Topic Course on Probabilistic Methods
(Week 3)
Alterations

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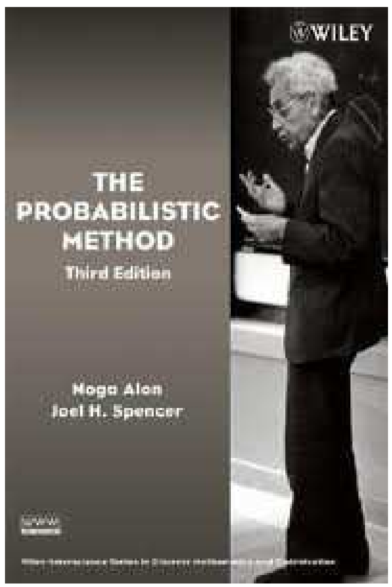
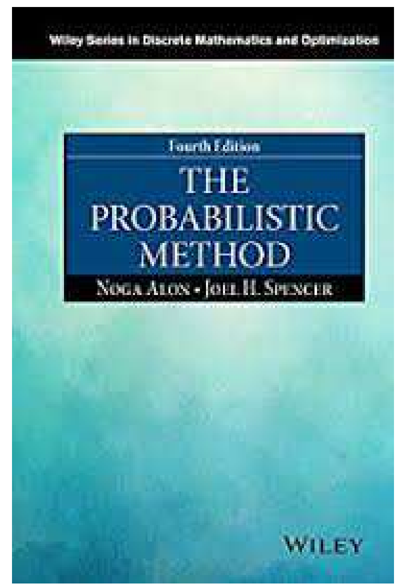
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Alteration

- Ramsey number $R(r, r)$
- Combinatorial geometry
- Ramsey number $R(k, r)$
- Property B problem revisited

Alteration method

Suppose that the "random" structure does not have all desired properties but many have a few "blemishes". With a small alteration we remove the blemishes, giving the desired structures.

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This gives $R(r,r) > (1+o(1))\frac{1}{e}r2^{r/2}$. □

Combinatorial geometry

- S: a set of n points in the unit square $[0,1]^2$.
- $T(S)$: the minimum area of a triangle whose vertices are three distinct points of S.

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Proof: Select $2n$ random points uniformly and independently from $[0,1]^2$.

- P, Q, R: three random points.
- $\mu := \Delta PQR$: the area of PQR.

Proof

$$\Pr(x \le |PQ| \le x + \Delta x) \le \pi(x+\Delta x)^2 - \pi x^2 \approx 2\pi x \Delta x.$$

If $\mu \le \epsilon$, then R is in the region of a rectangle of width $\frac{4\epsilon}{x}$ and length at most $\sqrt{2}$.

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Delete one vertex from each small triangle and leave at least n vertices. Now no triangle has area less that $\frac{1}{100n^2}$. $\square$

Ramsey number $R(k,t)$

Theorem: For any $0 < p < 1$, we have

$$R(k,t) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}.$$

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Proof: Color each edge independently in red or blue; the probability of being red is p while the probability of being blue is $1-p$. Let X be the number of red K_k and Y be the number of blue K_t.

$$\begin{aligned} \mathrm{E}(X) &= \binom{n}{k} p^{\binom{k}{2}} \\ \mathrm{E}(Y) &= \binom{n}{t} (1-p)^{\binom{t}{2}}. \end{aligned}$$

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Best lower bound: **Kim (1995)** and best upper bound: **Shearer (1983)**.

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Before Shearer's result, **Ajtai-Komlós and Szemerédi (1980)** proved $R(3, t) \leq \frac{c't^2}{\ln t}$.

Recoloring

Property B problem revisited:
Let $m(r)$ denote the minimum possible number of edges of an r-uniform hypergraph that does not have property B.

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Theorem [Radhakrishnan-Srinivasan 2000]:

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Proof: For a fixed r-uniform hypergraph $H = (V, E)$ with $|E| = k2^{r-1}$. Let $p \in [0, 1]$ satisfying $k(1-p)^r + k^2 p < 1$.

Coloring process

Here is a two-round coloring process.

- **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected k monochromatic edges. Let U be the set of vertices in some monochromatic edges.

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- **First round:** Color each vertex independently in red or blue with equal probability. It ends with a coloring with expected k monochromatic edges. Let U be the set of vertices in some monochromatic edges.
- **Second round:** Consider vertices in U sequentially in the (random) order of V. A vertex $u \in U$ is **still dangerous** if there is some monochromatic edge in the first coloring and for which no vertex has yet changed color.
    - If u is not dangerous, do nothing.
    - If u is still dangerous; with probability p, flip the color of u.

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- e was red in the first coloring and remained red through the final coloring; call this event A_e.

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$$2\sum_{e \in E(H)} \Pr(A_e) = k(1-p)^r.$$

Estimating $\Pr(C_e)$

For two edge e, f, we say e **blames** f if

- $e \cap f = \{v\}$ for some v.
- In the first coloring f was blue and in the final coloring e was red.
- v was the last vertex of e that changed color from blue to red.
- When v changed its color f was still entire blue.

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Call this event B_{ef}. Then

$$\sum_e \Pr(C_e) \leq \sum_{e \neq f} \Pr(B_{ef}).$$

continue

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$$\Pr(B_{ef} \mid \sigma) \leq \frac{p}{2} 2^{-r+1} (1-p)^j 2^{-r+1+i} \left(\frac{1+p}{2} \right)^i .$$

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We have

$$\begin{aligned} \Pr(B_{ef}) &\leq 2^{1-2r} p \mathrm{E}[(1+p)^i (1-p)^j]. \\ &\leq 2^{1-2r} p. \end{aligned}$$

Estimating k

The failure probability is at most

$$2 \sum_{e \in E(H)} (\Pr(A_e) + \Pr(C_e)) \leq k(1-p)^r + k^2 p < k e^{-pr} + k^2 p.$$

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The function $f(p) = k e^{-pr} + k^2 p$ reaches its minimum at $p = \frac{\ln(r/k)}{r}$. The minimum value is less than 1 if

$$k < (1 + o(1)) \sqrt{\frac{2r}{\ln r}}.$$

□

Continuous time

Spencer modified the Radhakrishnan-Srinivasan's proof slightly. To assign a random ordering of the vertex in V, it is sufficient to assign each vertex v a birth time $x_v \in [0, 1]$. The birth time x_v is assigned uniformly and independently.

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The rest of proof is the same.