# Topic Course on Probabilistic Methods
## (Week 11)
## Random Graphs (I)

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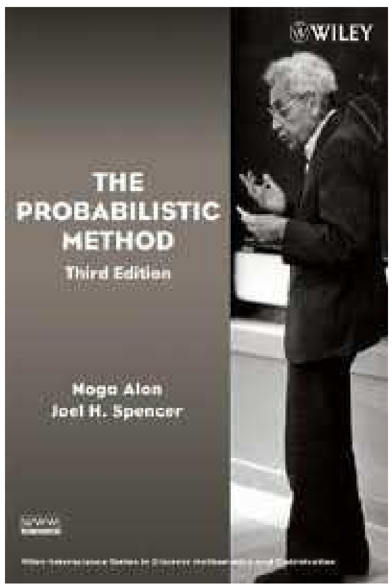
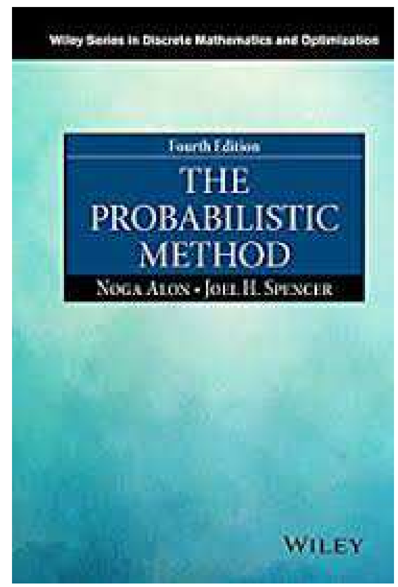
# Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)



# Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

# Subtopics

Random graphs

- Erdős-Rényi model
- Evolution of $G(n, p)$
- Galton-Watson process
- Graph branching process
- Barely subcritical regimes

# Erdős-Rényi model

$G(n, p)$: Erdős-Rényi random graphs

- n nodes

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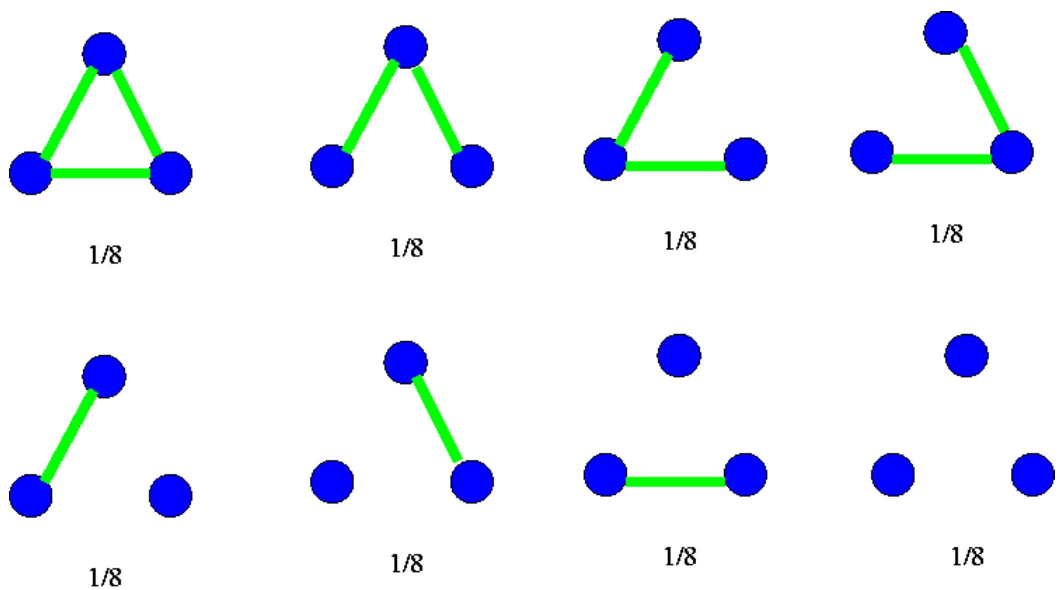
An example $G(3, \frac{1}{2})$:



# The birth of random graph theory

Paul Erdős and A. Rényi, On the evolution of random graphs
*Magyar Tud. Akad. Mat. Kut. Int. Kozl.* **5** (1960) 17-61.

# The birth of random graph theory

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDŐS and A. RÉNYI
*Institute of Mathematics*
*Hungarian Academy of Sciences, Hungary*

## *1. Definition of a random graph*

Let $E_{n,N}$ denote the set of all graphs having $n$ given labelled vertices $V_1, V_2, \cdots, V_n$ and $N$ edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing $N$ out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly

# Evolution of $G(n,p)$

$p$

| | |
|---|---|
| $0$ | the empty graph. |
| | disjoint union of trees. |
| $\frac{c}{n}$ | cycles of any size. |
| $\frac{1}{n}$ | the double jumps. |
| $\frac{c'}{n}$ | one giant component, others are trees. |
| $\frac{\log n}{n}$ | $G(n,p)$ is connected. |
| $\Omega(\frac{\log n}{n})$ | connected and almost regular. |
| $\Omega(n^{\epsilon-1})$ | finite diameter. |
| $\Theta(1)$ | dense graphs, diameter is 2. |
| $1$ | the complete graph. |

# Evolution of $G(n, p)$

**Range I** $p = o(1/n)$

The random graph $G_{n,p}$ is the disjoint union of trees. In fact, trees on $k$ vertices, for $k = 3, 4, \ldots$ only appear when $p$ is of the order $n^{-k/(k-1)}$.

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Furthermore, for $p = cn^{-k/(k-1)}$ and $c > 0$, let $\tau_k(G)$ denote the number of connected components of $G$ formed by trees on $k$ vertices and $\lambda = c^{k-1}k^{k-2}/k!$. Then,

$$\Pr(\tau_k(G_{n,p}) = j) \to \frac{\lambda^j e^{-\lambda}}{j!}$$

for $j = 0, 1, \ldots$ as $n \to \infty$.

# Evolution of $G(n, p)$

**Range II** $p \sim c/n$ for $0 < c < 1$

- In this range of $p$, $G_{n,p}$ contains cycles of any given size with probability tending to a positive limit.

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- All connected components of $G_{n,p}$ are either trees or unicyclic components. Almost all (i.e., $n - o(n)$) vertices are in components which are trees.
- The largest connected component of $G_{n,p}$ is a tree and has about $\frac{1}{\alpha}(\log n - \frac{5}{2}\log\log n)$ vertices, where $\alpha = c - 1 - \log c$.

# Evolution of $G(n, p)$

**Range III** $p \sim 1/n + \mu/n$, the double jump

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- If $\mu = 0$, the largest component has size of order $n^{2/3}$.
- If $\mu > 0$, there is a unique giant component of size $\alpha n$ where $\mu = -\alpha^{-1}\log(1-\alpha) - 1$.
- Bollobás showed that a component of size at least $n^{2/3}$ in $G_{n,p}$ is almost always unique if $p$ exceeds $1/n + 4(\log n)^{1/2} n^{-4/3}$.

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- The total number of vertices in components which are trees is approximately $n - f(c)n + o(n)$.
- The largest connected component of $G_{n,p}$ has approximately $f(c)n$ vertices, where

$$f(c) = 1 - \frac{1}{c}\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}(ce^{-c})^k.$$

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- The graph $G_{n,p}$ almost surely becomes connected.
- If

$$p = \frac{\log n}{kn} + \frac{(k-1)\log\log n}{kn} + \frac{y}{n} + o(\frac{1}{n}),$$

then there are only trees of size at most $k$ except for the giant component. The distribution of the number of trees of $k$ vertices again has a Poisson distribution with mean value $\frac{e^{-ky}}{k \cdot k!}$.

# Evolution of $G(n,p)$

**Range VI** $p \sim \omega(n) \log n / n$ where $\omega(n) \to \infty$.

In this range, $G_{n,p}$ is not only almost surely connected, but the degrees of almost all vertices are asymptotically equal.

# Galton-Watson process

**Galton-Watson branching process:** Let $Z$ be a distribution over the non-negative integers. Starting with a single node, it gives $Z$ children nodes. Each of children nodes have $Z$ children independently. The process continues, each new offspring having an independent number $Z$ of children.

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- $Z_1, Z_2, \ldots, Z_t, \ldots$ : a countable sequence of independent identically distributed variables, each have distribution $Z$.

- $Y_t$: the number of living children at time $t$.

$$Y_0 = 1$$
$$Y_t = Y_{t-1} + Z_t - 1.$$

# Galton-Watson process

Let $T$ be the total number of nodes in Galton-Watson process. There are two essentially different cases.

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- $Y_t > 0$ for all $t \geq 0$. In this case the Calton-Watson process goes on forever and $T = \infty$.
- $Y_t = 0$ for some $t \geq 0$. In this case, $T$ is the least integer for which $Y_T = 0$. The Galton-Watson process stops with $T$ nodes.

# Poisson branching process

Let $Z$ be the Poisson distribution with the expectation $c$.
Write $T = T_c^{po}$.
**Theorem:** If $c \leq 1$, then $T$ is finite with probability one. If $c > 1$, then $T$ is infinite with probability $y = y(c)$, where $y$ is the unique positive real satisfying

$$e^{-cy} = 1 - y.$$

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**Proof:** Suppose $c < 1$.

$$\Pr(T > t) \leq \Pr(\sum_{i=1}^{t} Z_i \geq t) < e^{-kt},$$

for some constant $k$. $\lim_{t\to\infty} \Pr(T > t) = 0$.

# Continue

Suppose $c \geq 1$. Let $z = 1 - y = \Pr(T < \infty)$. Then

$$z = \sum_{i=0}^{\infty} \Pr(Z_1 = i) z^i = \sum_{i=0}^{\infty} e^{-c} \frac{c^i}{z^i} i! = e^{c(z-1)}.$$

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Hence $1 - y = e^{-cy}$. When $c = 1$, this equation has a unique solution $y = 0$. When $c > 1$, there are two solutions $1$ and $y(c)$. By Chernoff's equality, for any $t$

$$\Pr(\sum_{i=1}^{t} Z_i \leq t) < e^{-\frac{(c-1)^2 t}{2c}}.$$

There is a $t_0$ so that $\sum_{t \geq t_0} e^{-\frac{(c-1)^2 t}{2c}} < 1$. Thus,

$$y > \Pr(T = \infty \mid T \geq t_0) \Pr(T \geq t_0) > 0.$$

□

# Graph branching process

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Let $C(v)$ denote the component of $G(n,p)$, containing a vertex $v$. Explore $C(v)$ using Breadth First Search (BFS). In this procedure all vertices will be live, dead, or neutral. The live vertices will be contained in a queue $Q$.

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**Algorithm for computing $C(v)$:**
Push $v$ into $Q$. Mark all vertices but $v$ neutral.
**while**( $Q$ is not empty){
  Pop $Q$ and get $w$, mark $w$ dead
  **foreach**($w'$ neutral){
    **if** ($ww'$ is an edge of $G(n,p)$){
      mark $w'$ live and push it into $Q$
    }
  }
} Return the set of all dead vertices.

# Analysis

In the graph branching process, let $Y_t$ be the size of the queue at time $t$ and $N_t$ be the set of neutral vertices. Let $N_t$ be the set of neutral vertices.

$$Z_t \sim B(N_{t-1}, p).$$

$$N_t \sim B(n-1, (1-p)^t).$$

If $T = t$ it is necessary that $N_t = n - t$. We have

$$\Pr(|C(v)| = t) \leq \Pr(B(n-1, (1-p)^t) = n - t).$$

Or, equivalently,

$$\Pr(|C(v)| = t) \leq \Pr(B(n-1, 1-(1-p)^t) = t - 1).$$

# Comparison

**Theorem:** For any positive real $c$ and any fixed integer $k$

$$\lim_{n\to\infty} \Pr(|C(v)| = k \text{ in } G(n, \frac{c}{n})) = \Pr(T_c^{po} = k).$$

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**Proof:** Let $\Gamma$ be the set of $k$-tuples $\vec{z} = (z_1, z_2, \ldots, z_k)$ of nonnegative integers such that the recursion $y_0 = 1$, $y_t = y_{t-1} + z_t - 1$ has $y_t > 0$ for $t < k$ and $y_k = 0$.

$$\Pr(T^{gr} = k) = \sum \Pr(Z_i^{gr} = z_i, 1 \le i \le k)$$
$$\Pr(T^{po} = k) = \sum \Pr(Z_i^{po} = z_i, 1 \le i \le k).$$

Here both sums are over $\vec{z} \in \Gamma$.

# Continue

Since $Z_{i-1} = n - O(1)$ and $B(Z_i, p)$ approaches the Poisson distribution, we have

$$\lim_{n\to\infty} \Pr(B(N_{i-1}^{gr}, p) = z_i) = \Pr(Z_i^{po} = z_i).$$

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$$\begin{aligned}
\Pr(T^{gr} = k) &= \sum \Pr(Z^{gr}_i = z_i, 1 \leq i \leq k) \\
&= \sum \prod_{i=1}^{k} \Pr(B(N^{gr}_{i-1}, p) = z_i) \\
&\to \sum \prod_{i=1}^{k} \Pr(B(Z^{po}_i = z_i) \\
&= \Pr(T^{po} = k). \qquad \square
\end{aligned}$$

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**Proof:** We have $\Pr(T_c^{po} = k) = \lim_{n\to\infty} \Pr(|C(v)| = k)$ in $G(n,p)$ with $p = c/n$.

$$\begin{aligned} \Pr(C(v) = k) &\approx \binom{n}{k-1} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \\ &\to \frac{e^{-ck}(ck)^{k-1}}{k!}. \qquad \square \end{aligned}$$

# $p = \frac{c}{n}$, $0 \leq c < 1$

With Poisson approximation,

$$\Pr(|C(v)| \geq u) \leq (1+o(1))\Pr(T_c^{po} \geq u) \approx \sum_{k=u}^{\infty} e^{-ck} \frac{(ck)^{k-1}}{k!}.$$

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Thus, the size of largest component in $G(n,p)$ is at most $(c-1-\ln c)^{-1}\ln n + O(\ln\ln n)$.

Most of them are trees. Then number of trees of size $k$ is

$$(1+o(1))e^{-ck}\frac{(ck)^{k-1}}{k!}n.$$

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The size of the largest component approaches $Kn^{2/3}\lambda^{-2}\ln n$.