Math777: Graph Theory (II)
Spring, 2018
Homework 4, Solution

1. **[page 305, #1**] An oriented complete graph is called a *tournament.* Show that every tournament contains a (directed) Hamilton path.

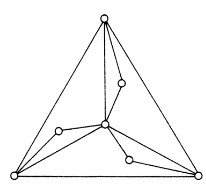
   Proof: Let $P = v_1 v_2 \cdots v_k$ be a longest directed path. If P contains all vertices, then we are done. Otherwise, there is a vertex u not on the path P. Since P can not be extended, we must have edge directions $v_1 \to u$ and $u \to V_k$. There is an index i so the directions are changing: $v_i \to u$ and $u \to v_{i+1}$. But now $v_1 \cdots v_i u v_{i+1} \cdots v_k$ is a longer path. Contradiction.

2. **[page 305, #2**] Show that every uniquely 3-edge-colorable cubic graph is hamiltonian. ("Unique" means that all 3-edge-colorings induce the same edge partition.)

   Proof: Suppose G is a uniquely 3-edge-colorable cubic graph. Let $E(G) = E_1 \cup E_2 \cup E_3$ are the color partition. Note that each E_i is a perfect matching. Thus $E_1 \cup E_2$ is a disjoint union of even cycles. If $E_1 \cup E_2$ consists of only one cycle, then this cycle is the Hamilton cycle. Otherwise, say $E_1 \cup E_2$ contains more than one cycle. Say one of the cycles is C, we can exchanges the edges in C: Let $E_1' = (E_2 \cap E(C)) \cup (E_1 - E(C))$ and $E_2' = (E_1 \cap E(C)) \cup (E_2 - E(C))$. Then $E_1' \cup E_2' \cup E_3$ is another color partition. Contradiction.

3. **[page 305, #5**] Find a graph that is 1-tough but not hamiltonian.

   Solution: The following graph is 1-tough but not Hamiltonian.



4. **[page 306, #7**] Find a hamiltonian graph whose degree sequence is not hamiltonian.

   Solution: C_n for $n \geq 6$. $(2, 2, \ldots, 2)$ is not hamiltonian.

5. **[page 306, #9**] Prove that the square G^2 of a k-connected graph G is k-tough.

   Proof: For any set S, if $G^2 - S$ has $t \geq 2$ components, say $C_1, C_2, \ldots, C_t$, then we need show that $|S| \geq kt$. Let $\Gamma_G(C_i)$ be the vertex boundary of C_i in G, then $\Gamma_G(C_i) \subset S$. Since each pair C_i, C_j are at least distance 3 away in G. These sets are disjoint. Furthermore, since G is k-connected. we have

$$|\Gamma_G(C_i)| \geq k.$$

Thus $|S| \geq kt$. This proves that G^2 is k-tough.

6. [page 306, #11] Find a connected graph G whose square G^2 has no Hamilton cycle.

Solution: Let G be the graph obtained by subdividing each edge of S_4 once. Then G^2 is the graph shown in Problem 3, which is not Hamiltonian.