Math777: Graph Theory (II)
Spring, 2018
Homework 1 Solutions

1. **[page 165, #6**] Let H be an abelian group, $G = (V, E)$ a connected graph, T a spanning tree, and f a map from the orientations of the edges in $E - E(T)$ to H that satisfies (F1). Show that f extends uniquely to a circulation on G with values in H.

Solution: Designate a vertex r as a root and orient edges on T from the root toward leaves. Pick a non-root leaf vertex of T, called v_n, and the edge connecting to v_n in T is called e_{n-1}. Let $\vec{e}_{n-1}$ be the orientation received from the root toward leaves. Define

$$f(\vec{e}_{n-1}) = -\sum_{\vec{e}\in\vec{E}(v_n,V),e\neq e_{n-1}} f(\vec{e}).$$

We continue this process to assign values on all edges of T. At i-th iteration, let T_i be the remaining tree, v_i be a non-root leave of T_i, and e_{i-1} be the edge of T_i connecting to v_i. We define

$$f(\vec{e}_{i-1}) = -\sum_{\vec{e}\in\vec{E}(v_i,V),e\neq e_{i-1}} f(\vec{e}).$$

Note that the right expression does not contain any edge e_j with $j < i$. Thus $f(\vec{e}_{i-1})$ is well-defined. When the process is finished, we define the values on all edges of T. For $i \geq 2$, $f(v_i, V) = 1$. We need to show that $f(v_1, V) = 0$ as well. This is because

$$f(v_1, V) = f(V, V) - \sum_{i=2}^{n} f(v_i, V) = 0.$$

Here $f(V, V) = 0$ since f satisfies (F1).

2. **[page 166, #15**] Show that every graph with a Hamilton cycle has a 4-flow.

Solution: Assume that graph H has a Hamilton cycle C. We only need to prove that H has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Assign $(0, 1)$ to each edge not in C. Since it is over the field $\mathbb{Z}_2$, the orientation of edges do not matter. By Problem 1, we can extend this assignment to a circulation of H. The only problem is that some edges on C might receive 0 value. But this can be fixed by adding $(1, 0)$ to each edge of C. Finally we get a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-circulation on H. By Tutte's theorem, H admits a 4-flow.

3. **[page 166, #17**] Determine the flow number of $C_5 * K_1$, the wheel with 5 spokes.

Solution: Note that the 5-wheel is self-dual planar graph. Thus $\phi(C_5 * K_1) = \chi(C_5 * K_1)$. Since $\chi(C_5) = 3$, we get $\chi(C_5 * K_1) = 4$. Thus, $\phi(C_5 * K_1) = 4$.

4. **[page 166, #18**] Find bridgeless graph G and $H = G - e$ such that $2 < \varphi(G) < \varphi(H)$.

   Solution: Let G be the graph obtained by connecting a pair of non-adjacent vertices in the wheel $C_5 * K_1$. Call this edge e. Then $H = G - e = C_5 * K_1$. Note that G is still planar graph. One can easy to show the dual of G has chromatic number 3. Thus $\phi(G) = 3$. From the problem 3, we have shown that $\phi(H) = 4$.

5. **[page 166, #20**] Prove Heawood's theorem that a plane triangulation is 3-colorable if and only if all its vertices have even degree.

   Solution: Let G be a plane triangulation. Then the dual graph G^* is 3-regular. G is 3-colorable if and only if G^* has a 3-flow. By Proposition 6.4.2, a cubic graph has a 3-flow if and only if it is bipartite. It is equivalent to say G^* is bipartite, or G has a 2-flow. It is necessary and sufficient that all its degrees of G have even degree.

6. **[page 167, #23**] Show that a graph $G = (V, E)$ has a k-flow if and only if it admits an orientation D that directs, for every $X \subset V$, at least $\frac{1}{k}$ of the edges in $E(X, \bar{X})$ from X towards $\bar{X}$.

   Solution: One direction is easy. Suppose that G has a k-flow f. One may orient an edge e so that $f(\vec{e}) > 0$. For any partition of $C = X \cup \bar{X}$. We have $f(X, \bar{X}) = 0$. Let a be the number of edges oriented from X to $\bar{X}$ and a be the number of edges oriented from $\bar{X}$ to X. We have

$$0 = f(X, \bar{X}) \leq (k-1)a - b.$$

   This implies that $a \geq \frac{1}{k}(a+b) = \frac{1}{k}|E(X, \bar{X})|$.

   For the other direction, we need to prove a stronger statement for induction. Suppose that G admits an orientation D that directs, for every $X \subset V$, at least $\frac{1}{k}$ of the edges in $E(X, \bar{X})$ from X towards $\bar{X}$. We assign each directed edge $\vec{e}$ a capacity $c_{\vec{e}}$. Initially all capacities are set to be $k-1$. We say the capacity $\{c_{\vec{e}}\}$ is *balanced*, if for any vertex partition $V = X \cup \bar{X}$, we have

$$\sum_{\vec{e} \in \vec{E}_D(X, \bar{X})} c_{\vec{e}} \geq |\vec{E}_D(\bar{X}, X)|.$$

   The property "for every $X \subset V$, at least $\frac{1}{k}$ of the edges in $E(X, \bar{X})$ from X towards $\bar{X}$" implies that the initial capacity is balanced.

   Claim: If D has a balanced capacity $\{c_{\vec{e}}\}$, then G has an integer flow f so that for each directed edge $\vec{e}$, $1 \leq f(\vec{e}) \leq c_{\vec{e}}$.

We will prove the Claim by induction on the sum of all capacities. It is trivial if G is an empty graph, otherwise, we assume the claim holds for smaller graphs or smaller capacities. Note that if G has a balanced capacity then every non-isolated vertex has positive indegree and outdegree. G must contain a directed cycle C. Now we push a flow f_C on C by 1 unit along the direction of C and decreasing each capacity of the edge of C by 1. If some edge has 0 capacity, delete that edge. Note that the balanced property still holds after updating the capacity. By induction, the new graph and new capacity has a flow g. Now let $f = g + f_C$. This is the k-flow of G.