Math576: Combinatorial Game Theory

Lecture note I

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Artificial Intelligence

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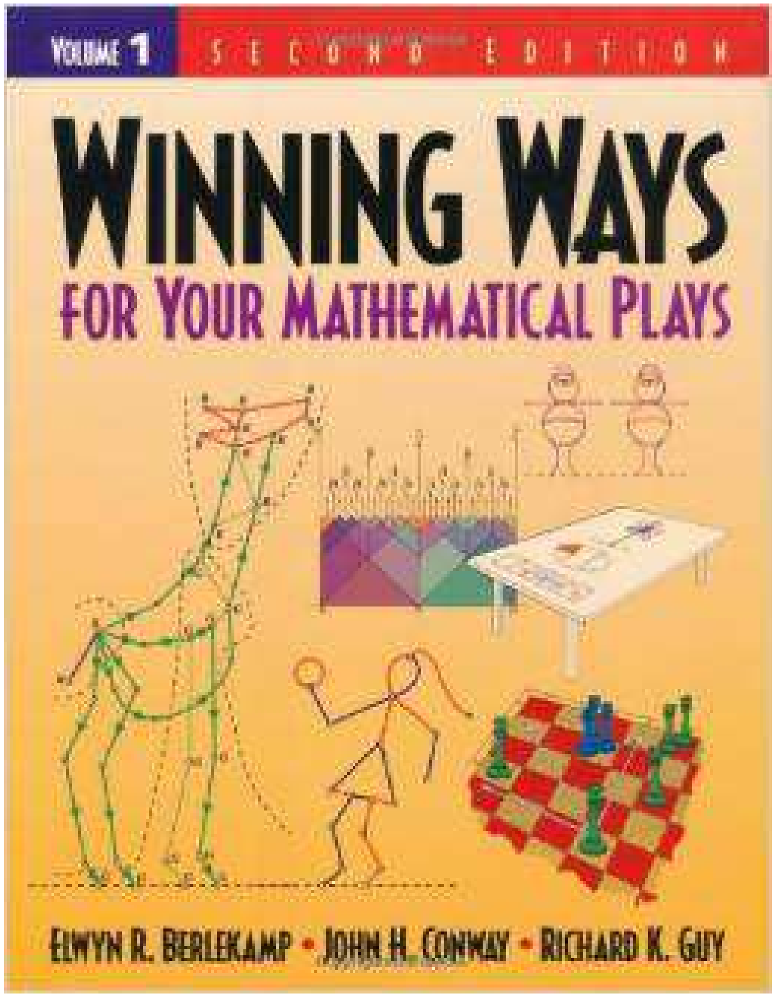
Will AI outsmart human being? How soon? How to play games smarter?

About the course

Textbook:



Course Material

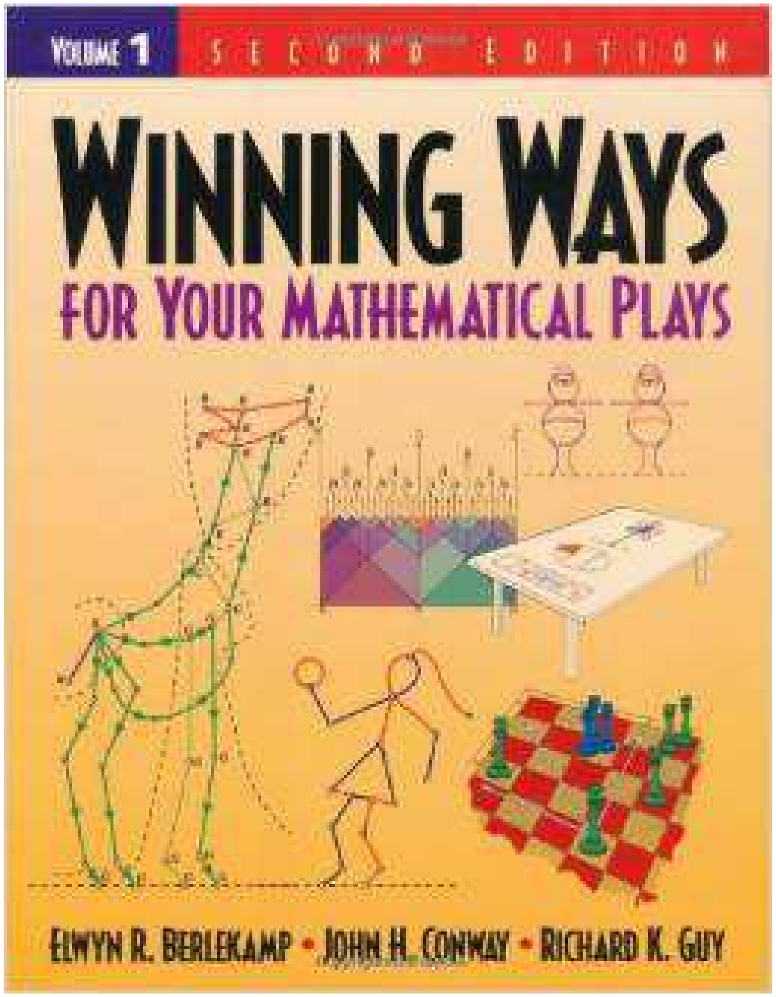
- Chapter 1-5, part of Chapter 7.
- Conways' Game of Life
- Puzzles

Assessment

- Homework
- Two midterm exam
- Final project

Disclaimer

The slides are solely for the convenience of the students who are taking this course. The students should buy the textbook. The copyright of many figures in the slides belong to the authors of the textbook: **Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy.**

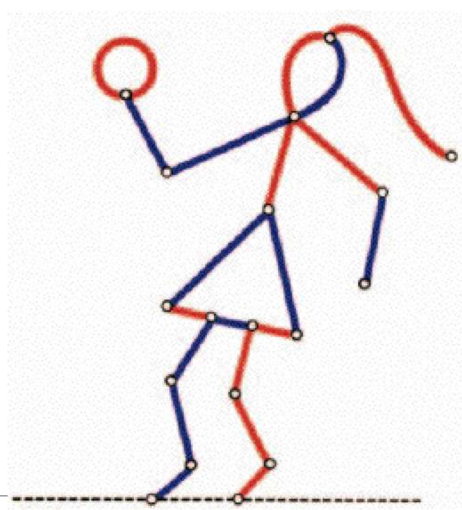


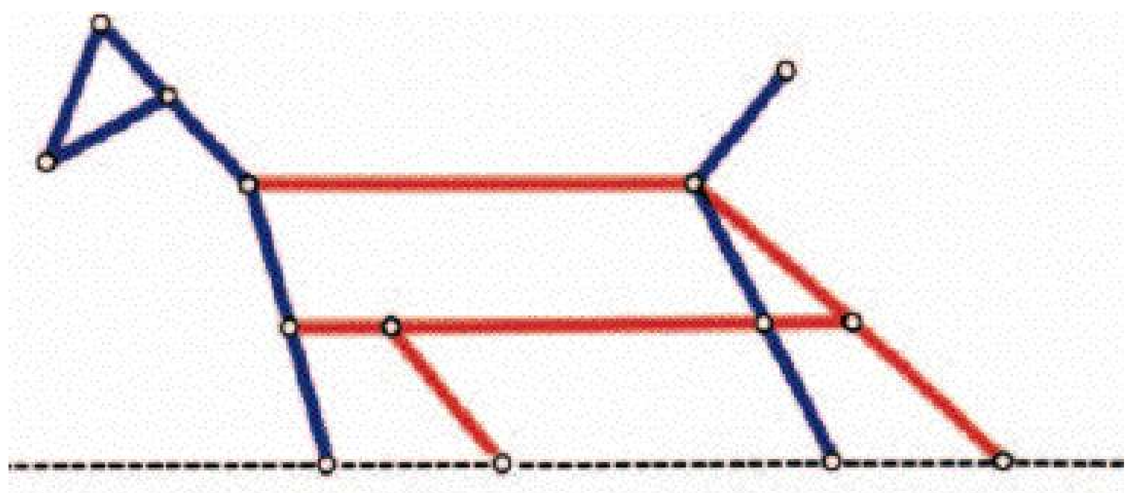
What Games?

- Number of players?
- Type of Games?
- Rules?
- Ending positions?
- Winning Strategies?

Blue-Red Hackenbush

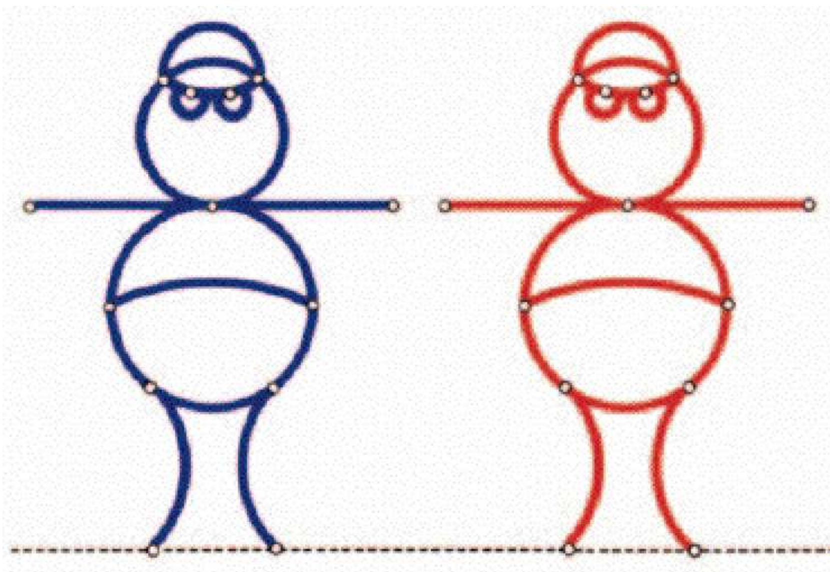
- **Two players:** "Left" and "Right".
- **Game board:** blue-red graphs connected to the ground.
- **Rules:** Two players take turns. Right deletes one red edge and also remove any piece no longer connected to the ground. Left does the similar move but deletes one blue edge.
- **Ending positions:** Whoever gets stuck is the loser.



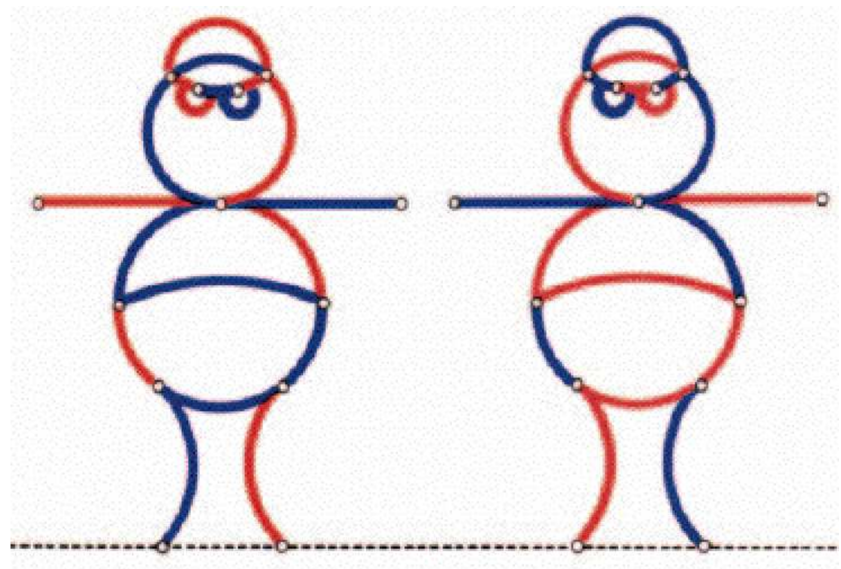


Copy strategy

Who wins?



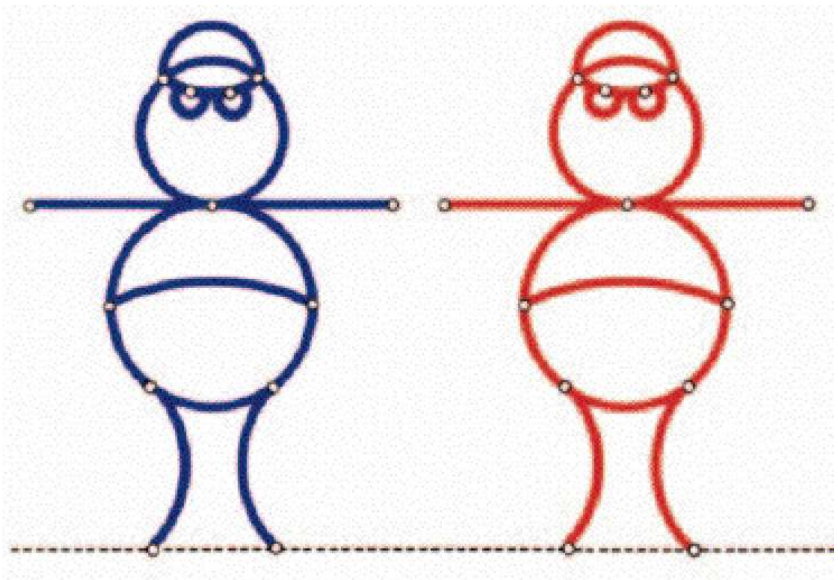
Tweedledum Tweedeldee (I)

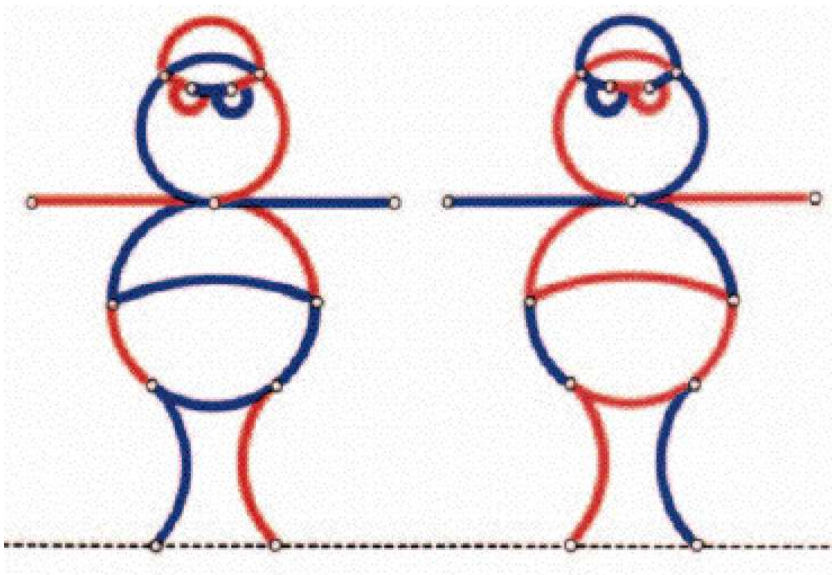


Tweedledum Tweedeldee (II)

Copy strategy

Who wins?





Tweedledum Tweedeldee (I)   Tweedledum Tweedeldee (II)

The second player has a winning strategy: copy the move of the first player.

Zero Game

A game have the value 0 if the second player has a winning strategy.

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The sum of two games G and H, denoted by $G + H$, is a game that player can choose one of the game board to play at his/her turn.

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For any game G, let $-G$ be the mirror image of G. Then

$$G + (-G) = 0.$$

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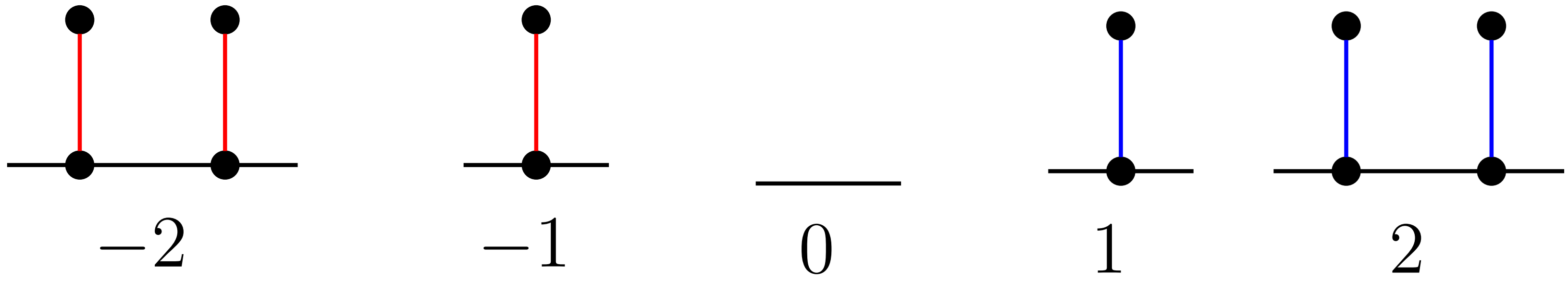
For any game G, let $-G$ be the mirror image of G. Then

$$G + (-G) = 0.$$

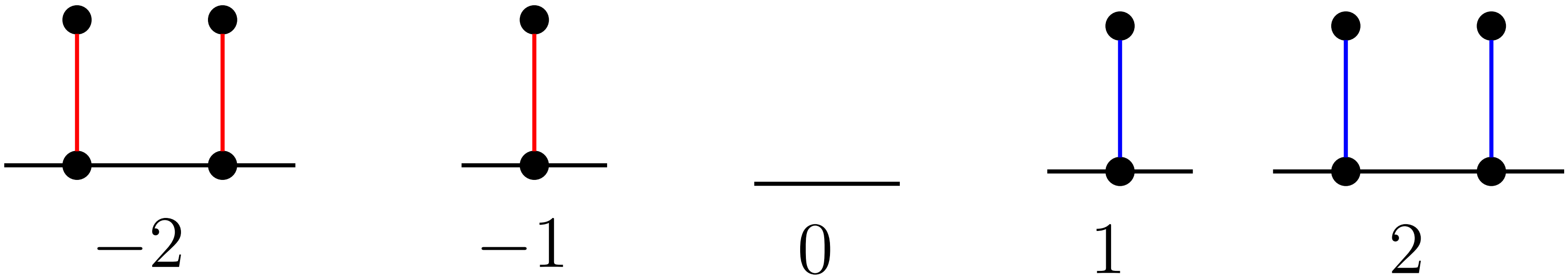
Two games G and H have the same value if

$$G + (-H) = 0.$$

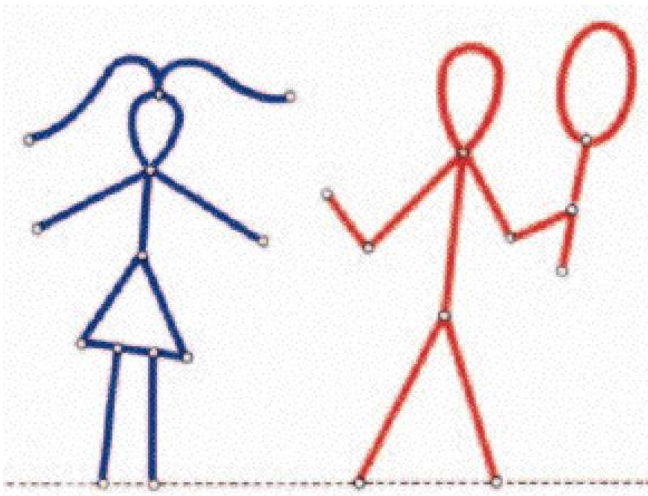
Game Values

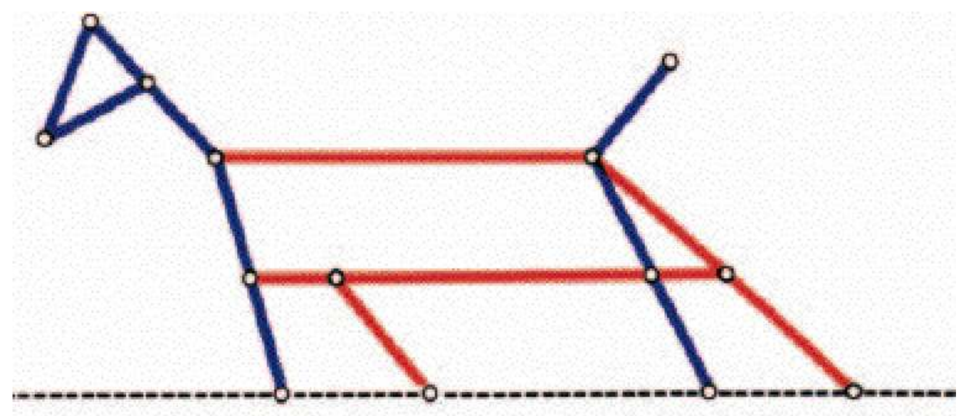


Game Values



What are the values of the following games?





Answers

Observation: If each edge in a red-blue Hackenbush game G is connected to the ground via its own color, then the other player cannot delete its opponent's edges. Therefore the value of G is the number of blue edges minus the number of red edges.

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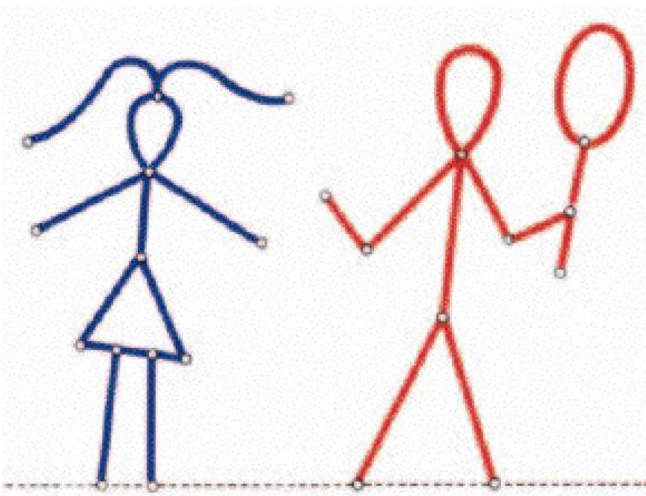
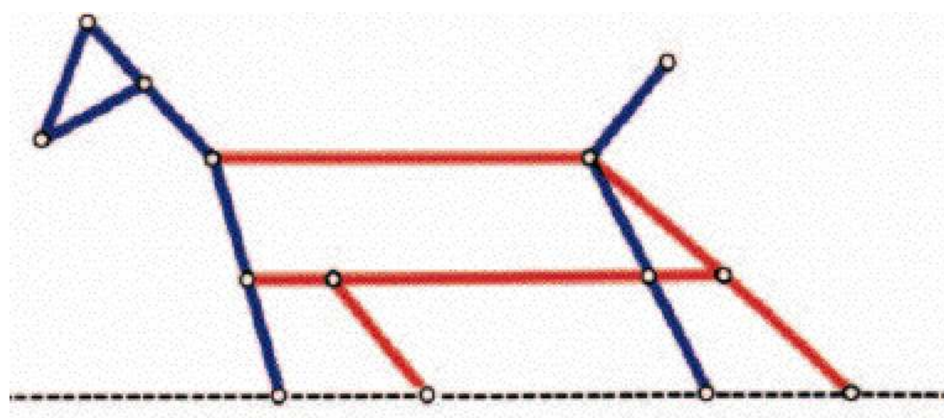
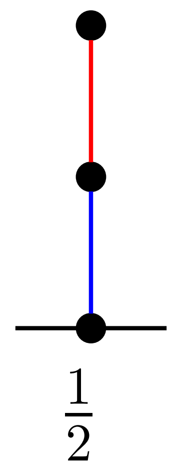


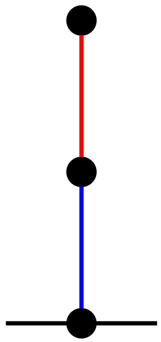


$$14 - 11 = 3 \qquad 9 - 7 = 2$$

Half move

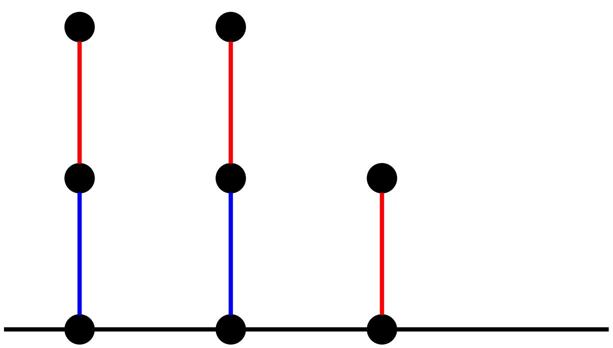


Half move

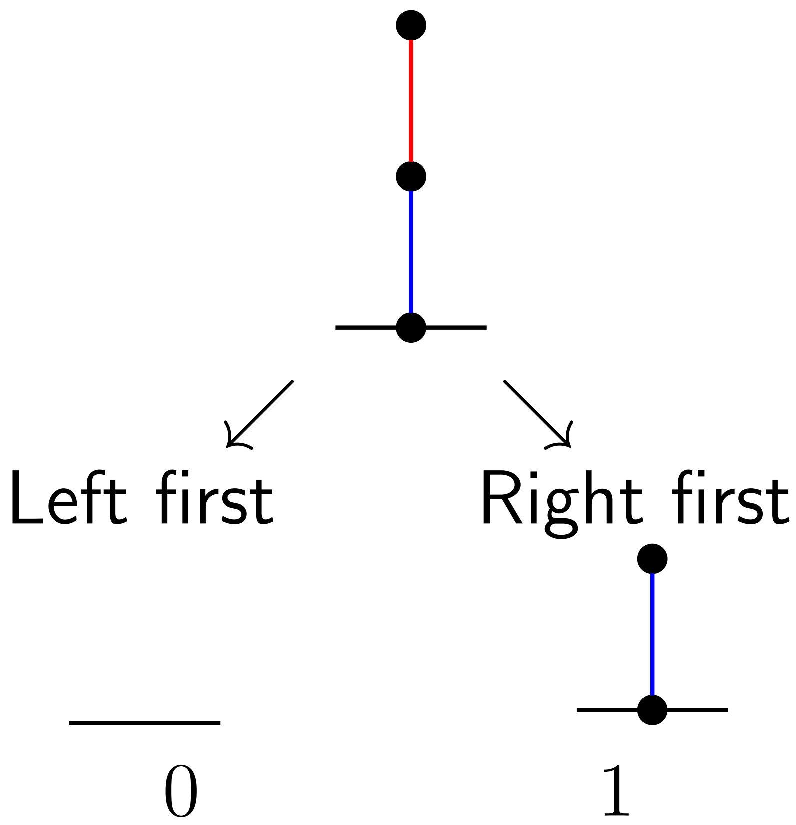


$\frac{1}{2}$

Show that the following game is a zero-game.



Brace notation



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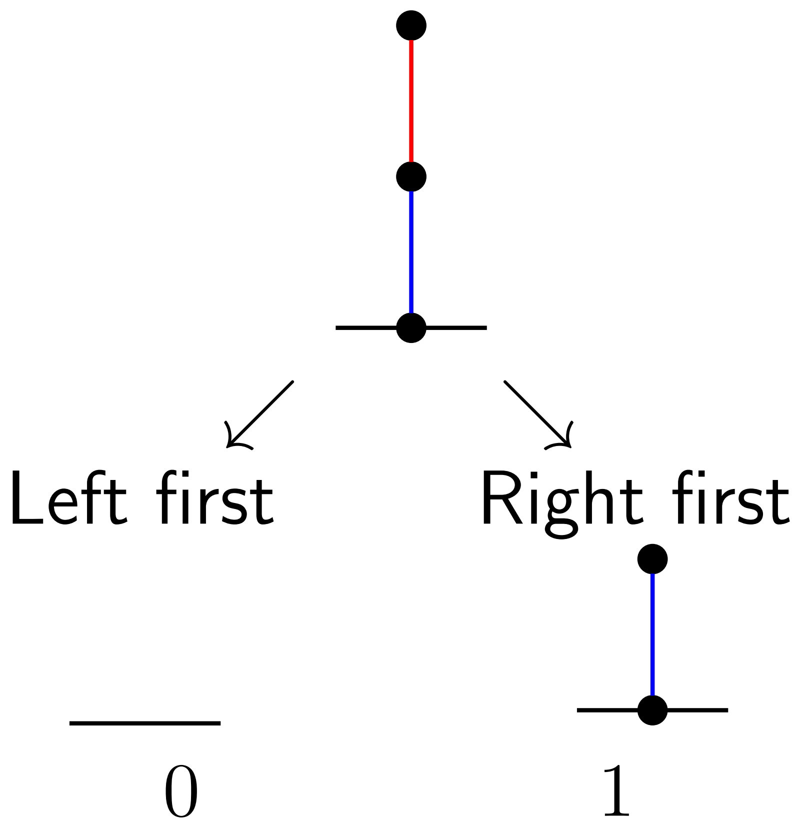
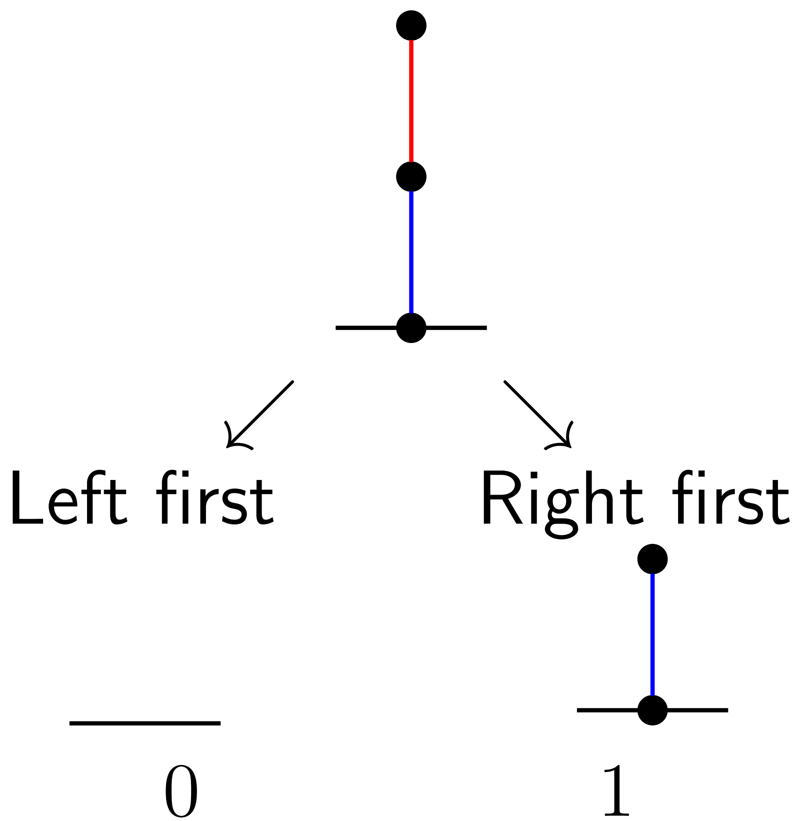


$$\{0 \mid 1\} = \frac{1}{2}$$

Brace notation



$$\{0 \mid 1\} = \frac{1}{2}$$

More notation:

$$\{\ \mid\ \} = 0$$

$$\{0 \mid\ \} = 1$$

$$\{1 \mid\ \} = 2$$

$$\{\ \mid 0\} = -1$$

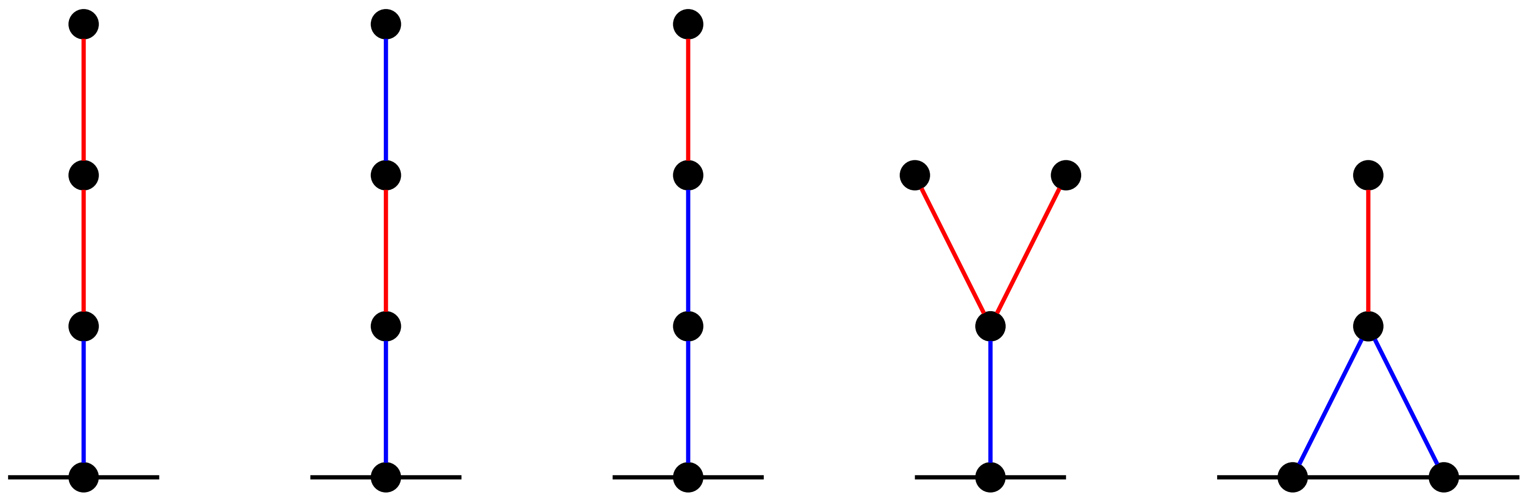
$$\{\ \mid -1\} = -2$$

$$\{n \mid\ \} = n + 1$$

$$\{n \mid n + 1\} = n + \frac{1}{2}$$

More game values

What are the values of the following games?



More game values

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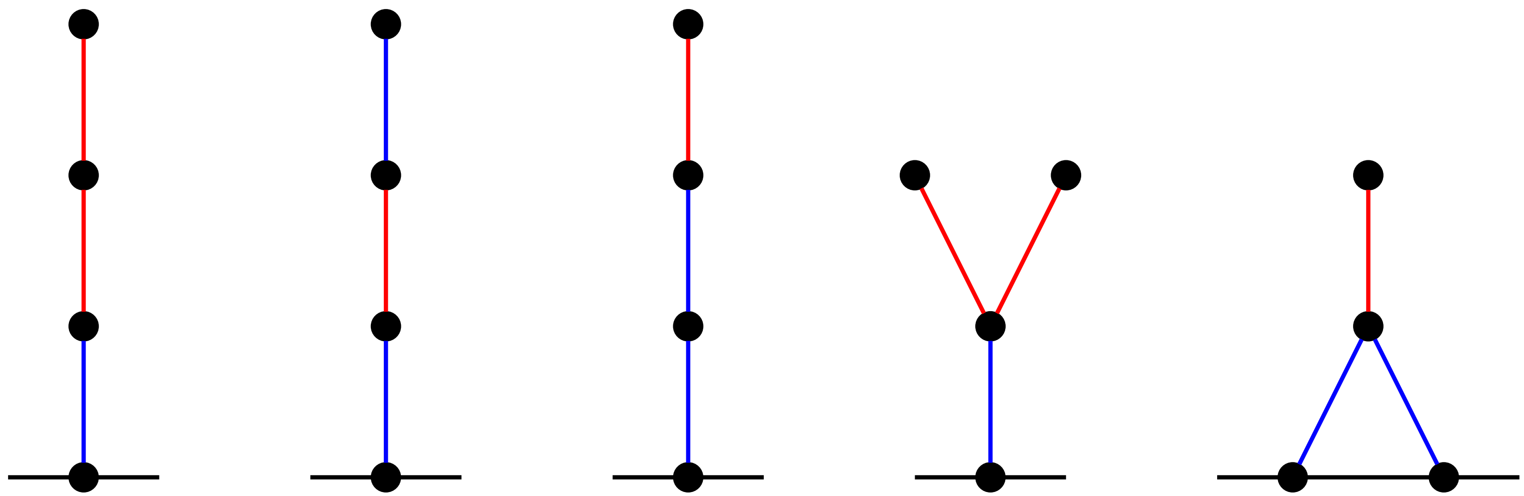
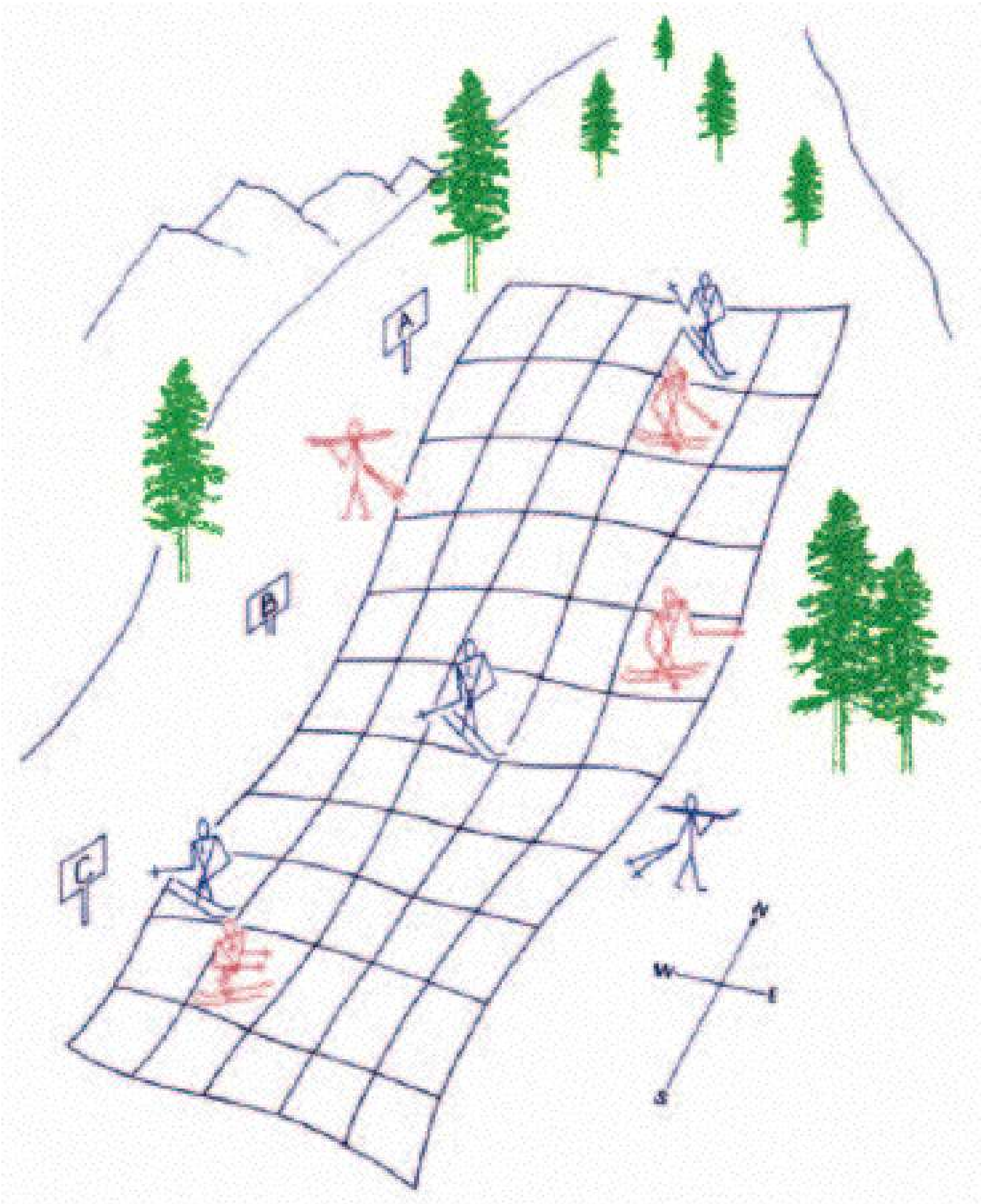


$\frac{1}{4}$ $\quad$ $\frac{3}{4}$ $\quad$ $1\frac{1}{2}$ $\quad$ $\frac{1}{4}$ $\quad$ 1

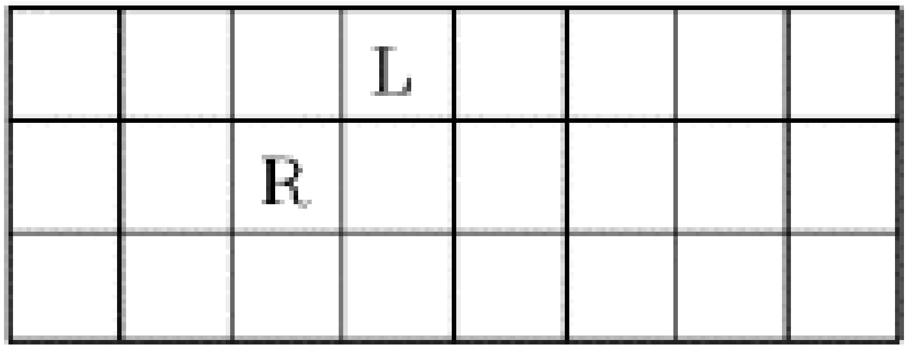
A game of Ski-Jumps

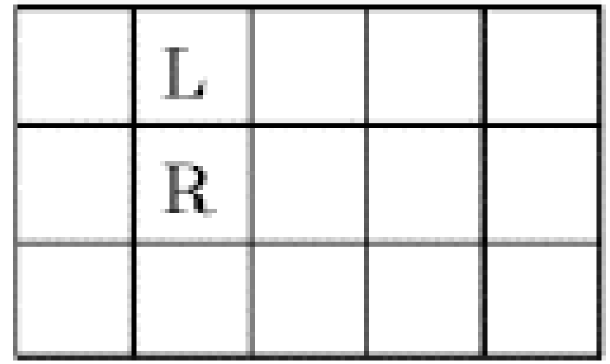


A game of Ski-Jumps

- **Two players:** "Left" and "Right".
- **Game board:** several skiers on a rectangular board
- **Rules:** Two players take turns. Left may move any skier a square or more Eastwards, or Right any one of his, Westwards, provided there is no active skier in the way. Such a move may take a skier off the slope; in this case he takes no further part in the game. Alternatively a skier on the square immediately above one containing a skier of the opposing team, may jump over him on the the square immediately below, provided this is empty. A man jumped over will never jump over anyone else.
- **Ending positions:** Whoever gets stuck is the loser.

Some examples of Ski-Jumps





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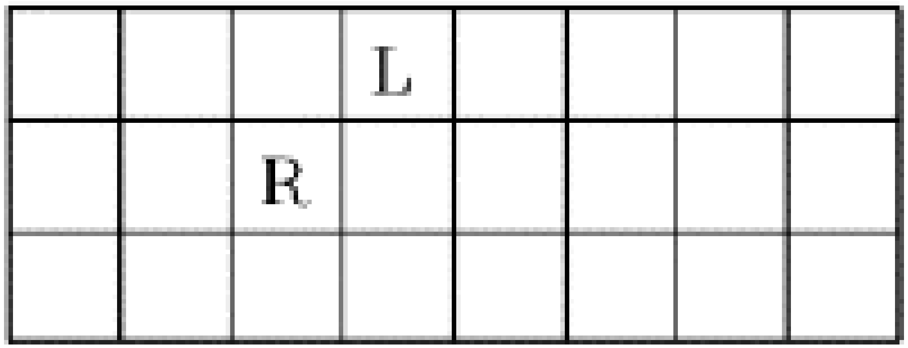
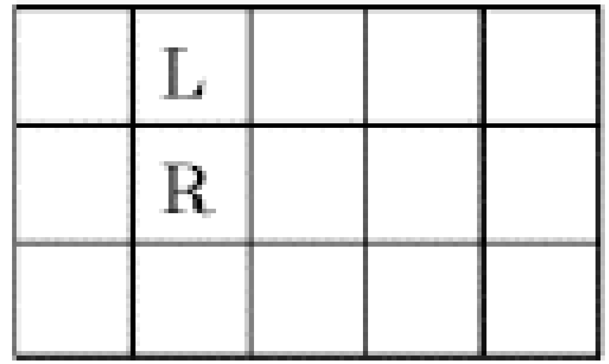


$$5 - 3 = 2$$



$$\{2 \mid 3\} = 2\tfrac{1}{2}$$

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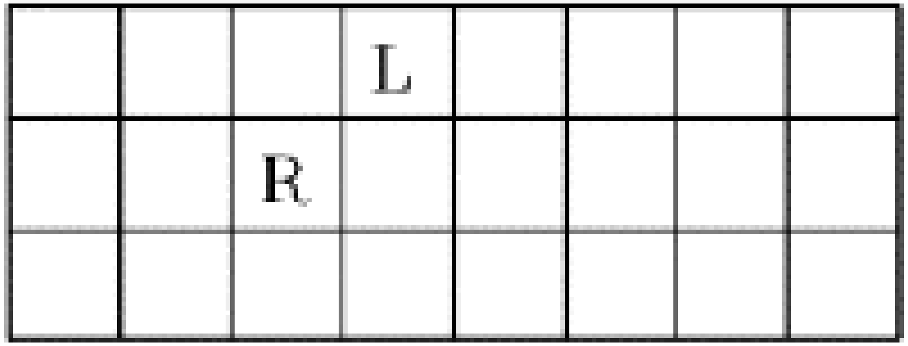

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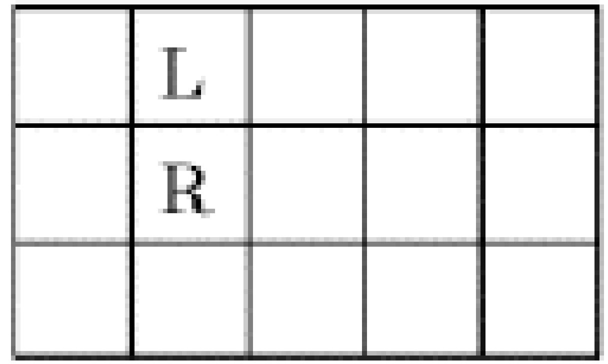

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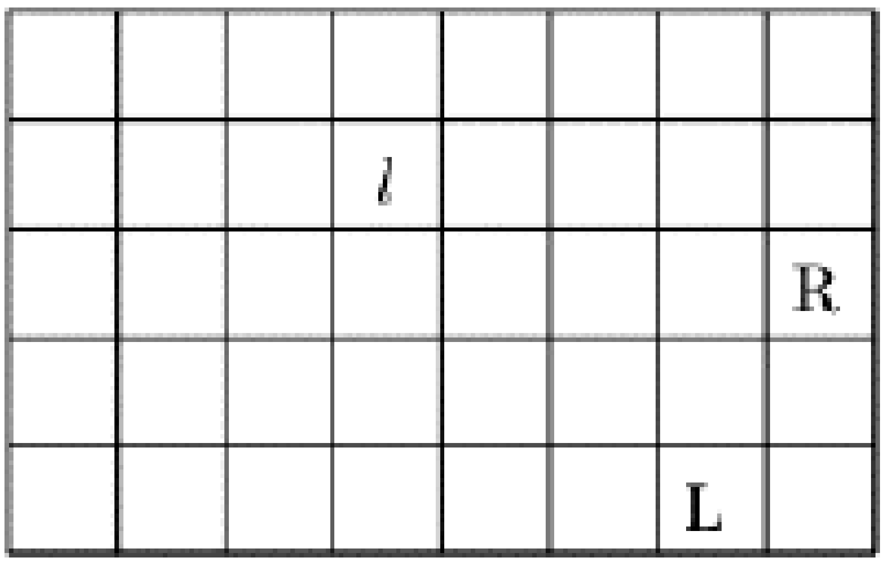

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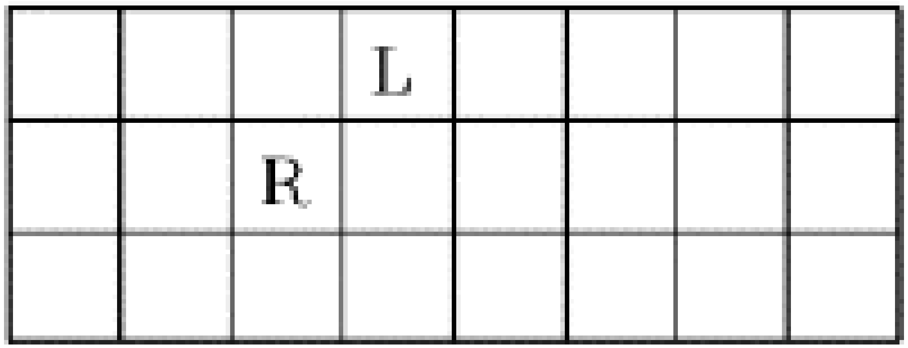

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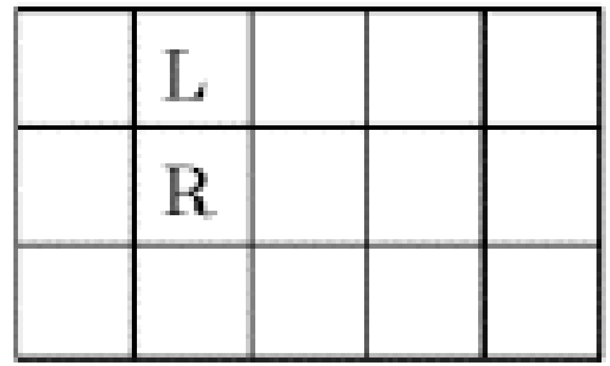

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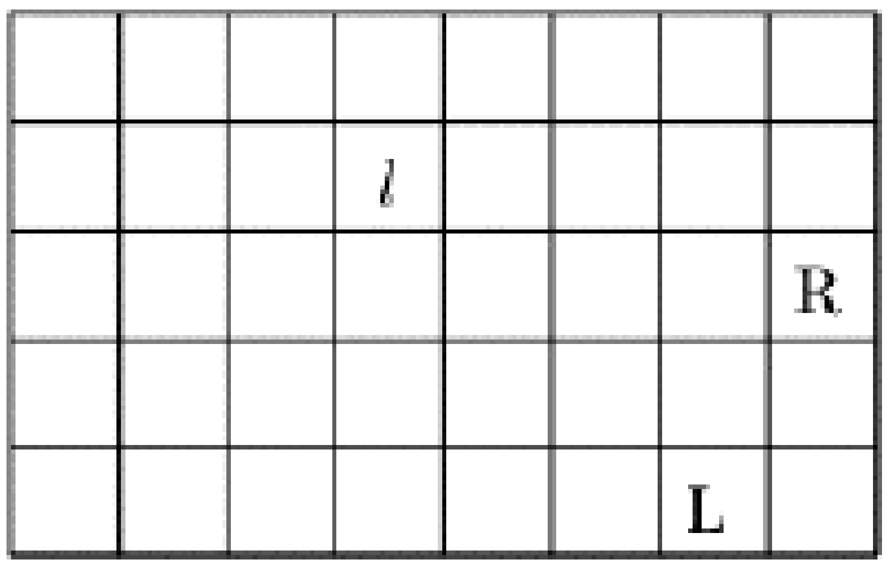
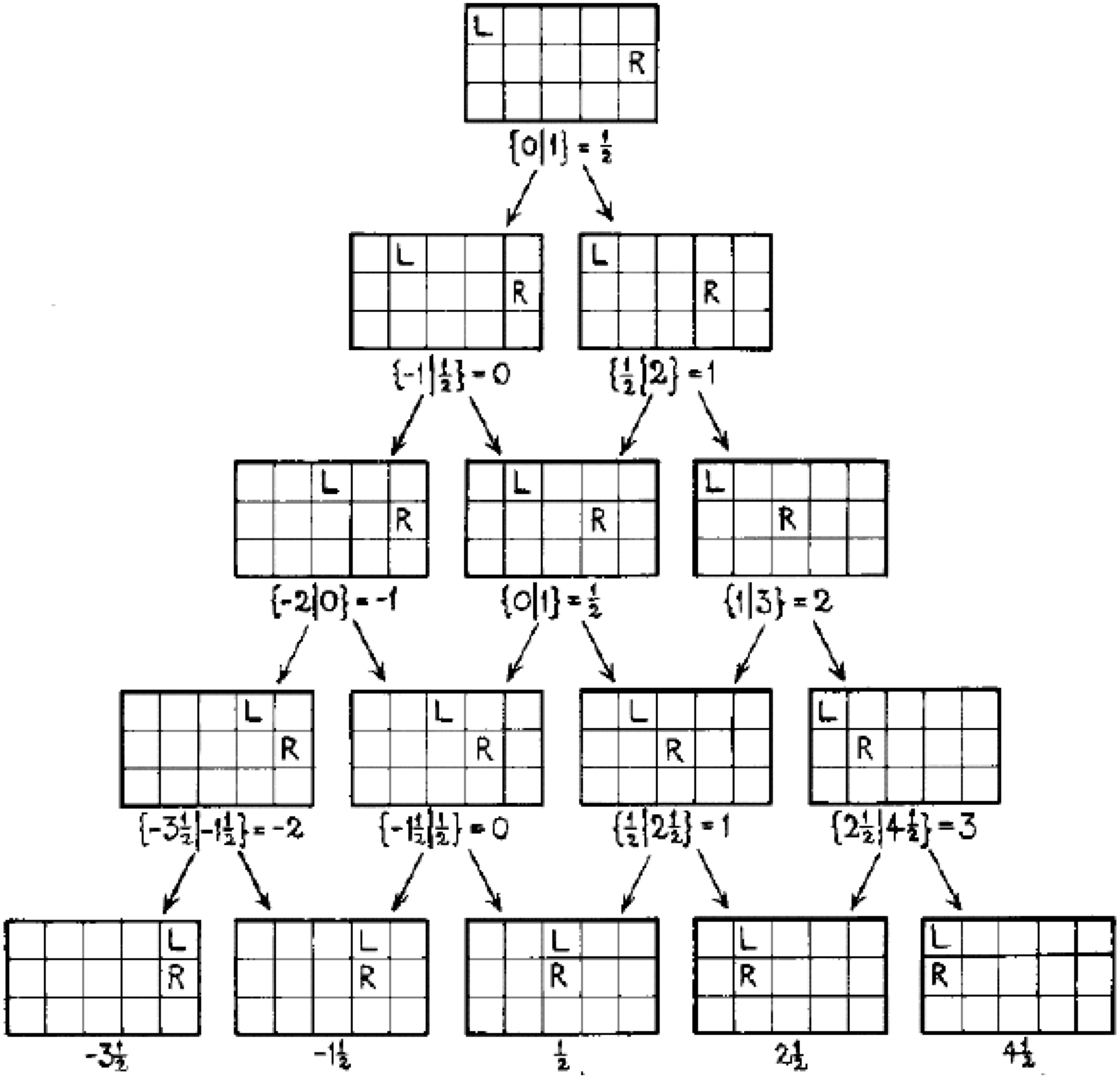


5+2-8=-1

A 3×5 board



Don't take the average!

$$\{2\frac{1}{2} \mid 4\frac{1}{2}\} = 3$$

Why?

Simplicity Rule

If the options in

$$\{a, b, c, \ldots \mid d, e, f, \ldots\}$$

are all numbers, we say the number x fits if x is greater than each of $a, b, c, \ldots$ and less than each of $d, e, f, \ldots$.

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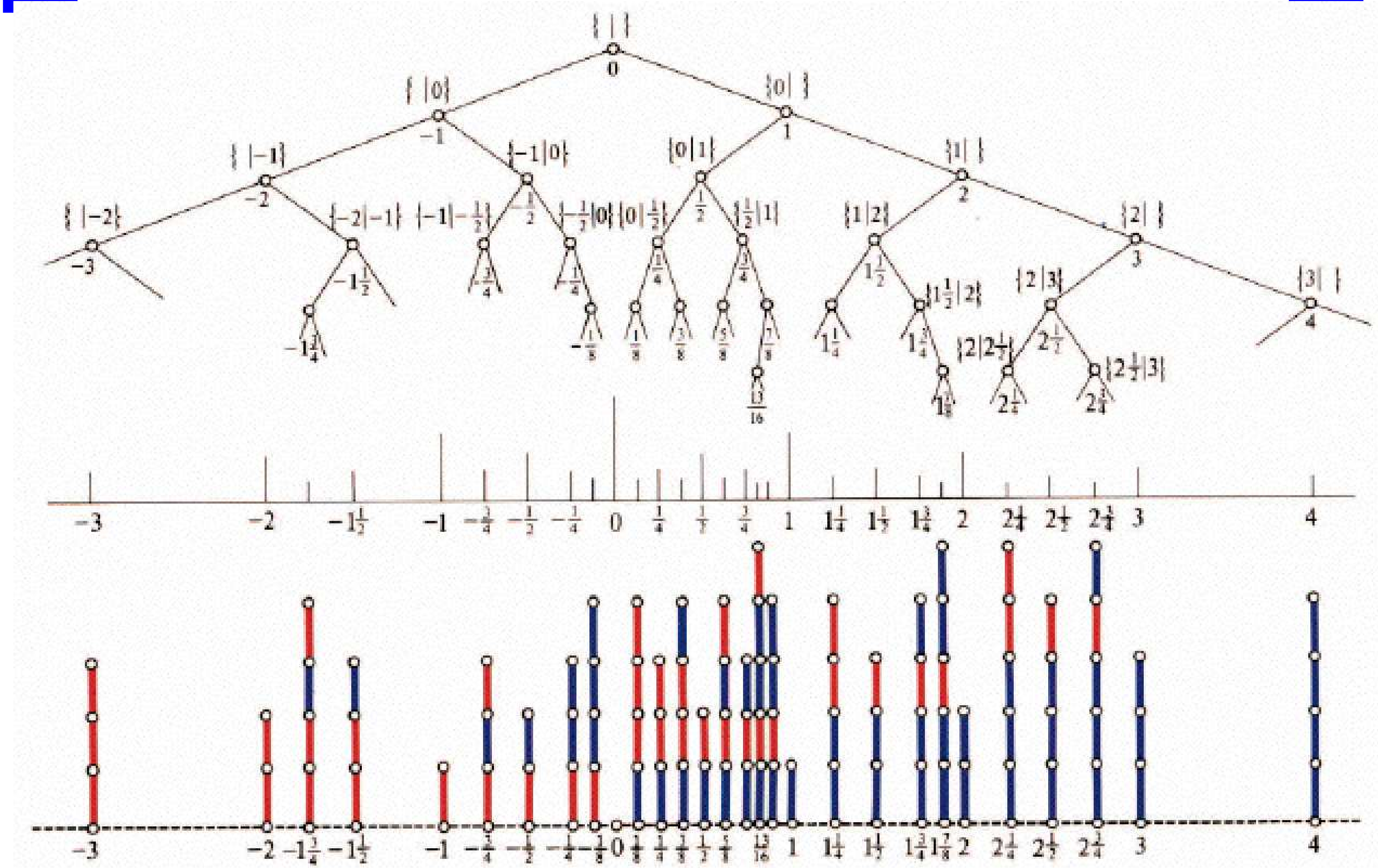
Simplicity Rule: If there's any number that fits, the answer's the simplest number that fits.

For example,

$$\{0 \mid 1\} = \frac{1}{2}, \qquad \{\frac{1}{2} \mid 1\} = \frac{3}{4},$$

$$\{-1\frac{1}{2} \mid 3\} = 0, \qquad \{2\frac{1}{2} \mid 4\frac{1}{2}\} = 3.$$

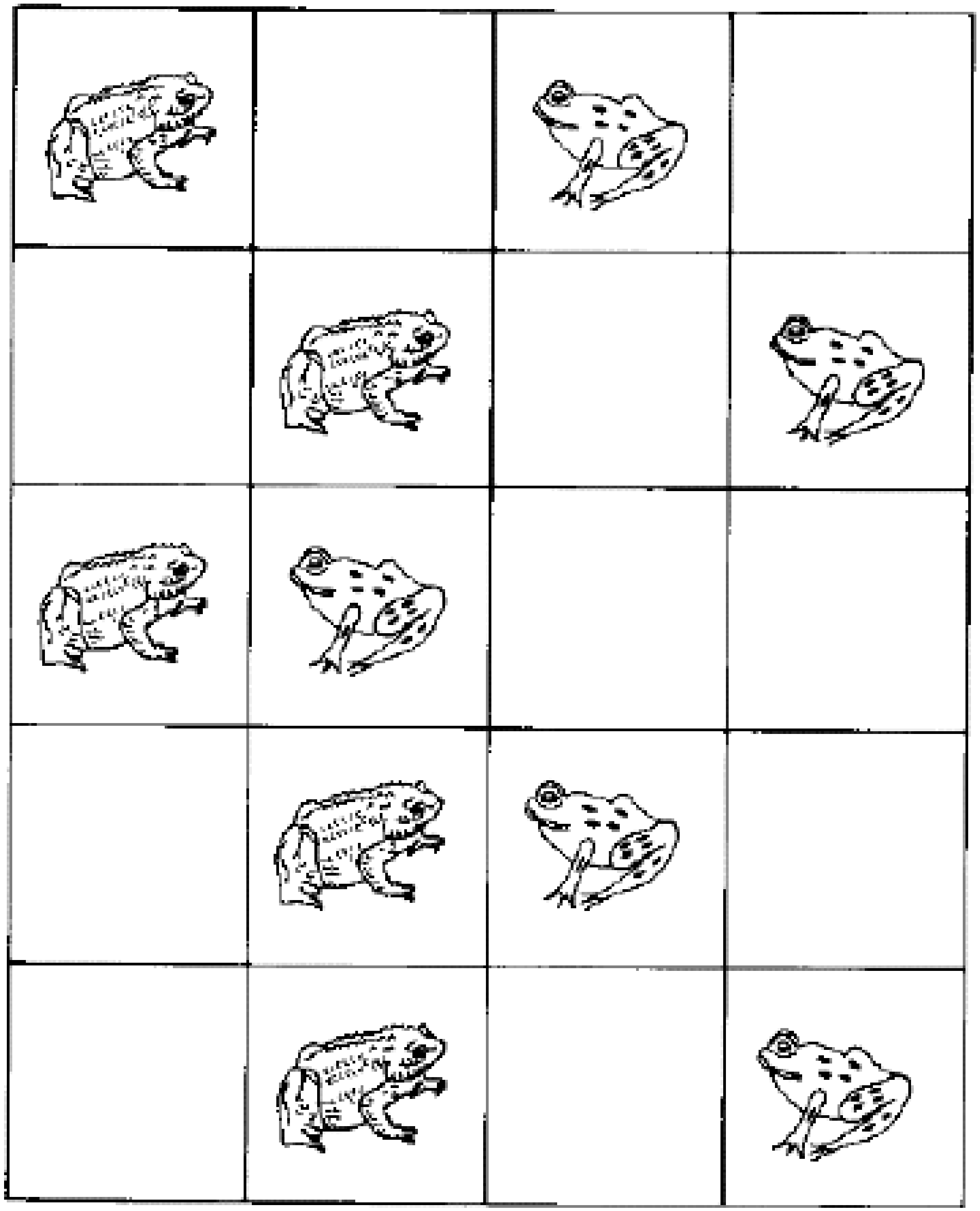
Simplest Forms for Numbers



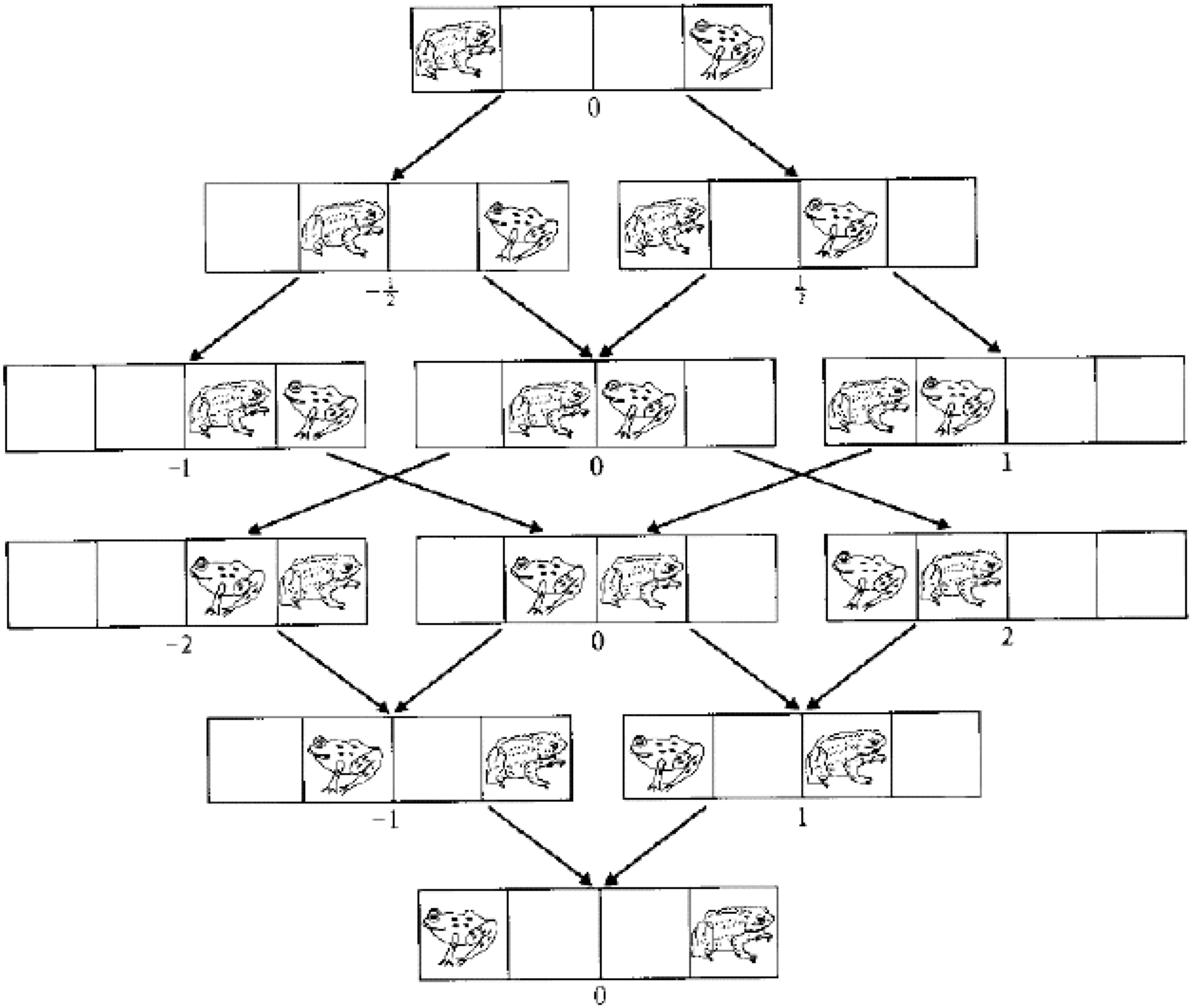
Toads-and-Frogs

- **Two players:** "Left" and "Right".
- **Game board:** Some Toads and Frogs on a rectangular board.
- **Rules:** Two players take turns. Left moves one of Toads Eastwards. Right moves one of Frogs Westwards. The creature (Toad or Frog) may jump over an opposing creature, onto an empty square.
- **Ending positions:** Whoever gets stuck is the loser.

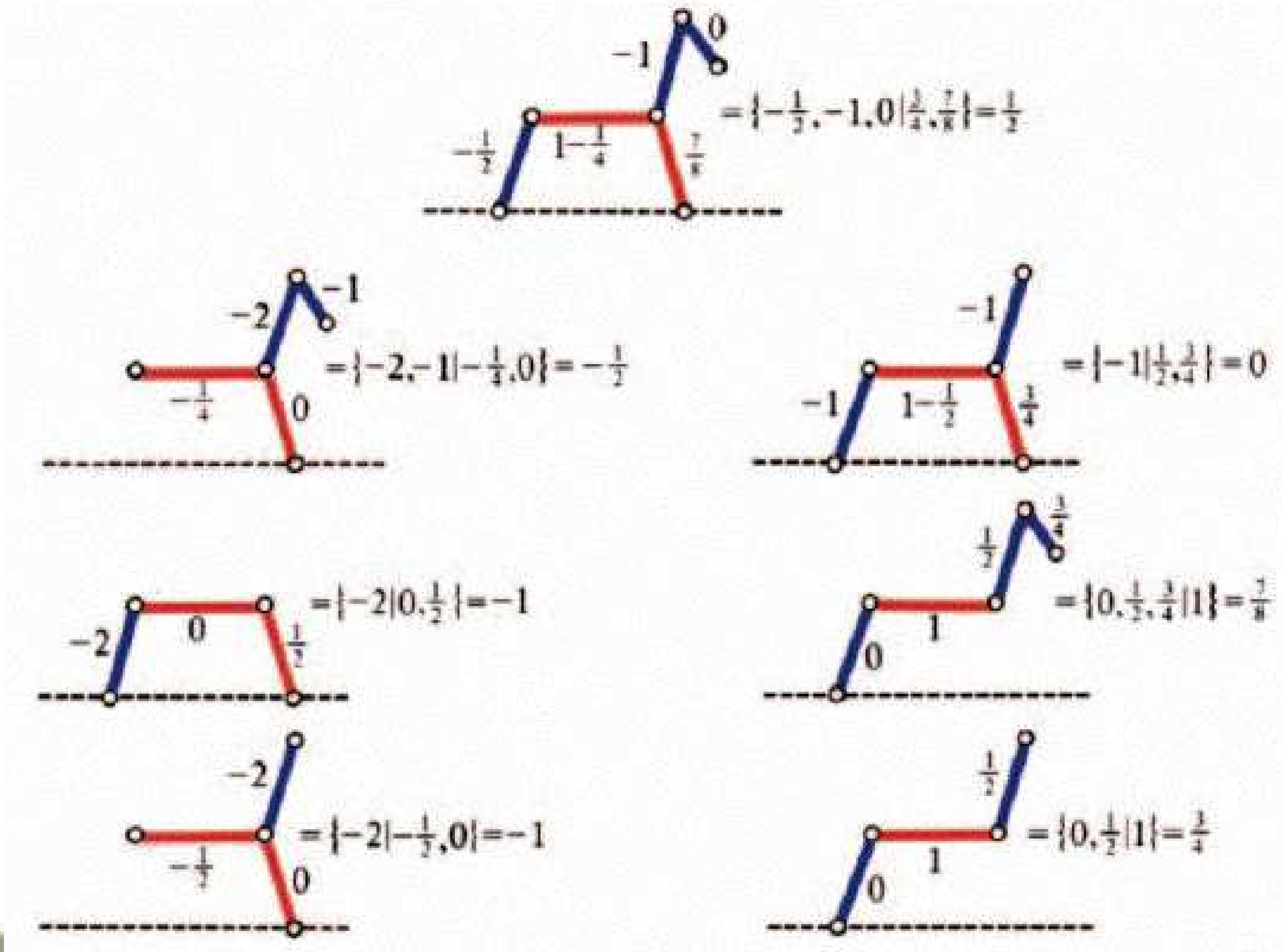
An example



Game Values

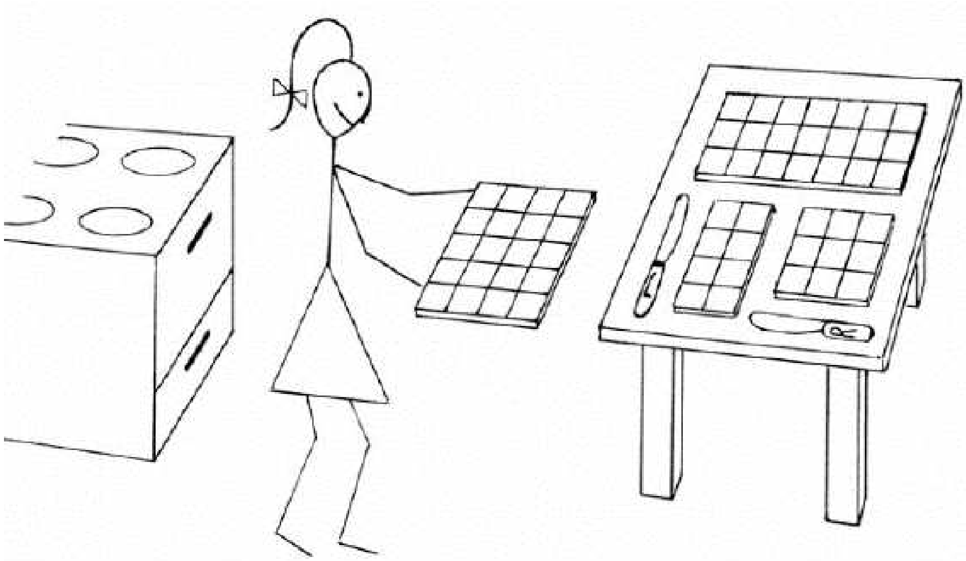


Working out a horse



Game of CutCake

- **Two players:** "Left" and "Right".
- **Game board:** A rectangular cake.
- **Rules:** Two players take turns. Left may cut any rectangle into two smaller ones along the North-South lines while Right cut it along the the East-West lines.
- **Ending positions:** Whoever gets stuck is the loser.



Game Values in Cutcake

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th><th>16</th></tr>
<tr><td>1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td></tr>
<tr><td>2</td><td>-1</td><td colspan="2" rowspan="2">0</td><td colspan="2" rowspan="2">1</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">3</td><td colspan="2" rowspan="2">4</td><td colspan="2" rowspan="2">5</td><td colspan="2" rowspan="2">6</td><td rowspan="2"></td></tr>
<tr><td>3</td><td>-2</td></tr>
<tr><td>4</td><td>-3</td><td colspan="2" rowspan="2">-1</td><td colspan="4" rowspan="4">0</td><td colspan="4" rowspan="4">1</td><td colspan="4" rowspan="4">2</td><td rowspan="4"></td></tr>
<tr><td>5</td><td>-4</td></tr>
<tr><td>6</td><td>-5</td><td colspan="2" rowspan="2">-2</td></tr>
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<tr><td>8</td><td>-7</td><td colspan="2" rowspan="2">-3</td><td colspan="4" rowspan="4">-1</td><td colspan="8" rowspan="6">0</td><td rowspan="6"></td></tr>
<tr><td>9</td><td>-8</td></tr>
<tr><td>10</td><td>-9</td><td colspan="2" rowspan="2">-4</td></tr>
<tr><td>11</td><td>-10</td></tr>
<tr><td>12</td><td>-11</td><td colspan="2" rowspan="2">-5</td><td colspan="4" rowspan="2">-2</td></tr>
<tr><td>13</td><td>-12</td></tr>
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Maundy Cake

- **Two players:** "Left" and "Right".
- **Game board:** A rectangular cake.
- **Rules:** Two players take turns. Left may cut any rectangle into any number of smaller equal ones along the North-South lines while Right cut it along the the East-West lines.
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For example, a 4×9 cake may be cut into

- nine 4×1 or three 4×3 by Left;
- four 1×9 or two 2×9 by Right.

Maundy Cake Values

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	1	3	1	4	1	7	4	6	1	10	1	8	6	15	1	13
2	-1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
3	-1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
4	-3	-1	-1	0	-1	0	-1	1	0	0	-1	1	-1	0	0	3	-1	1
5	-1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
6	-4	-1	-1	0	-1	0	-1	1	0	0	-1	1	-1	0	0	3	-1	1
7	-1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4
8	-7	-3	-3	-1	-3	-1	-3	0	-1	-1	-3	0	-3	-1	-1	1	-3	0
9	-4	-1	-1	0	-1	0	-1	1	0	0	-1	1	-1	0	0	3	-1	1
10	-6	-1	-1	0	-1	0	-1	1	0	0	-1	1	-1	0	0	3	-1	1
11	-1	0	0	1	0	1	0	3	1	1	0	4	0	1	1	7	0	4

Working out Maundy Cake

Let $M(r, l)$ be the value of Maundy Cake of dimension $r \times l$.

Working out Maundy Cake

Let $M(r,l)$ be the value of Maundy Cake of dimension $r \times l$.

$r = 999:\quad 333:\quad 111:\quad 37:\quad 1$

$l = 1000:\quad 500:\quad 250:\quad 125:\quad 25:\quad 5:\quad 1$

$$M(999, 1000) = 5 + 1.$$

Working out Maundy Cake

Let $M(r,l)$ be the value of Maundy Cake of dimension $r \times l$.

$r = 999:$ $333:$ $111:$ $37:$ 1

$l = 1000:$ $500:$ $250:$ $125:$ $25:$ $5:$ 1

$$M(999, 1000) = 5 + 1.$$

$r = 1000:$ $500:$ $250:$ $125:$ $25:$ $5:$ 1

$l = 1001:$ $143:$ $13:$ 1

$$M(1000, 1001) = (-25) + (-5) + (-1) = -31.$$

Four possible game outputs

		If Left starts	
		Left wins	Right wins
If Right starts	Left wins	positive (L wins)	zero (2 wins)
	Right wins	fuzzy (1 wins)	negative (R wins)

$G > 0$ or G is **positive** if player Left can always win.
$G < 0$ or G is **negative** if player Right can always win.
$G = 0$ or G is **zero** if second player can always win.
$G \| 0$ or G is **fuzz** if first player can always win.

Combination of outputs

- $G \geq 0$ means that Left has a winning strategy provided Right plays first.

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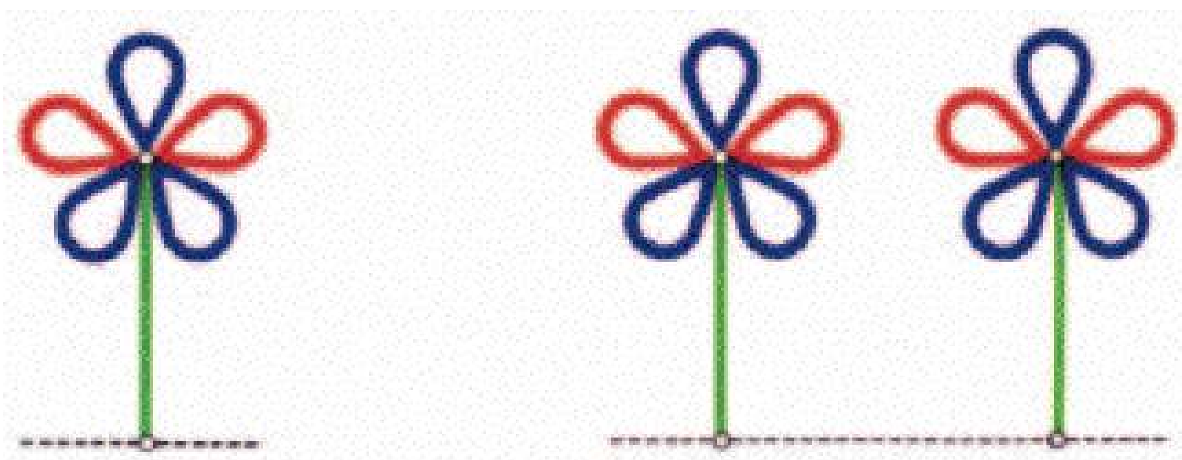
- $G \mathrel{\triangleleft\!|} 0$ means that Right has a winning strategy provided Right plays first.

Hackenbush Hotchpotch

- **Two players:** "Left" and "Right".
- **Game board:** blue-red-green graphs connected to the ground.
- **Rules:** Two players take turns. Right deletes one red edge or one green edge and also remove any piece no longer connected to the ground. Left does the similar move but deletes one blue edge or one green edge.
- **Ending positions:** Whoever gets stuck is the loser.



Sum of arbitrary games

Let G^L be the typical left options and G^R be the typical right options. Then

$$G = \{G^L \mid G^R\}.$$

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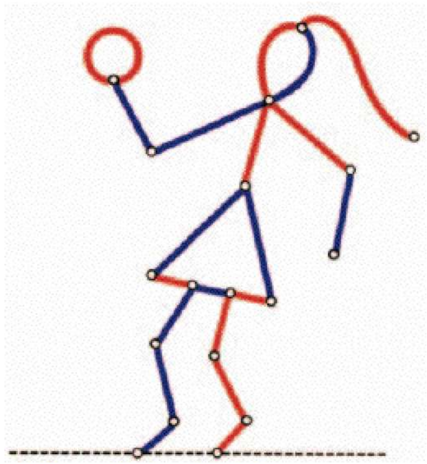
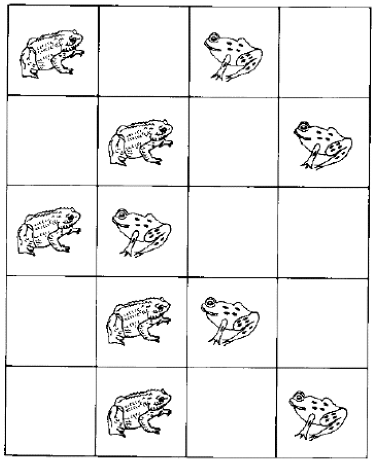
For two arbitrary games $G = \{G^L \mid G^R\}$ and $H = \{H^L \mid H^R\}$, the sum of the games is defined as

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}.$$





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- If $G \geq 0$ and $H \geq 0$ then $G + H \geq 0$.

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- If $G \mathrel{\triangleleft\!|} 0$ and $H \leq 0$ then $G + H \mathrel{\triangleleft\!|} 0$.

We only prove the first property here. Assume Right plays first. If Right plays on G, then Left responds in G since Left has a winning strategy in G. If Right plays on H, then Left responds in H since Left has a winning strategy in H. In ether case, Left can win. Thus, $G + H \geq 0$.

Outcome of sum of games

	$H = 0$	$H > 0$	$H < 0$	$H \| 0$
$G = 0$	$G + H = 0$	$G + H > 0$	$G + H < 0$	$G + H \| 0$
$G > 0$	$G + H > 0$	$G + H > 0$	$G + H ? 0$	$G + H \rhd 0$
$G < 0$	$G + H < 0$	$G + H ? 0$	$G + H < 0$	$G + H \lhd 0$
$G \| 0$	$G + H \| 0$	$G + H \rhd 0$	$G + H \lhd 0$	$G + H ? 0$

Here $G + H ? 0$ are unrestricted.

Comparing two games

Use the negative game $-H = \{-H^R \mid -H^L\}$. One can define $G = H$, $G > H$, $G < H$, and $G \| H$. For example, define $G > H$ if $G + (-H) > 0$.

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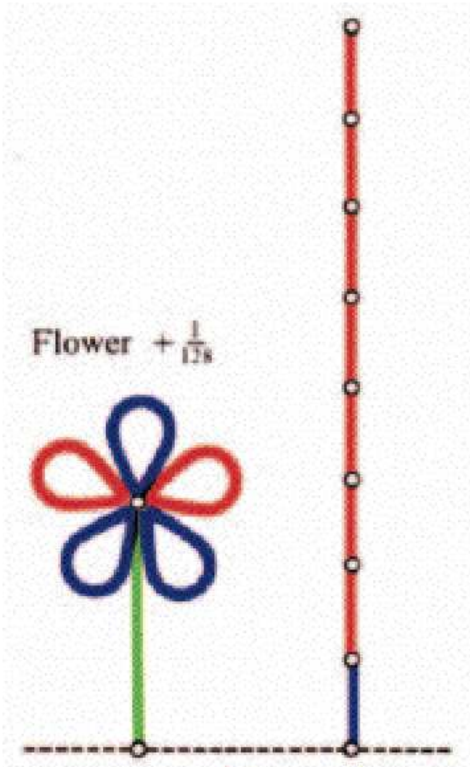
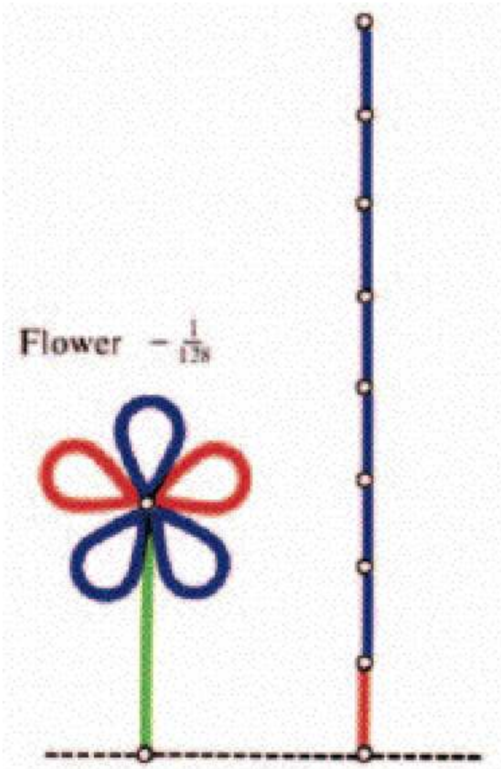
We have

	$H = K$	$H > K$	$H < K$	$H \| K$
$G = H$	$G = K$	$G > K$	$G < K$	$G \| K$
$G > H$	$G > K$	$G > K$	$G?K$	$G \,\vert\!\triangleright K$
$G < H$	$G < K$	$G?K$	$G < K$	$G \triangleleft\!\vert\, K$
$G \| H$	$G \| K$	$G \,\vert\!\triangleright K$	$G \triangleleft\!\vert\, K$	$G?K$

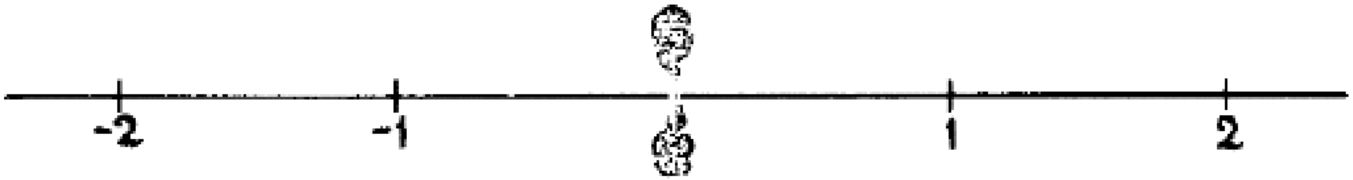
Small Hackenbush Positions



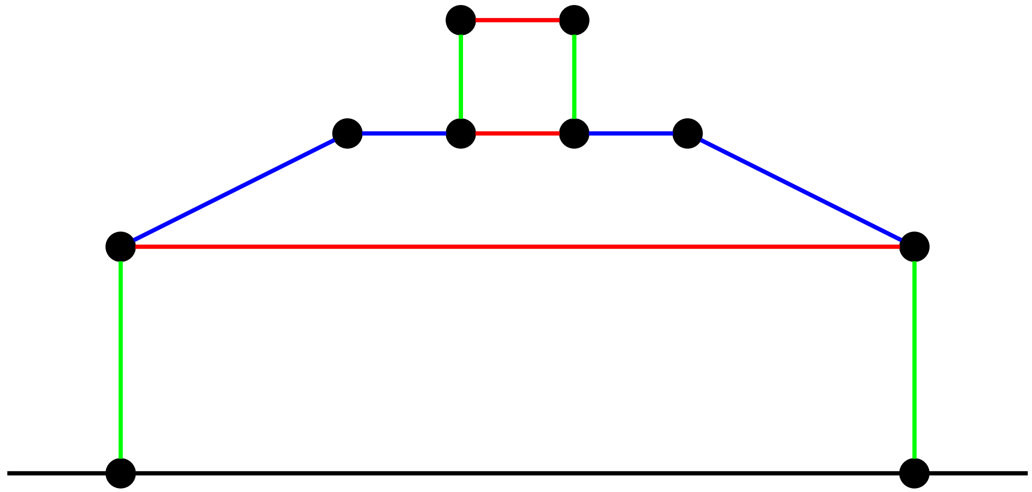


Flower is dwarfed by Very Small Hollyhocks of Either Sign.

$$-\frac{1}{2^n} < flower < \frac{1}{2^n} \quad \text{for any } n.$$



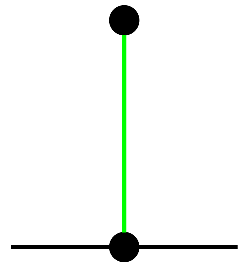
Small positive Hackenbush



This house has a positive value but is smaller than any positive number.

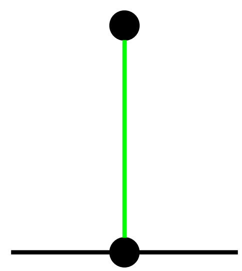
$$0 < house < \frac{1}{2^n} \quad \text{for any } n.$$

The game value $*$



$$* = \{0 \mid 0\}.$$

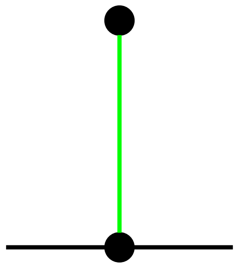
The game value $*$



$$* = \{0 \mid 0\}.$$

How big is the star $*$?

The game value $*$



$$* = \{0 \mid 0\}.$$

How big is the star $*$?

$*$ is less than any positive number, greater than any negative number, and fuzz with 0.

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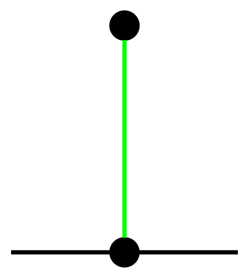


$$* = \{0 \mid 0\}.$$

How big is the star $*$?
$*$ is less than any positive number, greater than any negative number, and fuzz with 0.

For any number x, let $x* = x + *$. We have

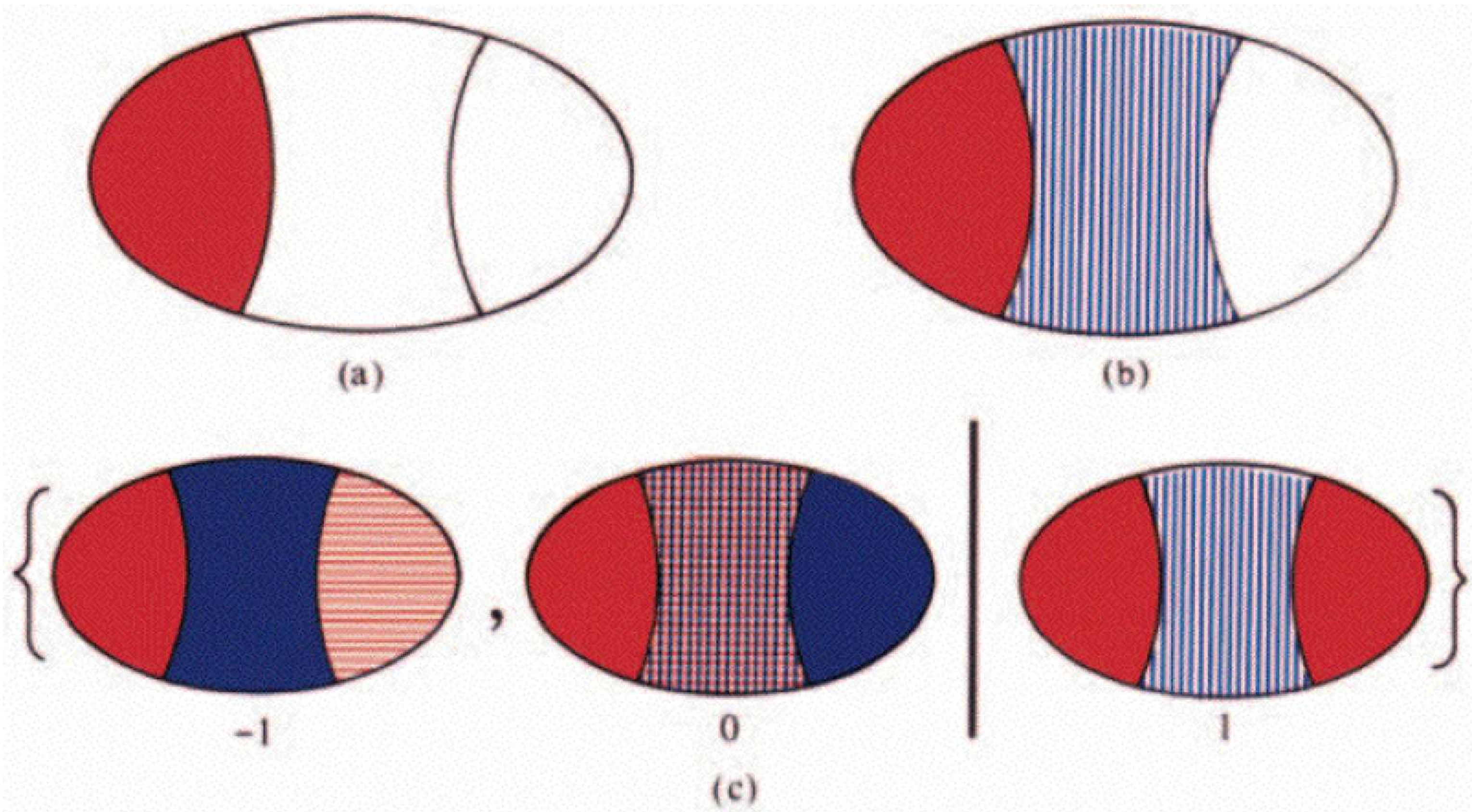
$$\{x \mid x\} = x * .$$

Game of Col

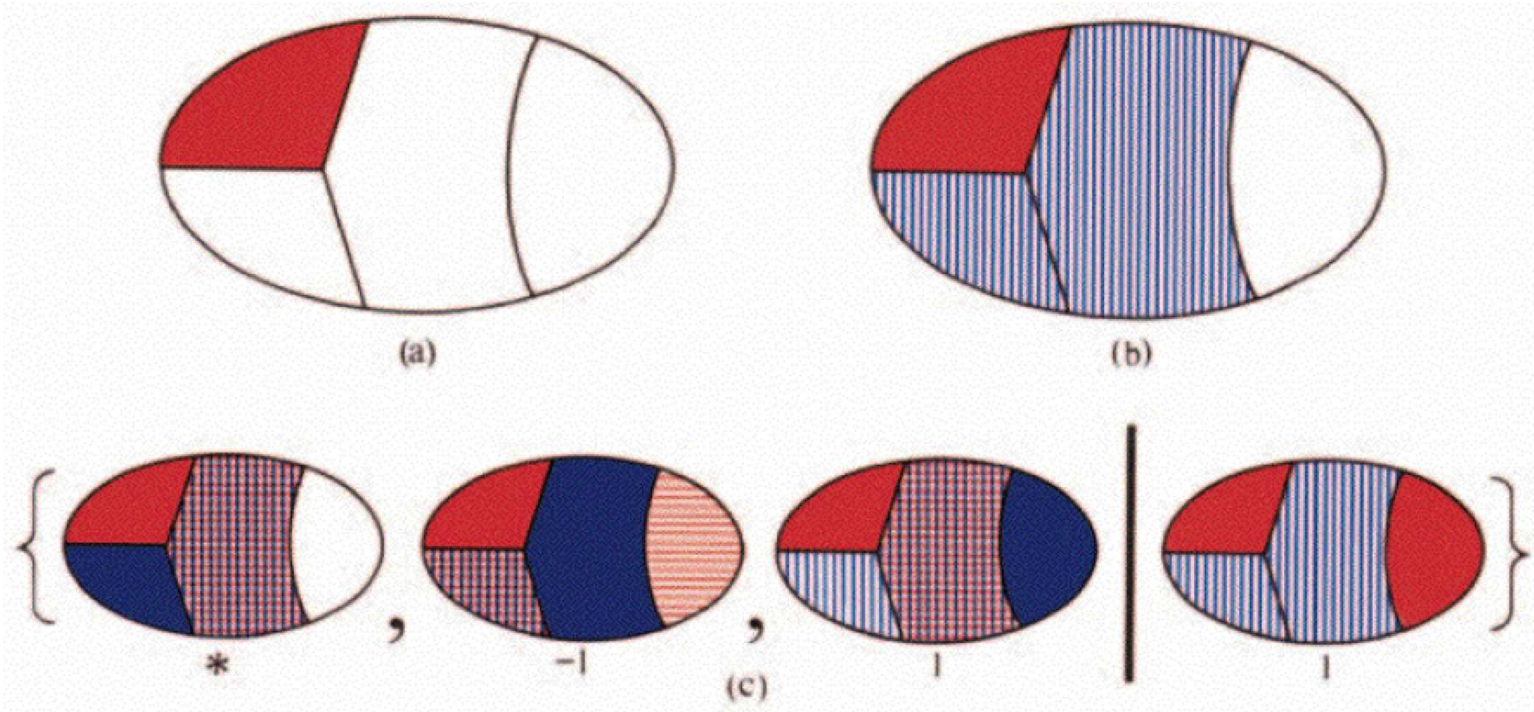
- **Two players:** "Left" and "Right".
- **Game board:** a partial colored planar map.
- **Rules:** Two players take turns to paint regions of the map. Each player, when in his turn to move, paint one region of the map, Left using the color blue and Right using the color red. No two regions having a common frontier may be painted in the same color.
- **Ending positions:** Whoever gets stuck is the loser.

Example 1 of Col Game

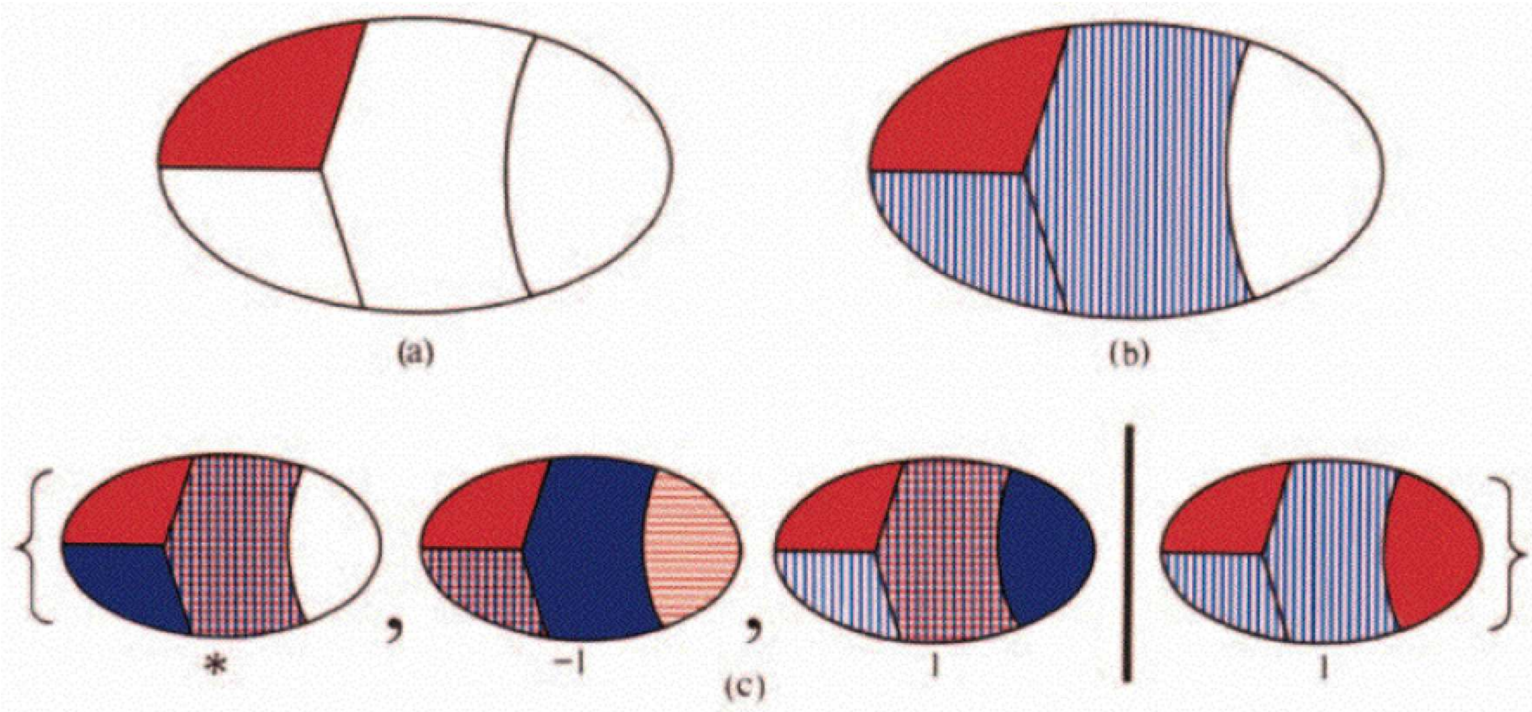
The region belonging to Left only is blue-tinted while the one belonging to Right only is red-tinted.



Example 2 of Col Game



Example 2 of Col Game



$$\{*, -1, 1 \mid 1\} = \{1 \mid 1\} = 1* .$$

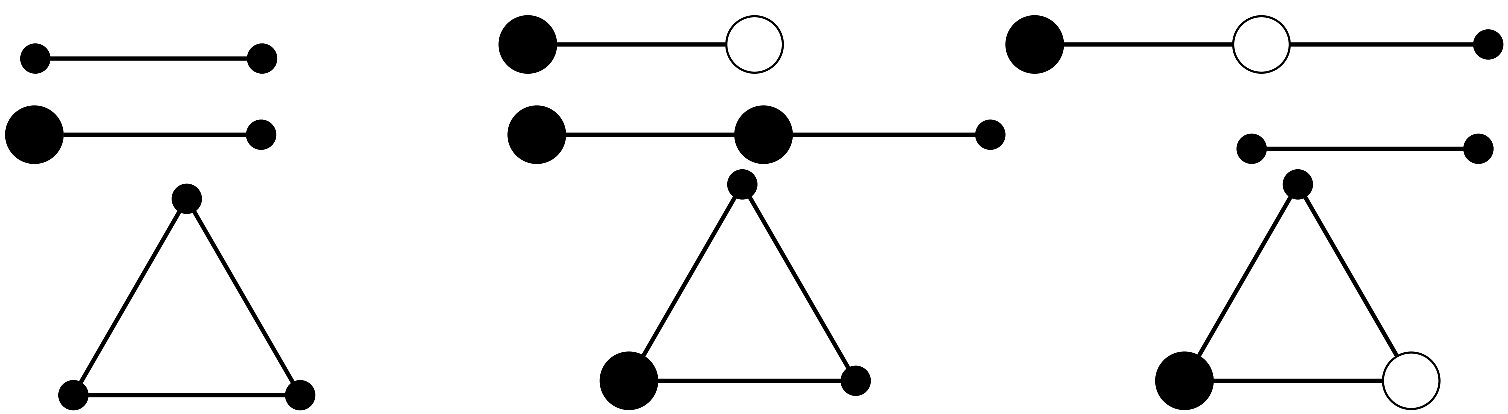
Alternative version of Col

We can also represent regions by nodes and adjacency by edges.

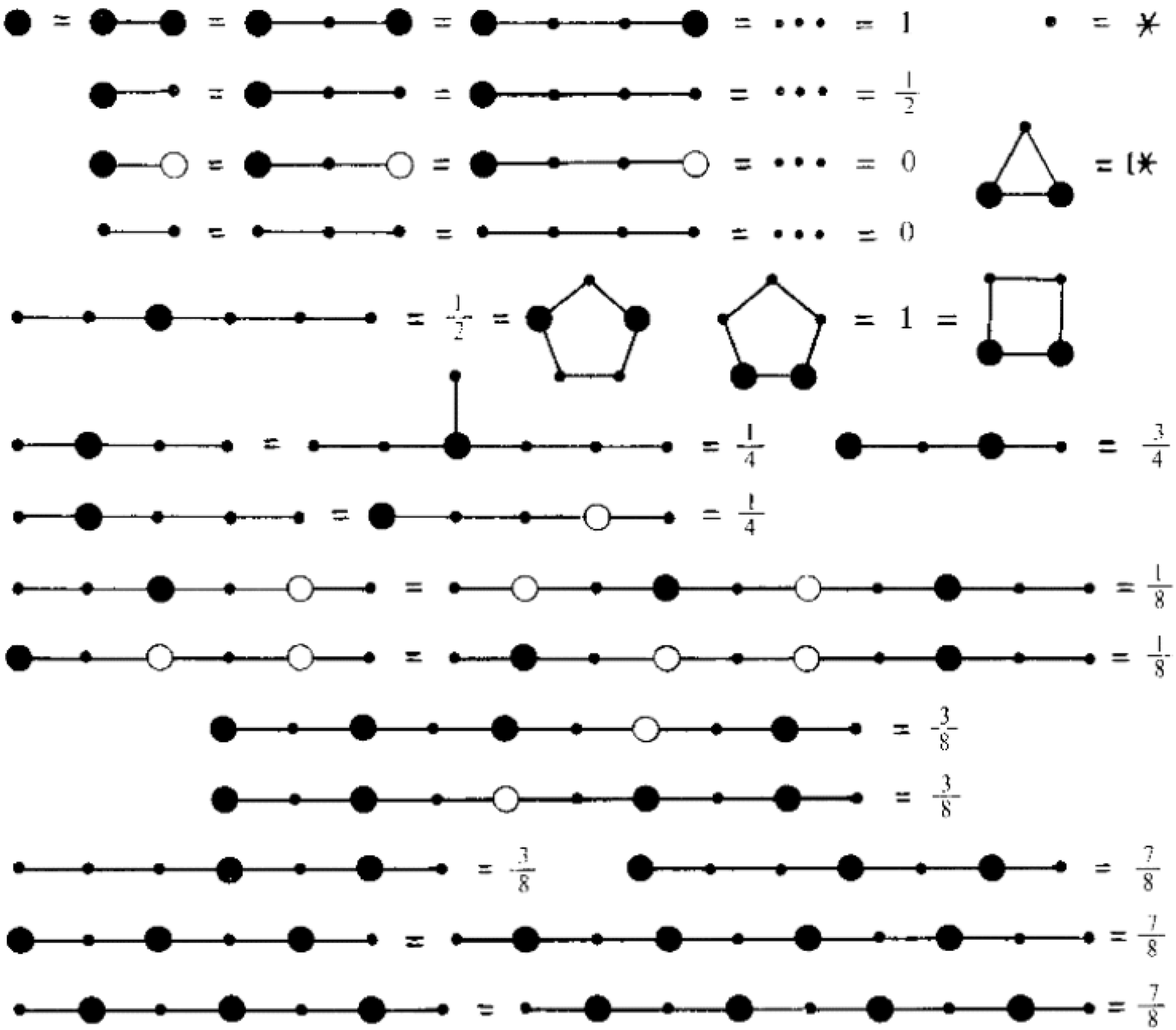
•: nodes available for both Left and Right.

●: nodes available for Left only.

○: nodes available for Right only.



Some Col Values



A theorem of Col

Theorem Every position of Col has the value z or $z*$ for some number z.

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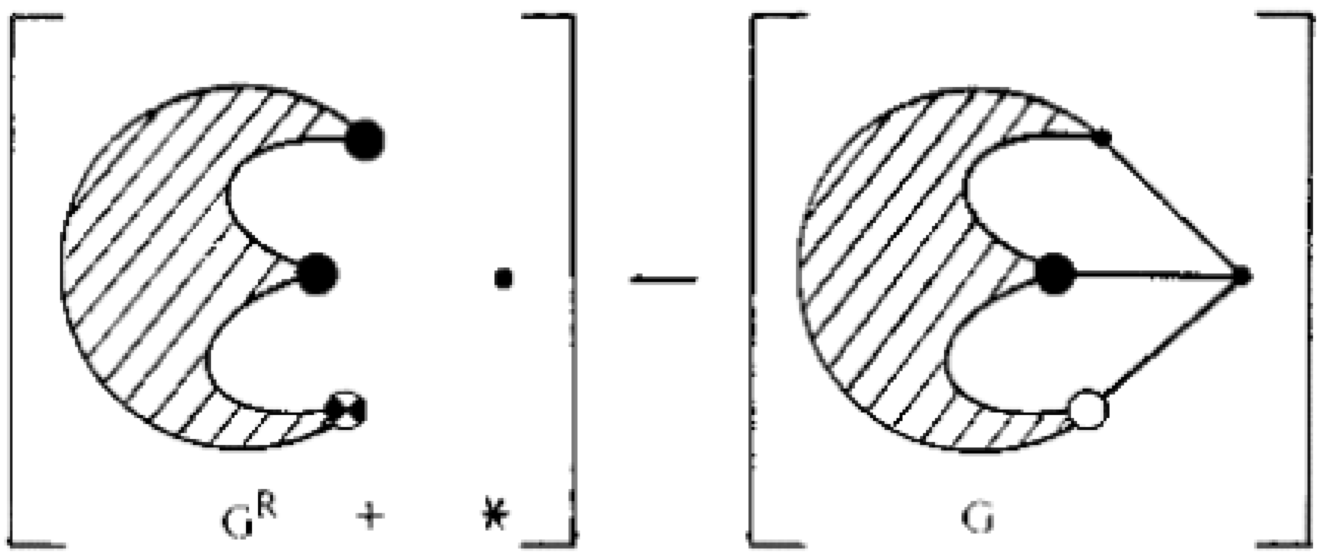
Proof: It is sufficient to show

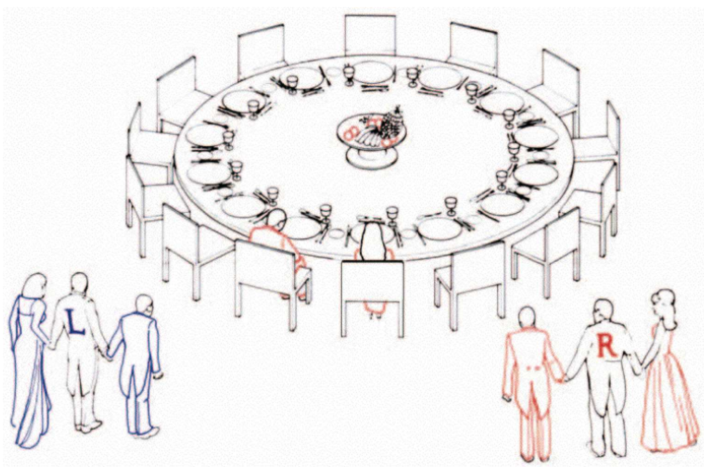
$$G^L + * \leq G \leq G^R + *.$$



The statement follows from induction.

Seating Couples

- **Two players:** "Left" and "Right".
- **Game board:** some dinning tables of various sizes.
- **Rules:** Two players take turns to seat couples for a dinner. Left prefers to seat a lady to the left of her partner, while Right thinks it proper only to seat her to the right. No gentleman may be seated next to a lady other than his own partner.
- **Ending positions:** Whoever gets stuck is the loser.



Values of seating couples

LnL, a row of n empty chairs between two of Left's guests, RnR, a row of n empty chairs between two of Right's, and LnR or RnL, a row of n empty chairs between one of Left's guests and one of Right's.

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Recursive formular:

$$LnL = \{LaL + LbL \mid LaR + RbL\}$$
$$RnR = \{RaL + LbR \mid RaR + RbR\} = -LnL$$
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n	0	1	2	3	4	5	6	7	8	9	10	11	...
LnL	$-$	0	-1	-1	$*$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$*$	$-\frac{1}{8}$	...
LnR	0	0	0	$*$	$*$	$*$	0	0	0	$*$	$*$	$*$	...
RnR	$-$	0	1	1	$*$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$*$	$\frac{1}{8}$	...