Math576 Combinatorial Game Theory
Homework 5 Solutions

1. Two players play Treblecross game. At certain moment where there are three rows of empty spaces left: 9, 10, 11 between four X's. What is the winning move for the next player?

   Solution: Treblecross game is equivalent to the Take-and-Break Game .007. The game value is

   $$\mathcal{G}(7) + \mathcal{G}(8) + \mathcal{G}(9) = *2 + 0 + *3.$$

   The winning move is to insert an X to the row of 11 empty spaces and split the 8 empty spaces and 2 empty spaces.

2. Two players play a substraction game $S(2, 5, 6)$ of three heaps of sizes $7, 8, 9$. Who wins this game? What is the winning move for the first player?

   Solution: The subtraction game $S(2, 5, 6)$ has the nim sequence

   $$\overline{00110213021}$$

   with period 11. Thus, the game value of given three heaps is

   $$*3 + 0 + *2 = *1.$$

   The first player has a winning strategy. His/her best move is taking 2 away from the heap of size 7, which re-set the game value to

   $$*2 + 0 + *2 = 0.$$

3. Find the nim sequence for the substraction game $S(2, 3, 6, 8)$. What is the period of this nim sequence?

   Solution: The subtraction game $S(2, 3, 6, 8)$ has the nim sequence

   $$\overline{00112031220312}$$

   with period 14.

4. Explain what rules is for the Take-and-Break Games .34, then find the nim sequence.

   Solution: The Take-and-Break Game .34 means

   - The player can take away the heap consisting of one bean.
   - The player can take away one bean away from the top of the heap more than one bean.
   - The player can take away 2 beans from any heap more than 4 beans and split the remaining of the heap into two non-empty heaps.

The nim sequence is

$$0.1012010\overline{03121203}$$

with period 8 starting at $n = 7$.

5. In a nim-like game, the legal move is to take away at least half chips from a heap. Discover the pattern of its nim squence.

   Solution: It is easy to see that $g(0) = 0$, $g(1) = 1$, $g(2) = 2$. For $n \geq 3$, by induction, we have

   $$g(n) = Mex(g(a) \colon a \leq \lfloor n/2 \rfloor).$$

   It turns outs that the nim sequence starts with one 0, one 1, two 2's, four 3's, eight 4's, and so on:

   $$0.1223333444444445555555555555555 \cdots.$$

   Equivalently, we have

   $$\mathcal{G}(n) = \lceil \log_2(n+1) \rceil.$$

   This statement can be proved by induction on n.

   Initial cases, it holds for $n = 0, 1, 2$. Assume that the statement holds for smaller instance up to n. Now we consider the case n. Let $k = \lceil \log_2(n+1) \rceil$. Then $n + 1 \leq 2^k$. This implies $\lfloor \frac{n}{2} \rfloor + 1 \leq 2^{k-1}$. Thus $\mathcal{G}(a)$ consists of $0, 1, 2, \ldots, k-1$ except k. By the MEX rule, $\mathcal{G}(n) = k$. Done!