Math777: Graph Theory (II)
Spring, 2018
Homework 3, solution

1. [**page 289, #10**] Prove the following result of Schur: for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for every partition of $\{1, 2, \ldots, n\}$ into k sets, at least one of the subsets contains numbers x, y, z such that $x + y = z$.

Proof: Choose $n = R_k(3)$, the Ramsey number such that any k-coloring of K_n contains a monochromatic triangle. Given a color $c\colon [n] \to [k]$, we construct an edge coloring of K_n: color edge ij by $c(|i - j|)$. By the Ramsey theorem for graphs, there is a monochromatic triangle $\{i, j, k\}$; assume $i < j < k$. Then we set $x = ji$, $y = kj$ and $z = ki$. We have $c(x) = c(y) = c(z)$ and $x + y = z$.

2. [**page 289, #11**] A family of sets is called a Δ-system if every two of the sets have the same intersection. Show that every infinite family of sets of the same finite cardinality contains an infinite Δ-system.

Proof: This is the **Erdos-Rado**'s theorem: There is a function $f(k, r)$ so that every family $\mathcal{F}$ of k-sets with more than $f(k, r)$ members contains a Δ-system of size r.

Let $\mathcal{F}$ be a family of k-sets without a Δ-system of size r. Let $A_1, A_2, \ldots, A_t$ be a maximum subfamily of pairwise disjoint sets in $\mathcal{F}$. Since a family of pairwise disjoint sets is a Δ-system, we must have $t < r$. Now let $A = \cup_{i=1}^{t} A_i$. For every $a \in A$ consider the family $\mathcal{F}_a = \{S \backslash \{a\} : S \in \mathcal{F}, a \in S\}$. Now, the size of A is at most $(r-1)k$ and the size of each $\mathcal{F}_a$ is at most $f(k-1, r)$. We get that $f(k, r) \leq (r-1)kf(k-1, r)$. This gives $f(k, r) \leq (r-1)^k \times k!$.

3. [**page 290, #14**] Prove that $2^c < R(2, c, 3) \leq 3c!$ for every $c \in \mathbb{N}$.

Proof: Lower bound: let $n = 2^c$ and consider the c-coloring of K_n so that an edge ij receives the color l if $2^{l-1} \leq |i - j| < 2^l$. Since each interval $[2^{l-1}, 2^l - 1]$ contains no triple $x < y < z$ so that $x + y = z$. There is no monochromatic triangle in this coloring. Thus, $R(2, c, 3) > 2^c$.

Upper bound: For each vertex v and a fixed color i, the neighbors of v in the color i can have at most $R(2, c - 1, 3)$ vertices. Thus we have a recursive formula:

$$R(2, c, 3) \leq cR(2, c - 1, 3).$$

Since $R(2, 1, 3) = 3$, we have

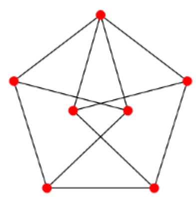
$$R(2, c, 3) \leq 3c!.$$

4. [**page 290, #18**] Show that any Kuratowski set $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of a given collection $\mathcal{C}$ of non-trivial graph properties is unique up to equivalence.

Proof: Let $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_l\}$ be another Kuratowski set. Assume Q_l is a minimum element under the partial ordering. By the definition, there is a P_i with $P_i \leq Q_l$. For this P_i, there is a Q_j such that $Q_j \leq P_i$. We get $Q_j \leq P_i \leq Q_1$. Since Q_1 is minimum, we must have $j = 1$. Thus P_i and Q_1 are equivalent. Delete these two elements from the two sets the do the induction. We conclude that $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ are equivalent to $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_l\}$.

5. Let us 3-color the points of the plane. Prove that there will be two points at distance 1 with the same color.

Proof: The Moser Spindle graph G is the 7-node unit-distance graph shown below:



It is know that $\chi(G) = 4$. Thus any 3-color of the 7-nodes contains a monochromatic edge, which has distance 1 in the plane.

6. Let us k-color all non-empty subsets of an n-element set. Prove that if n is large enough, there are two disjoint non-empty subsets X and Y such that X, Y, $X \cup Y$ have the same color.

Proof: Let $n = R(2, k, 3)$. Assume that all non-empty subsets of $[n]$ are k-colored with colors $1, 2, \ldots, k$. Now we construct a k-edge coloring of K_n. Color each edge ij by the color of the interval $[i, j-1]$. By the definition of Ramsey number $R(2, k, 3)$, there is a monochromatic triangle ijl with $i < j < l$. Let $X = [i, j-1]$, $Y = [j, l-1]$, and $X \cup Y = [i, l-1]$. These three sets are in the same color.