Math777: Graph Theory (II)
Homework 2 Solutions

1. [page 195, #4] Determine the value of $ex(n, K_{1,r})$ for all $r, n \in \mathbb{N}$.

Solution: We would like to determine how many edges a graph G on n vertices can have before a $K_{1,r}$ subgraph is forced. Note that G has a $K_{1,r}$ subgraph if and only if $\Delta(G) \geq r$. Thus, we must determine the maximum number of edges G can have and still maintain $\Delta(G) < r$. We consider two cases:

Case 1: $n \leq r$. Clearly, if $n \leq r$, we must have $\Delta(G) < r$. In this case, a complete graph on n vertices has no $K_{1,r}$ subgraph and $ex(n, K_{1,r}) = \binom{n}{2}$.

Case 2: $n > r$. In this case, we try to draw edges on the vertices of G so that $d(v) = r - 1$ for all $v \in V(G)$. While it may not be possible to draw an $r-1$-regular graph on n vertices, we are able to draw $\lfloor \frac{(r-1)\cdot n}{2} \rfloor$ edges. Thus $ex(n, K_{1,r}) = \lfloor \frac{(r-1)\cdot n}{2} \rfloor$.

2. [page 195, #5] Given $k > 0$, determine the extremal graphs without a matching of size k.

Solution: Let $n \in \mathbb{N}$ and $k > 0$. We will consider two cases:

Suppose $n < 2k$. The complete graph K_{2n} will certainly have no matching of size k. This graph K_{2n} has $\binom{n}{2}$ edges, and we certainly cannot better. Thus K_{2n} is the extremal graph in this case.

Suppose $n \geq 2k$. We construct the extremal graph in the following way: First, construct a K_{k-1}. Then, draw an edge from each of the remaining $(n - k + 1)$ vertices to each of the edges in the K_{k-1}. Then every edge in a maximal matching will be incident to a vertex in the K_{k-1}. Thus, the size of the maximal matching on this graph is $k - 1$. This graph has $\binom{k-1}{2} + (n - k + 1)(k - 1)$ edges, and it is the extremal graph.

3. [page 195, #9] Show that deleting at most $(m - s)(n - t)/s$ edges from a $K_{m,n}$ will never destroy all its $K_{s,t}$ subgraphs.

Solution: Let $M \cup N$ be the partition of the graph G obtained from $K_{m,n}$ by deleting these vertices. On the average, a vertex in M is losing $(m-s)(n-t)/(sm)$ edges. Picking a set S of s vertices with most degrees from M. Consider the induced subgraph $G[S \cup N]$. We have

$$|E(G[S\cup N])| \geq s(n-(m-s)(n-t)/(sm)) = (s-1)n+t-\frac{s}{m}(n-t) > (s-1)n+t.$$

Thus in $G[S \cup N]$, there are a set T of t vertices from N with degree equal s. The induced subgraph $G[S \cup T]$ is a complete bipartite graph $K_{s,t}$.

4. **[page 195, #11**] Let $1 \leq r \leq n$ be integers. Let G be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$, and assume that $K_{r,r} \not\subset G$.

Show that

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1)\binom{n}{r}.$$

Use it to deduce $ex(n, K_{r,r}) \leq cn^{2-1/r}$.

Solution: Let $1 \leq r \leq n$ be integers. Let G be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$. Assume $K_{r,r} \not\subset G$. Let $d(x)$ denote the degree of vertex $x \in A$, and let $N(x)$ denote the neighborhood of x. Note that $N(x)$ contains $\binom{d(x)}{r}$ r-tuples of vertices. If we take the sum of all such r-tuples over the neighborhoods of all $x \in A$, we get $\sum_{x \in A} \binom{d(x)}{r}$. Note that any r-tuple can be counted at most $r-1$ times. Otherwise, we would get a $K_{r,r}$ subgraph. Thus,

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1)\binom{n}{r}.$$

Due to the convexity of $\binom{d(x)}{r}$ (for $d(x) > r-1$), $\sum_{x \in A} \binom{d(x)}{r}$ is minimized if the degrees of $x \in A$ are as even as possible. Thus,

$$\sum_{x \in A} \binom{d(x)}{r} \geq n \cdot \binom{|E(G)|/n}{r} \geq n \cdot \frac{(|E(G)|/n - r)^r}{r!}$$

Also,

$$(r-1)\binom{n}{r} \leq (r-1)\frac{n^r}{r!}$$

Therefore,

$$n \cdot \frac{(|E(G)|/n - r)^r}{r!} \leq (r-1)\frac{n^r}{r!}$$

If we solve this for $|E(G)|$, we conclude that $ex(n, K_{r,r}) \leq cn^{2-1/r}$.

5. **[page 196, #11**] Given a graph G with $\epsilon(G) \geq k \in \mathbb{N}$, find a minor $H \prec G$ such that $\delta(H) \geq k \geq |H|/2$.

Solution: Let $k = 1$. Let G be a graph with $\epsilon(G) \geq 1 = k$. That means G has at least one edge. If we let H be a path P_1, H is certainly a minor of G, and $\delta(H) = k = |H|/2 = 1$.

We proceed by induction. Let $n \in \mathbb{N}$. Suppose for all $k \leq n-1$, for every graph G with $\epsilon(G) \geq k$, we can find a minor $H \prec G$ such that $\delta(H) \geq k \geq |H|/2$. Let G be a graph with $\epsilon(G) \geq n$. Pick the minimal minor $H \prec G$ such that $\delta(H) \geq k$, and let $x \in H$. Let us create a new graph H' from H by removing x. Since $\delta(H) \geq k$, x is not isolated, and the neighbors of x will have degree at least $k-1$ when x is removed. Since $\epsilon(H') \geq k-1$, by the inductive hypothesis, we can find a minor H'' of H' that satisfies $\delta(H'') \geq k-1 \geq |H''|/2$. When we add x back in, we still get $\delta(H) \geq k$. Since we are adding only one vertex, $|H''|$ goes up by at most $\frac{1}{2}$, so $k \geq |H|/2$.

6. If a graph G_n contains no K_4 and only contains $o(n)$ independent vertices, then $||G_n|| < (\frac{1}{8} + o(1))n^2$. (Hint: apply Szemerédi's Regularity Lemma.)

Solution: For any $\epsilon > 0$, we apply Szemerédi's Regularity Lemma to G to get a regularity partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$. We define an auxiliary graph R with the vertex set $\{V_1, \ldots, V_k\}$. A pair (V_i, V_j) forms an edge of R if it is a regular pair with edge density at least 3ϵ. We claim:

Claim a: No regularity pair has density $d > \frac{1}{2} + 2\epsilon$.

Claim b: R is triangle-free.

Proof of Claim a: If a regular pair (V_i, V_j) has density $d > \frac{1}{2} + 2\epsilon$. We claim that we can find a K_4 in G. Call a vertex $v \in V_i$ is good if for any $B \subset V_j$ with $|B| > \epsilon |V_j|$, v has at least $(d - \epsilon)$ neighbors in B. All vertices in V_i but a ϵ-fraction are good. Since the independent number of G is $o(n)$, there is an edge xy in V_i so that both x and y are good. This implies $|N(x) \cap N(y) \cap V_j| > (d-\epsilon)^2 |V_j|$. Thus inside $N(x) \cap N(y) \cap V_j$ contains an edge st. The induced graph on $\{x, y, s, t\}$ is a K_4. Contradiction.

Proof of Claim b: Suppose that R contains a triangle $V_i V_j V_s$. We can define a vertex $v \in V_i$ is good in a similarly way. At least $(1-2\epsilon)V_i$ vertices are good. Pick an edge xy so that both x and y are good in V_i. Consider $N(x) \cap N(y) \cap V_j$ and $N(x) \cap N(y) \cap V_s$. Both sets have size at least $(d-\epsilon)^2 |V_i|$. Thus we can select an edge zw so that $z \in N(x) \cap N(y) \cap V_j$ and $w \in N(x) \cap N(y) \cap V_s$. Once again, we found a K_4. Contradiction.

Let $l = |V_i| \approx \frac{n}{k}$. Since R is triangle-free, R has at most $k^2/4$ edges. The total number of edge in G can be bounded by

$$\begin{aligned} ||G_n|| &\leq ||R||(\frac{1}{2} + 2\epsilon)l^2 + \left(\binom{k}{2} - ||R|| \right) 3\epsilon l^2 + \epsilon k^2 l^2 + \epsilon k l^2 \\ &\leq (\frac{1}{8} + 20\epsilon)n^2. \end{aligned}$$

Now let $\epsilon \to 0$, we have $||G_n|| < (\frac{1}{8} + o(1))n^2$.