Topic Course on Probabilistic Methods
(Week 10)
Poisson Paradigm

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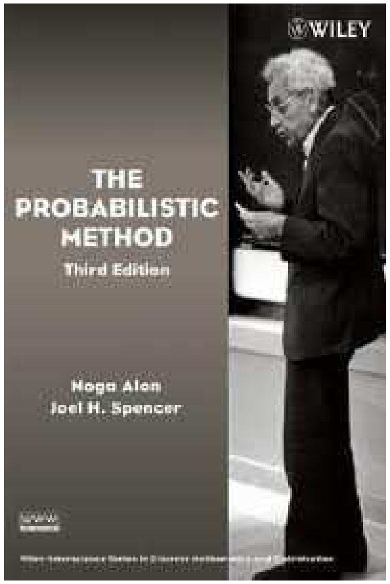
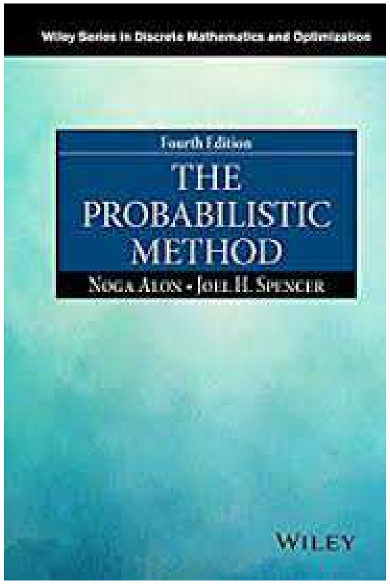
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Poisson paradigm

- Poisson Paradigm
- Janson's inequality
- Brun's sieve
- An application–EPIT
- Large deviation (sparse cases)
- Suen's theorem

Poisson Paradigm

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- Let X_i be the random indicator of the event B_i and $X = \sum_{i\in I} X_i$. If $\Pr(B_i)$'s are small and "mostly independent", then one may expect X follows "Poisson-like distribution". In particular,

$$\Pr(X = 0) \approx e^{-\mathrm{E}(X)}.$$

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- $M = \prod_{i \in I} \Pr(\bar{B}_i)$.

Janson inequality

The Janson inequality: Assume all $\Pr(B_i) \leq \epsilon$. Then

$$M \leq \Pr(\wedge_{i \in I} \bar{B}_i) \leq M e^{\frac{\Delta}{2(1-\epsilon)}},$$

and, further,

$$\Pr(\wedge_{i \in I} \bar{B}_i) \leq e^{-\mu + \frac{\Delta}{2}}.$$

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The Extended Janson inequality: If further $\Delta \geq \mu$, then

$$\Pr(\wedge_{i\in I}\bar{B}_i) \leq e^{\frac{-\mu^2}{2\Delta}}.$$

Proof

Proof given by Boppana and Spencer: We will use the following correlation inequality.

- For all $J \subset I$, $i \notin J$,

$$\Pr(B_i \mid \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i).$$

- For $J \subset I$, $i, k \notin J$,

$$\Pr(B_i \mid B_k \wedge \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).$$

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Order the index set $I = \{1, 2, \ldots, m\}$.

$$\Pr(\wedge_{i \in I} \bar{B}_i) = \prod_{i=1}^{m} \Pr\left(\bar{B}_i \mid \wedge_{1 \leq j < i} \bar{B}_j\right) \geq \prod_{i=1}^{m} \Pr(\bar{B}_i).$$

Continue

For a given i renumber, for convenience, so that $i \sim j$ for $1 \leq j \leq d$ and not for $d+1 \leq j < i$. Let $A = B_i$, $B = \bar{B}_1 \wedge \cdots \wedge \bar{B}_d$, and $C = \bar{B}_{d+1} \wedge \cdots \wedge \bar{B}_{i-1}$,

$$\begin{aligned} \Pr(B_i \mid \wedge_{1 \leq j < i} \bar{B}_j) &= \Pr(A \mid B \wedge C) \\ &\geq \Pr(A \wedge B \mid C) \\ &= \Pr(A \mid C) \Pr(B \mid A \wedge C). \end{aligned}$$

Note $\Pr(A \mid C) = \Pr(A)$ and

$$\Pr(B \mid A \wedge C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i \wedge C) \geq 1 - \sum_{j=1}^{d} \Pr(B_j \mid B_i).$$

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$$\Pr(B_i \mid \wedge_{1\leq j<i}\bar{B}_j) \geq \Pr(B_i) - \textstyle\sum_{j=1}^{d} \Pr(B_j \wedge B_i).$$

Continue

$$
\begin{aligned}
\Pr(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j) &\le \Pr(\bar{B}_i) + \sum_{j=1}^{d} \Pr(B_j \wedge B_i) \\
&\le \Pr(\bar{B}_i)\left(1 + \frac{1}{1-\epsilon}\sum_{j=1}^{d} \Pr(B_j \wedge B_i)\right) \\
&\le \Pr(\bar{B}_i) e^{\frac{1}{1-\epsilon}\sum_{j=1}^{d} \Pr(B_j \wedge B_i)}.
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\end{aligned}$$

Plug it into $\Pr(\wedge_{i\in I}\bar{B}_i) = \prod_{i=1}^{m} \Pr\left(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j\right)$; we get the first inequality. The second inequality use the following estimation.

$$\begin{aligned}
\Pr(\bar{B}_i \mid \wedge_{1\le j<i}\bar{B}_j) &\le \Pr(\bar{B}_i) + \sum_{j=1}^{d} \Pr(B_j \wedge B_i) \\
&\le e^{-\Pr(B_i)+\sum_{j=1}^{d} \Pr(B_j \wedge B_i)}.
\end{aligned}$$

Proof of second Theorem

From the Jansen inequality, we have

$$-\ln(\Pr(\wedge_{i\in I}\bar{B}_i)) \geq \sum_{i\in I}\Pr(B_i) - \frac{1}{2}\sum_{i\sim j}\Pr(B_i \wedge B_j).$$

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For any set $S \subset I$, the same inequality applied to $\{B_i\}_{i\in S}$:

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Now take S be a random subset of I given by $\Pr(i \in S) = p$, and take the expectation.

$$E\left[-\ln(\Pr(\wedge_{i\in S}\bar{B}_i))\right] \geq p\mu - p^2\frac{\Delta}{2}.$$

Continue

Now choose $p = \mu/\Delta$.

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Brun's sieve

- X_i: the indicator random variable for B_i, for $i \in I$.
- $X := \sum_{i=1}^{m} X_i$.
- $m = m(n)$, $B_i = B_i(n)$, and $X = X(n)$.
- Let
$$S^{(r)} = \sum \Pr(B_{i_1} \wedge \cdots \wedge B_{i_r}),$$
where the sum is over all sets $\{i_1, \ldots, i_r\} \subset \{1, 2 \ldots, m\}$.
- Let
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By inclusion-exclusion principle,

$$\Pr(X=0) = \Pr(\bar{B}_1 \wedge \cdots \wedge \bar{B}_m) = \sum_{r\geq 0} (-1)^r S^{(r)}.$$

Brun's sieve

Theorem: Suppose there is a constant μ so that for every fixed r,

$$\mathrm{E}\binom{X}{r} = S^{(r)} \to \frac{\mu^r}{r!}.$$

Then

$$\Pr(X=0) \to e^{-\mu},$$

and for every t

$$\Pr(X=t) \to \frac{\mu^t}{t!}e^{-\mu}.$$

Proof

Proof: We only prove the case $t = 0$. Fix $\epsilon > 0$. Choose s so that

$$\left|\sum_{r=0}^{2s}(-1)^r\frac{\mu^r}{r!} - e^{-\mu}\right| \leq \frac{\epsilon}{2}.$$

Select n_0 so that for $n \geq n_0$,

$$|S^{(r)} - \frac{\mu^r}{r!}| \leq \frac{\epsilon}{2s(2s+1)}$$

for $0 \leq r \leq 2s$.

Continue

For such n,

$$\begin{aligned}
\Pr[X=0] &\leq \sum_{r=0}^{2s}(-1)^r S^{(r)} \\
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\end{aligned}$$

Similarly, taking the sum to $2s+1$, we can find n_0 so that for $n \geq n_0$,

$$\Pr[X=0] \geq e^{-\mu} - \epsilon.$$

As ϵ was arbitrary $\Pr(X=0) \to e^{-\mu}$. $\square$

An application

Let $G = G(n,p)$, and EPIT represent the statement that every vertex lies in a triangle.

Theorem (a special case of Spencer's Theorem): Let $c > 0$ be fixed and let $p = p(n)$, $\mu = \mu(n)$ satisfy

$$\begin{aligned} \binom{n-1}{2} p^3 &= \mu, \\ e^{-\mu} &= \frac{c}{n}. \end{aligned}$$

Then

$$\lim_{n\to\infty} Pr(G(n,p) \text{ satisfies } EPIT) = e^{-c}.$$

Proof

First fix $x \in V(G)$. For each unordered $y, z \neq x$ let B_{xyz} be the event that $\{x, y, z\}$ is a triangle of G. Let C_x be the event $\wedge_{y,z} \bar{B}_{xyz}$ and X_x the corresponding indicator random variable. Apply Janson's Inequality to bound $\mathrm{E}(X_x) = \Pr(\wedge_{y,z} \bar{B}_{xyz})$.

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$$\Delta = \sum_{y,z,z'} \Pr(B_{xyz} \wedge B_{xyz'}) = O(n^3 p^5) = o(1)$$

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since $p = n^{-2/3+o(1)}$. Thus

$$\mathrm{E}(X_x) \approx e^{-\mu} = \frac{c}{n}.$$

continue

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$$\mathrm{E}(X) = \sum_x \mathrm{E}(X_x) \to c.$$

We need to show that the Poisson Paradigm applies to X.

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where the sum is over all sets $\{x_1, \ldots, x_r\}$.

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where the sum is over all sets $\{x_1, \ldots, x_r\}$. Note

$$C_{x_1} \wedge \cdots \wedge C_{x_r} = \wedge_{1 \le i \le r, y, z} \overline{B_{x_i y z}}.$$

Continue

We apply Janson's Inequality again.

$$\sum \Pr(B_{x_i yz}) = p^3 \left(r\binom{n-1}{2} - O(n) \right) = r\mu + O(n^{-1+o(1)}).$$

As before Δ is p^5 times the number of pairs $x_i yz \sim x_j yz$; $\Delta = O(n^3 p^5) = o(1)$.

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$$\Pr(C_{x_1} \wedge \cdots \wedge C_{x_r}) \sim e^{-r\mu}$$

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Applying Brun's Sieve method, we have $\Pr(X = 0) \to e^{-c}$.

Large deviations

Let $X = \sum_{i,j} t_{xi} t_{xj} t_{ij}$ be the number of triangles containing x in $G(n,p)$. Let $\mu = \mathrm{E}(X)$.

- Kim-Vu's inequality implies " if $\mu \gg \ln^6 n$, then with probability $1 - o(1)$, $(1-\epsilon)\mu \le X \le (1+\epsilon)\mu$."

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- Kim-Vu's inequality implies " if $\mu \gg \ln^6 n$, then with probability $1 - o(1)$, $(1-\epsilon)\mu \leq X \leq (1+\epsilon)\mu$."
- With disjoint family (of Poisson paradigm), one can lower the condition to $\mu \gg \ln n$.

Disjoint family

For a fixed random set R, an index $J \subseteq I$ is a **disjoint family** (disfam) if

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Lemma 8.4.1:
$\Pr(\text{there exists a maxdisfam } J, |J| = s) \leq \frac{\mu^s}{s!}$.

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$\Pr(\text{there exists a maxdisfam } J, |J| = s) \leq \frac{\mu^s}{s!}$.

Lemma 8.4.2: Let $\nu = \max_{j \in I} \sum_{i \sim j} \Pr(B_i)$. Then
$\Pr(\text{there exists a maxdisfam } J, |J| = s) \leq \frac{\mu^s}{s!} e^{-\mu} e^{s\nu} e^{\Delta/2}$.

Proof of Lemma 8.4.1

Let $\sum^*$ denote the sum over all s-sets $J \subseteq I$ with no $j \sim j'$.
Let $\sum^o$ denote the sum over ordered distinct s-tuples. Then

$$\begin{aligned}
&\Pr(\text{there exists a maxdisfam } J, |J| = s) \\
&\leq \sum\nolimits^* \Pr\left(\wedge_{j\in J} B_j\right) \\
&= \sum\nolimits^* \prod_{j\in J} \Pr(B_j) \leq \frac{1}{s!} \sum\nolimits^o \Pr(B_{j_1}) \cdots \Pr(B_{j_s}) \\
&\leq \frac{1}{s!} \left(\sum_{i\in I} \Pr(B_i) \right)^s = \frac{\mu^s}{s!}.
\end{aligned}$$

Proof of Lemma 8.4.2

Let μ_s denote the minimum, over all $j_1, \ldots, j_s \in I$ of $\sum \Pr(B_i)$, the sum over all $i \in I$ except those i with $i \sim j_l$ for some $1 \leq l \leq s$. We have $\mu_s \geq \mu - s\nu$.

$$\Pr(J \text{ maxdisfam}) = \Pr(J \text{ disfam})\Pr(\wedge^* \bar{B}_i).$$

Applying Janson's inequality, we get

$$\Pr(\wedge^* \bar{B}_i) \leq e^{-\mu_s} e^{\Delta/2}.$$

$$\begin{aligned} \sum{}^* \Pr(J \text{ maxdisfam}) &\leq e^{-\mu_s} e^{\Delta/2} \sum{}^* \Pr(J \text{ disfam}) \\ &\leq \frac{\mu^s}{s!} e^{-\mu_s} e^{\Delta/2} \leq \frac{\mu^s}{s!} e^{-\mu} e^{s\nu} e^{\Delta/2}. \end{aligned}$$

Conclusion

Let P be the Poisson distribution with mean μ. When $\mu \gg \ln n$, $\Delta = o(1)$, and $\nu\mu = o(1)$, then

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$$\Pr(\text{there exists a maxdisfam } J, |J| \le \mu(1-\epsilon)) \\ \le (1+o(1))\Pr(P \le \mu(1-\epsilon));$$

$$\Pr(\text{there exists a maxdisfam } J, \mu(1+\epsilon) \le |J| \le 3\mu) \\ \le (1+o(1))\Pr((1+\epsilon)\mu \le P \le 3\mu);$$

$$\Pr(\text{there exists a maxdisfam } J, |J| \ge 3\mu) \le \sum_{s=3\mu}^{\infty} \frac{\mu^s}{s!} = o(n^{-1}).$$

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$$\begin{aligned}
&\Pr(\text{there exists a maxdisfam } J, |J| \leq \mu(1-\epsilon)) \\
&\qquad \leq (1+o(1))\Pr(P \leq \mu(1-\epsilon)); \\
&\Pr(\text{there exists a maxdisfam } J, \mu(1+\epsilon) \leq |J| \leq 3\mu) \\
&\qquad \leq (1+o(1))\Pr((1+\epsilon)\mu \leq P \leq 3\mu);
\end{aligned}$$

$$\Pr(\text{there exists a maxdisfam } J, |J| \geq 3\mu) \leq \sum_{s=3\mu}^{\infty} \frac{\mu^s}{s!} = o(n^{-1}).$$

With probability $1 - o(n^{-1})$, every maxdisfam J has size between $(1-\epsilon)\mu$ and $(1+\epsilon)\mu$.

Application

Let X be the number of triangles containing x in $G(n, p)$.
Let $\mu = \mathrm{E}(X) \sim \frac{1}{2}n^2p^3$. Assume $\mu \gg \ln n$.

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We have $\Delta \leq n^3p^5 = o(1)$ and $\mu\nu = n^3p^5 = o(1)$. Thus, with probability $1 - o(n^{-1})$, every maxdisfam J has size between $(1-\epsilon)\mu$ and $(1+\epsilon)\mu$.

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Construct a graph $H = (V, E)$ with $V = \{$ all triangles containing $x\}$ and two triangles is adjacent if they share an edge. The with probability $1 - o(1)$, each vertex xyz has degree at most 9 and no set of four disjoint edges. This implies, for any J, $|J| \geq X - 27$. Thus,

$$X \leq (1+\epsilon)\mu + 27 \leq (1+\epsilon')\mu.$$

Generalization

A sufficient condition for Janson's Inequality:

- I: a dependency digraph; if for each $i \in I$ the event B_i is mutually independent of $\{B_j : i \not\sim j\}$.
- $\Delta := \sum_{i \sim j} \Pr(B_i \wedge B_j)$.
- For all $J \subset I$, $i \notin J$,

$$\Pr(B_i \mid \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i).$$

- For $J \subset I$, $i, k \notin J$,

$$\Pr(B_i \mid B_k \wedge \wedge_{j \in J} \bar{B}_j) \leq \Pr(B_i \mid B_k).$$

Then Janson's inequality holds.

Suen's theorem

An binary relation $\sim$ on I is **superdenpendency digraph** if the following holds:
Suppose that $J_1, J_2 \subset I$ are disjoint subsets so that there is no edge between J_1 and J_2. Let B^1 be any Boolean combination of the events $\{B_j\}_{j \in J_1}$ and B^2 be any Boolean combination of the events $\{B_j\}_{j \in J_2}$. Then B^1 and B^2 are independent.

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Theorem [Suen]: Under the above conditions,

$$\left|\Pr(\wedge_{i\in I}\bar{B}_i) - M\right| \leq M(e^{\sum_{i\sim j} y(i,j)} - 1),$$

where
$y_{i,j} = (\Pr(B_i \wedge B_j) + \Pr(B_i)\Pr(B_j)) \prod_{l\sim i \text{ or } l\sim j}(1 - \Pr(B_l))^{-1}.$