Topic Course on Probabilistic Methods
(Week 7)
Large deviation inequalities (I)

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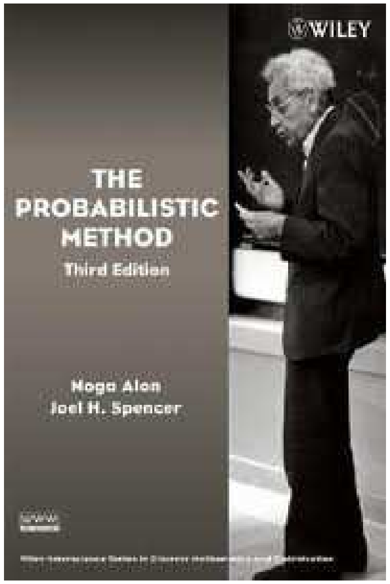
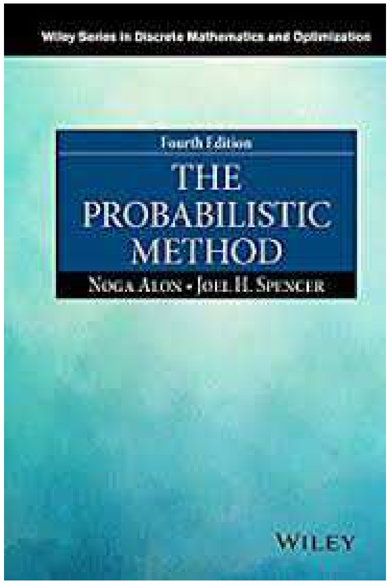
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Large deviation inequality

- Chernoff inequalities
- Weighted version
- McDiarmid's theorem
- Another generalization
- Lower tail versus upper tail
- More general versions

Large deviation inequality I

Chernoff inequalities: Suppose $X = \sum_{i=1}^{n} X_i$, where X_i are independent 0-1 random variables with

$$\Pr(X_i = 1) = p_i, \quad \Pr(X_i = 0) = 1 - p_i.$$

Then we have

$$\begin{aligned} Pr(X < E(X) - \lambda) &\leq e^{-\frac{\lambda^2}{2E(X)}} \\ Pr(X > E(X) + \lambda) &\leq e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}} \end{aligned}$$

Large deviation inequality II

A weighted version of Chernoff's inequality:

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Theorem [Chung,Lu] We have

$$Pr(X < E(X) - \lambda) \leq e^{-\lambda^2/2\nu} \tag{1}$$

$$Pr(X > E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\nu + M\lambda/3)}}. \tag{2}$$

Large deviation inequality III

Theorem [McDiarmid]: Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i - E(X_i) \leq M$ for a positive constant M. Let $X = \sum_{i=1}^n X_i$. Then

$$Pr(X - E(X) > \lambda) \leq e^{-\frac{\lambda^2}{2(Var(X)+M\lambda/3)}}.$$

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Note: If $Pr(X_i = a_i) = p_i$ and $Pr(X_i = 0) = 1 - p_i$, then $\mathrm{Var}(X) = a_i^2 p_i(1 - p_i) \leq \nu$. Thus

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This theorem implies inequality of upper tail in previous Theorem.

Large deviation inequality IV

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \leq M$, for $1 \leq i \leq n$. Let $X = \sum_{i=1}^n X_i$ and $\|X\| = \sqrt{\sum_{i=1}^n \mathrm{E}(X_i^2)}$. Then we have

$$\Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\|X\|^2 + M\lambda/3)}}.$$

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$$\text{Let } X_i' = X_i - \mathrm{E}(X_i), \text{ and } X' = X - \mathrm{E}(X).$$

$$X - \mathrm{E}(X) = X' - \mathrm{E}(X')$$

$$\|X'\|^2 = \sum_{i=1}^n \mathrm{E}(X_i'^2) = \mathrm{Var}(X).$$

Lower tail

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \geq 0$, for $1 \leq i \leq n$. Let $X = \sum_{i=1}^n X_i$ and $\|X\| = \sqrt{\sum_{i=1}^n \mathrm{E}(X_i^2)}$. Then we have

$$\Pr(X \leq E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2\|X\|^2}}.$$

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$$\Pr(X \leq E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2\|X\|^2}}.$$

Proof: Let $X_i' = -X_i$ and $X' = -X$. Applying the upper tail to X' with $M = 0$, we get

$$\begin{aligned}\Pr(X \leq E(X) - \lambda) &= \Pr(X' \geq E(X') + \lambda) \\ &\leq e^{-\frac{\lambda^2}{2\|X'\|^2}} = e^{-\frac{\lambda^2}{2\|X\|^2}}.\end{aligned}$$

A special function

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- $g(y) \leq 1$, for $y < 0.$
- $g(y)$ is monotone increasing, for $y \geq 0.$
- For $y < 3$, we have

$$g(y) = 2\sum_{k=2}^{\infty} \frac{y^{k-2}}{k!} \leq \sum_{k=2}^{\infty} \frac{y^{k-2}}{3^{k-2}} = \frac{1}{1 - y/3}.$$

Proof of upper tail

$$\mathrm{E}(e^{tX}) = \prod_{i=1}^{n} \mathrm{E}(e^{tX_i})$$

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\mathrm{E}(e^{tX}) &= \prod_{i=1}^{n} \mathrm{E}(e^{tX_i}) \\
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Choose $t = \frac{\lambda}{\|X\|^2+M\lambda/3}$. We have $1 - \frac{Mt}{3} = \frac{\|X\|^2}{\|X\|^2+M\lambda/3}$.

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Large deviation inequality V

Theorem [Chung, Lu] Let X_i denote independent random variables satisfying $X_i \leq \mathrm{E}(X_i) + a_i + M$, for $1 \leq i \leq n$. For, $X = \sum_{i=1}^n X_i$, we have

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It remains to verify

$$X' - \mathrm{E}(X') = X - \mathrm{E}(X).$$

$$\|X'\|^2 = \mathrm{Var}(X) + \sum_{i=1}^n a_i^2.$$

$\square$

Large deviation inequality VI

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \leq \mathrm{E}(X_i) + M_i$, for $0 \leq i \leq n$. We order X_i's so that M_i are in an increasing order. Let $X = \sum_{i=1}^n X_i$. Then for any $1 \leq k \leq n$, we have

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Compared with McDiarmid's inequality

- M is replaced by M_k.
- Additional cost $\sum_{i=k}^n (M_i - M_k)^2$.

Large deviation inequality VI

Theorem [Chung, Lu] Suppose X_i are independent random variables satisfying $X_i \leq \mathrm{E}(X_i) + M_i$, for $0 \leq i \leq n$. We order X_i's so that M_i are in an increasing order. Let $X = \sum_{i=1}^n X_i$. Then for any $1 \leq k \leq n$, we have

$$\Pr(X \geq E(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\mathrm{Var}(X) + \sum_{i=k}^n (M_i - M_k)^2 + M_k\lambda/3)}}.$$

Compared with McDiarmid's inequality

- M is replaced by M_k.
- Additional cost $\sum_{i=k}^n (M_i - M_k)^2$.
- McDiarmid's inequality is a special case with $k = n$.

Proof

For fixed k, we choose $M = M_k$ and

$$a_i = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ M_i - M_k & \text{if } k \leq i \leq n \end{cases}$$

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$$\sum_{i=1}^{n} a_i^2 = \sum_{i=k}^{n} (M_i - M_k)^2.$$

Apply previous theorem with these a_i's. $\square$

An application

Example: Consider the sum $X = \sum_{i=1}^{n} X_i$.

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Expectation and Variance

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\mathrm{E}(X) &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\
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$$\Pr(X \geq \mathrm{E}(X) + \lambda) \leq e^{-\frac{\lambda^2}{2((2n-1)p(1-p)+(1-p)\sqrt{n}\lambda/3)}}.$$

In particular, for constant $p \in (0,1)$ and $\lambda = \Theta(n^{\frac{1}{2}+\epsilon})$, we have

$$\Pr(X \geq \mathrm{E}(X) + \lambda) \leq e^{-\Theta(n^{\epsilon})}.$$

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