Name: **Worksheet 6: Series- Integral Test and Remainder Estimate, p-Series**

Thoughts:

Integral Test: This test is rarely used, but is helpful for specific series. I keep this as a last resort most times. For use, the big thing here is to remember the 3 conditions necessary to use it! Here are the conditions considering a_n as the function $f(n)$:

1. f is continuous
2. f is positive
3. f is decreasing on the interval $[1, \infty]$.

This is the key thing for many instructors when grading these types of problems. Now let's go about using the integral test. Once the conditions are met, we find:

$$\int_1^\infty f(x)\,\mathrm{d}x.$$

The test tells us 2 things:

1. If $\int_1^\infty f(x)\,\mathrm{d}x$ is convergent, then so is the series
2. If $\int_1^\infty f(x)\,\mathrm{d}x$ is divergent, then so is the series.

The rest of this test is just things from the previous exams! That's all i have to say about this test!

p-series: This is an example of the test above. Let's look at the following series such that $p > 0$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

First we check all the conditions and this series satisfies those for the given p. For now, we will exclude the case when $p = 1$ and od it after since the integral changes a little bit. I will keep the infinity in the limit for simplicity and take the limit as we approach infinity when needed.

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \int_1^\infty x^{-p}\,\mathrm{d}x = \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty = \lim_{c\to\infty} \frac{c^{-p+1}}{-p+1} - \text{(a number)}$$

Notice if $-p + 1 < 0$, this will converge since the x will be in the denominator (ex: $p = 2$ gives the limit of $1/c(-p+1)$). If $-p + 1 > 0$, we have divergence of the integral for the opposite reason (give this one a try yourself with $p = 1/2$). Lastly, if $p = 1$ we have $ln(x)$ from the integral which continues to grow/diverge. Thus we have a nice condition:

Theorem 0.1 (p-Series Test). *If we have a series with $p > 0$ as defined above then we can say either of 2 things:*

1. *When $0 < p \leq 1$, the series is divergent.*
2. *When $p > 1$, the series is convergent.*

Remainder Estimate: Let S_n be the n-th partial sum of a series, s be the sum of the series and $R_n = s - S_n$ be the remainder estimate. **Note that the conditions of the integral test must still be met to use this**. First I'm going to state some things you need to know. Then I will give examples/explain their use.

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x \tag{1}$$

$$S_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq S_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x \tag{2}$$

The first thing that could be asked is to approximate the k-th partial sum and estimate the error. Approximating is easy since you'll just add the first k terms. Estimating the error is where our formula comes in. Equation (1) gives you both an upper bound and a lower bound on the error! If you need upper, use the rihgt side integral. If you need lower, use the left side integral!

The second type of question is how many terms are needed to be within B of the correct sum? The word "terms" here is referring to finding n (remmeber n is an integer or whole number). Thus, you take each integral and bound them below and above respectively to find an n (typically you will only need the upperbound integral). The last type of question is to estimate the sum of a series given $n = k$. Equations 2 helps us here since we can find an upperbound and a lower bound for the value of the series, s. To give an example would be tetious to all of us, but I encourage you to try all 3 of these questions with the infinite series of $1/n^4$.

All of these refer to the integral test. I do have a short proof of the integral test if anyone is interested. Please let me know in class if you'd like to see the proof to maybe understand the idea of the test more.

Problems: Determine if the series is convergent or divergent and give the name of the test used.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3n-4}{n^2-2n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

5. (Challenge) Find all c such that the series is convergent: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1}\right)$.

6. How many terms of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$ would you need to add to find its sum to within .001?

7. For the series show it is convergent, find an upperbound for the error in saying $s = S_n$ (find the upperbound in terms of n), find the smallest value of n such that the sum is accurate within .05 of the upperbound and find S_n for the value of n you found.

$$\sum_{n=1}^{\infty} \frac{\ln^2(n)}{n^2}$$