and an individual effort. Electronic devices are not allowed on your person (e.g., no cell phones or calculators). Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero. Answer each question. **Show all work to receive full credit.** Unless the question specifies, you may provide either an exact answer or round to two decimal places. Write your answers on the test. You have **24 houra** to finish the exam. Answer all questions to the best of your ability. Unless otherwise specified, you are required to **SHOW ALL YOUR WORK** to receive full credit. The exam has 110 possible points. You will be graded out of 100 points.

WRITE THIS PARAGRAPH ON WHAT YOU SUBMIT ALONG WITH A SIGNATURE AND DATE.

I, ________________________, will not under any circumstance use an online source, my peers, my notes, or any other resource besides my own knowledge to complete this exam. I will show all my work to demonstrate my knowledge on the topic. If I do break this honor code, I will accept a 0 on this assignment.

Questions	Possible	Score		Possible	Score
MC	20		Question 4	10	
Question 1	10		Question 5	10	
Question 2	20		Question 6	10	
Question 3	20		Question 7	10	
Extra Credit			Total		

Multiple Choice

Each question is worth 1 point!

1. When A, B, C are symmetric, then the transpose of ABC is CBA.
   (a) **A Duh! True.**
   (b) No way! False
   (c) Where's the rest of the alphabet?

2. If $AB = AC$ then $B = C$.
   (a) Are sloths good swimmers? True
   (b) **A Are elephants just big, hairless, wrinkly dogs? False**
   (c) Fun fact: The mantis shrimp has the world's fastest punch.

3. Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (2, 1, 2)$.
   (a) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (b) $\mathbf{u}$ and $\mathbf{v}$ are orthonormal.
   (c) $\mathbf{u}$ and $\mathbf{v}$ are of equal length.
   (d) **A Both a and c are true.**
   (e) None of the above.

4. $\begin{bmatrix} 3 \\ 9 \end{bmatrix} \in \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.
   (a) **A Yessir! True**
   (b) No way! False
   (c) Did you spell spam wrong?

5. The matrix $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & 0 \end{bmatrix}$ is invertible.
   (a) If that is not true, I don't know what is! True
   (b) **A That matrix hurts me! False**
   (c) What's a matrix?

6. The subspaces $V = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}\right)$ and $W = \text{span}\left(\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}\right)$ are orthogonal.

   (a) **A By thorough investigation, true!**

   (b) By thorough investigation, false!

   (c) By lack of investigation this case is still open.

7. If the columns of a matrix are dependent, then so are the rows.

   (a) Truer than true

   (b) **A You think you can trick me Tommy? False**

   (c) Is this the Krusty Krab?

8. A square matrix has no free variables.

   (a) By math, true.

   (b) **A Something is fishy... False**

   (c) No, this is Patrick.

9. Any 6 vectors in $\mathbf{R}^5$ are linearly dependent.

   (a) **A YES YES YES YES YES True**

   (b) NO NO NO NO NO False

   (c) IDK IDK IDK IDK IDK

10. A linear system of 3 variables has exactly 3 solutions.

    (a) why not? True

    (b) **A But why though? False**

    (c) 21

11. Three vectors in $\mathbf{R}^3$ will always span all of $\mathbf{R}^3$.

    (a) **A YAS True**

    (b) UHHHHH

12. $||\mathbf{u}|| * (\mathbf{u} \cdot \mathbf{v})$ where $\mathbf{u} = (1, 2, 2)$ and $\mathbf{v} = (0, 3, -4)$.

    (a) -18

    (b) **A -6**

    (c) 6

    (d) 18

    (e) 12

13. Let $\mathbf{u} = (\sqrt{2}/2, \sqrt{2}/2, 0, 0)$ and $\mathbf{v} = (0, 0, \sqrt{3}/2, 1/2)$.

    (a) They are both unit vectors only.

    (b) They are orthogonal only.

    (c) They are not orthogonal or unit vectors.

    (d) **A They are orthonormal.**

    (e) None of the above.

14. How many 5 x 5 permutation matrices are there?

    (a) 20

    (b) **A 120**

    (c) 5

    (d) 50

    (e) 60

15. $(AB)^{-1}(C+D)^T(C^T+D^T)^{-1}(AB) =$

    (a) A^TB^TAB

    (b) $C+D$

    (c) AB

    (d) $ABC+ABD$

    (e) **A Identity**

16. $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = LU$ is a factorization where L is:

    (a) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

    (b) **A** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

    (c) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

    (e) Identity

17. If A is a matrix, then

    (a) $\dim(C(A)) + \dim(N(A))$ is the number of rows.

    (b) **A** $\dim(N(A)) + \dim(C(A))$ **is the number of columns.**

    (c) $\dim(N(A)) + \dim(C(A))$ is the rank.

    (d) $\dim(N(A)) + \dim(C(A))$ is the sum of the number of columns and rank.

    (e) $\dim(N(A)) + \dim(C(A))$ is the sum of the number of rows and the rank.

18. The null space is just the zero vector exactly when the columns are independent vectors.

    (a) **A Truth has been spoken**

    (b) Lying is a bad habit! False

    (c) The null space is always empty no matter what!

19. The left null space is the orthogonal complement of

    (a) $N(A)$

    (b) **A** $C(A)$

    (c) $C(A^T)$

    (d) $N(A^T)$

    (e) None of the above.

20. If $\mathbf{p}$ is the projection of $\mathbf{b}$ on to the line $\mathbf{a}$, then $||\mathbf{p}|| = ||\mathbf{a}||$.

    (a) I got this one for sure! True

    (b) **A Uhhh I don't think so Tommy. False**

    (c) Are we done yet?

Open-ended

1. Determine the angle between the vectors $\mathbf{v} = (3, 2, 6)$ and $\mathbf{u} = (3, -1 - 3\sqrt{3}, -3 + \sqrt{3})$. Leave your answer in terms of $\cos^{-1}$.

   Solution 1. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{v}||||\mathbf{u}||} = \dfrac{9 - 2 - 6\sqrt{3} - 18 + 6\sqrt{3}}{\sqrt{49}\sqrt{49}} = \dfrac{-11}{49}$. Thus, $\theta = \cos^{-1}\left(\dfrac{-11}{49}\right)$.

2. Consider the following linear system of equations:

$$\begin{aligned} x + 2y - 3z &= -1 \\ 2x + 3y - 9z &= -7 \\ 3x + 7y - 7z &= 0 \end{aligned}$$

   a. Solve the system.

   Solution 2. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & -9 \\ 3 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$ is the system in matrix form. We use an augmented matrix to solve.

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 2 & 3 & -9 & -7 \\ 3 & 7 & -7 & 0 \end{bmatrix} R_2 - 2R_1 \to R_2 \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -1 & -3 & -5 \\ 3 & 7 & -7 & 0 \end{bmatrix} R_3 - 3R_1 \to R_3 \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -1 & -3 & -5 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_2 + R_3 \to R_3 \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & -2 \end{bmatrix} \text{ Can stop here and use elimination if wanted!}$$

$$R_1 + 2R_2 \to R_1 \begin{bmatrix} 1 & 0 & -9 & -11 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & -2 \end{bmatrix} -R_2, -R_3 \begin{bmatrix} 1 & 0 & -9 & -11 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 + 9R_3 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_2 - 3R_3 \to R_2 \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   Thus, $x = 7$, $y = -1$, $z = 2$.

   b. Use the process above to determine if $\begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$ is in span$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ -7 \end{bmatrix}\right)$. Explain.

   Solution 3. The above is in the span since we found a solution to the system!

   c. Let A be the matrix you form when writing the system as $A\mathbf{x} = \mathbf{b}$. Find the LU factorization of A(you may cite previous work to explain your process).

   Solution 4. We see from (a) that $U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$. Also from (a) L has multipliers, $l_{21} = -2$, $l_{31} = -3$, and $l_{32} = 1$. Thus,

$$L = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

d. Find A^{-1} if it exists. If it does not exist, explain why.

Solution 5. $\begin{bmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 3 & -9 & 0 & 1 & 0 \\ 3 & 7 & -7 & 0 & 0 & 1 \end{bmatrix}$ By the operations we did in (a), we get $A^{-1} = \begin{bmatrix} 42 & -7 & -9 \\ -13 & 2 & 3 \\ 5 & -1 & -1 \end{bmatrix}$.

3. Consider the linear system:

$$\begin{aligned} x + 4y + 2z + 3w &= -3 \\ 2x + 8y + z &= -3 \\ -3x - 12y - 10z \quad -17w &= 13 \end{aligned}$$

a. Write the system in the form $A\mathbf{x} = \mathbf{b}$. Then find the span of each of the four subspaces of A.

Solution 6. $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 8 & 1 & 0 \\ -3 & -12 & -10 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 13 \end{bmatrix}$

$C(A)$ is spanned by column 1 and 3 in the original matrix by the reduction done for (b).

$N(A)$ is spanned by $\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ by the work done for $\mathbf{x}_n$ in (b).

$C(A^T)$ is spanned by rows 1 and 2 by the row reduction done in (b).

For $N(A^T)$, we need to reduce the transpose.

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -12 \\ 2 & 1 & -10 \\ 3 & 0 & 17 \end{bmatrix} 4R_1 - R_2 \to R_2 \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 2 & 1 & -10 \\ 3 & 0 & 17 \end{bmatrix} R_3 - 2R_1 \to R_3 \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \\ 3 & 0 & 17 \end{bmatrix} 3R_1 - R_4 \to R_1$$

$$\begin{bmatrix} 0 & 6 & -26 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \\ 3 & 0 & 17 \end{bmatrix} R_1 - 2R_3 \to R_1 \begin{bmatrix} 0 & 0 & -18 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \\ 3 & 0 & 17 \end{bmatrix} R_1 / -18 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \\ 3 & 0 & 17 \end{bmatrix} R_4 - 17R_1 \to R_4$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \\ 3 & 0 & 0 \end{bmatrix} R_3 + 4R_1 \to R_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Thus, y_2 is the only free variable. All other y_i are 0. Therefore, we get the vector spanning $N(A^T)$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

b. Find the complete solution to the system, $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$.

Solution 7. $\begin{bmatrix} 1 & 4 & 2 & 3 & -3 \\ 2 & 8 & 1 & 0 & -3 \\ -3 & -12 & -10 & -17 & 13 \end{bmatrix} R_3 + 3R_1 \to R_3 \begin{bmatrix} 1 & 4 & 2 & 3 & -3 \\ 2 & 8 & 1 & 0 & -3 \\ 0 & 0 & -4 & -8 & 4 \end{bmatrix}$

$R_2 - 2R_1 \to R_2 \begin{bmatrix} 1 & 4 & 2 & 3 & -3 \\ 0 & 0 & -3 & -6 & 3 \\ 0 & 0 & -4 & -8 & 4 \end{bmatrix} 2R_1 + R_3 \to R_1 \begin{bmatrix} 2 & 8 & 0 & -2 & -2 \\ 0 & 0 & -3 & -6 & 3 \\ 0 & 0 & -4 & -8 & 4 \end{bmatrix}$

$R_1/2, R_2/-3, R_3/-4 \begin{bmatrix} 1 & 4 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} R_2 - R_3 \to R_3 \begin{bmatrix} 1 & 4 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

This is the reduced form of A with the augmented piece. Thus, we get y and w are free with pivots in columns 1 and 3. If $y = 1$ and $w = 0$ we get the null vector $\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

If $w = 1$ and $y = 0$ we get the null vector $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. Thus, $\mathbf{x}_n = y \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. If we make these free variables 0 and solve with the augmented piece we get $\mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Thus,

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

4. Create an orthonormal basis for $\mathbf{R}^2$ with the basis $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Solution 8. We will use Gram-Schmidt! $\mathbf{q}_1 = \dfrac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}{\sqrt{25}} = \dfrac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\mathbf{b'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1/5 \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1/25 \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}} 1/5 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 7/25 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{\frac{1}{25} \begin{bmatrix} 4 \\ -3 \end{bmatrix}}{1/5} = \frac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

5. Let (-1,4), (0,-4), (1,10), (2,6) be a set of points we want to fit to a parabola.

   a. Write a linear system in the following form to attempt to answer this question: $A\mathbf{x} = \mathbf{b}$.

   Solution 9.

$$\begin{aligned} (-1)^2 a + (-1)b + c &= 4 \\ (0)^2 a + (0)b + c &= -4 \\ (1)^2 a + (1)b + c &= 10 \\ (2)^2 a + (2)b + c &= 6. \end{aligned}$$

   Thus,

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 10 \\ 6 \end{bmatrix}.$$

   b. Use the above to determine the parabola of best fit to the 4 points.

   Solution 10. $A^T A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 38 \\ 18 \\ 16 \end{bmatrix}$$

   Now we solve $A^T A\mathbf{x} = A^T\mathbf{b}$.

$$\begin{bmatrix}18&8&6&38\\8&6&2&18\\6&2&4&16\end{bmatrix} R_1/2, R_2/2, R_3/2 \begin{bmatrix}9&4&3&19\\4&3&1&9\\3&1&2&8\end{bmatrix} 3R_3 - R_1 \to R_3 \begin{bmatrix}9&4&3&19\\4&3&1&9\\0&-1&3&5\end{bmatrix}$$

$$R_1{+}4R_3 \to R_1 \begin{bmatrix}9&0&15&39\\4&3&1&9\\0&-1&3&5\end{bmatrix} R_1/3 \begin{bmatrix}3&0&5&13\\4&3&1&9\\0&-1&3&5\end{bmatrix} R_2{+}3R_3 \to R_2 \begin{bmatrix}3&0&5&13\\4&0&10&24\\0&-1&3&5\end{bmatrix}$$

$$R_2{-}2R_1 \to R_2 \begin{bmatrix}3&0&5&13\\-2&0&0&-2\\0&-1&3&5\end{bmatrix} R_2/{-2} \begin{bmatrix}3&0&5&13\\1&0&0&1\\0&-1&3&5\end{bmatrix} \text{Swap } R_1 \text{ and } R_2 \begin{bmatrix}1&0&0&1\\3&0&5&13\\0&-1&3&5\end{bmatrix}$$

$$R_2 - 3R_1 \to R_2 \begin{bmatrix}1&0&0&1\\0&0&5&10\\0&-1&3&5\end{bmatrix} R_2/5 \begin{bmatrix}1&0&0&1\\0&0&1&2\\0&-1&3&5\end{bmatrix} R_3 - 3R_2 \to R_3 \text{ and swap}$$

$$\begin{bmatrix}1&0&0&1\\0&-1&0&-1\\0&0&1&2\end{bmatrix} -R_2 \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&2\end{bmatrix}$$

Thus, the best approximation is $y = x^2 + x + 2$.

6. Determine if the vectors $\begin{bmatrix}1\\-1\\4\end{bmatrix}$, $\begin{bmatrix}-1\\17\\-60\end{bmatrix}$, and $\begin{bmatrix}3\\1\\-2\end{bmatrix}$ are linearly independent or linearly dependent.

Solution 11. $\begin{bmatrix}1&-1&3\\-1&17&1\\4&-60&-2\end{bmatrix} R_1 + R_2 \to R_2 \begin{bmatrix}1&-1&3\\0&16&4\\4&-60&-2\end{bmatrix} R_2/4, R_3/2 \begin{bmatrix}1&-1&3\\0&4&1\\2&-30&-1\end{bmatrix}$

$$R_3 - 2R_1 \to R_3 \begin{bmatrix}1&-1&3\\0&4&1\\0&-28&-7\end{bmatrix} R_3/-7 \begin{bmatrix}1&-1&3\\0&4&1\\0&4&1\end{bmatrix}$$

Thus, not independent.

7. For each of the below, determine a system of linear equations for which the property is holds.

a. No solutions.

Solution 12. $\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

b. The only solution is $\begin{bmatrix}3\\-1\end{bmatrix}$.

Solution 13. $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$

c. The only solutions are vectors of the form $\begin{bmatrix}-a\\2a\end{bmatrix}$.

Solution 14. $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

8. Extra Credit: Suppose that the 3 x 4 matrix A has the vector $\mathbf{s} = (2, 3, 1, 0)$ as the only special solution to $A\mathbf{x} = \mathbf{0}$. What is the exact row reduced echelon form of A?

   Solution 15. First we note that $\dim(N(A)) = 4 - r = 1$. Thus, $r = 3$ implying 3 pivots. Thus, the 1 in $\mathbf{s}$ is the free variable. This will associate to column 3. Therefore we have the following so far:

   $$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

   This means the last column must be the negative of each of the coordiantes except the third to get (0,0,0) when multipled by $\mathbf{s}$. Therefore, our reduced matrix is:

   $$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$