Math776: Graph Theory (I)
Fall, 2017
Homework 5 solutions

Select any 5 problems to solve. The total score of this homework is 10 points. You get a bonus point if you solve all 6 problems correctly.

1. **[page 111, #4**] show that every planar graph is a union of three forests.

   Proof: Let G be a planar graph and U is a subset of vertices. We need to verify that $||G[U]|| \leq 3(|U|-1)$. This is trivial when $|U| = 1$ or 2. For $|U| \geq 3$, observe $G[U]$ is still planar. Thus $G[U]$ has at most $3|U|-6$ edges, which is less than $3(|U|-1)$. By Nash-William's theorem, G can decomposed into the union of three forests.

2. **[page 112, #20**] Show that adding a new edge to a maximal planar graph of order at least 6 always produces both a TK_5 and a $TK_{3,3}$ subgraph.

   Solution: Let G be a maximal planar graph of order at least 6 and the new edge is v_1v_2. Since every maximal planar graph of order at least 6 is 3-connected, there are 3 vertex-disjoint paths P_1, P_2, and P_3, connecting v_1 and v_2. We may assume that the lengths of P_i are minimized so that each P_i only contains one neighbor, say u_i of v_1. Since G is maximal planar graph, $G[N(v_1)]$ forms a cycle C, which is broken into three arc segments u_1Cu_2, u_2Cu_3, u_3Cu_1. Observe that v_1v_2, v_1u_1, v_1u_2, v_1u_3, u_1Cu_2, u_2Cu_3, u_3Cu_1, u_1P_1, u_2P_2, u_3P_3 are ten inner-vertex-disjoint paths connecting every pair of vertices in $\{v_1, v_2, u_1, u_2, u_3\}$. This is a TK_5. To get $TK_{3,3}$, there is another vertex w other than these 5 vertices. WLOG, say w falls in the region between P_1 and P_2. There are another 3 vertex-disjoint paths connecting w to u_1, u_2, and v_2. This forms a $TK_{3,3}$ with one part of branching vertices u_1, u_2, and v_2, and the other part of branching vertices v_1, u_3, and w.

3. **[page 112, #22**] A graph is called *outplanar* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outplanar if and only if it contains neither K_4 nor $K_{2,3}$ as a minor.

   Solution: ($\Rightarrow$) Let G be an outplanar graph. Since every vertex lies on the outer face we can add a new vertex v in the outer face and connect v to every vertex of G without crossing edges. This makes a new graph, call it G'. Note the G' is still planar because there are no crossed edges. By our theorems on planar graphs, G' has no K_5 or $K_{3,3}$ as a minor. This implies that G must not have K_4 or $K_{2,3}$ as a minor because it is G' minus one vertex.

   ($\Leftarrow$) Assume that G has no K_4 or $K_{2,3}$ as minor. If we construct G' the same was as we did in the previous direction, G' will have neither a K_5 or a $K_{3,3}$ as minor. This means that G' must be planar. So we can map G' to the sphere. On the sphere you can manipulate G' until the vertex

v we added is in the same face as ∞. Then project this modification of G' onto the plane. Now v will be in the outer face of G'. By removing v, you will expose every vertex in G to the outer face, making G outplanar. $\square$

4. **[page 140, #13**] Show that every critical k-chromatic graph is $(k-1)$-edge-connected.

Solution: Proof by contradiction: assume that a critical k-chromatic graph is not $(k-1)$-edge-connected. There is an edge set F of size at most $(k-2)$ separating G into two pieces U and V. Since G is critical k-chromatic graph, both $G[U]$ and $G[V]$ is $(k-1)$-colorable. We would like to construct a $(k-1)$-coloring of G by pair the $(k-1)$ coloring classes of $G[U]$ to those of $G[V]$ so that between each pair there is no crossing edge from F. Say the coloring classes of $G[U]$ are $C_1, C_2, \ldots, C_k$ and the coloring classes of $G[V]$ are $C'_1, C'_2, \ldots, C'_k$. Select a color class, say C_1 so that at least one edge of F coming out of this class. Since $|F| \leq k-2$, C_1 can reach at most $k-2$ other classes through the edges of F. We can find a class, say C'_1, so that $C_1 \cup C'_1$ is an independent set. We continue this process on to find C_2 and C'_2, and so on. The key observation is that at i-th iteration, at least $(i-1)$ edges of F come out of C_j for some $j < i$. Therefore the number of edges in F out of $\{C_i, \ldots, C_{k-1}\}$ is at most $|F| - (i-1) \leq k-i-1$, but the number of avaiable classes are $k-i$. Thus, we can find C'_i so that $C_i \cup C'_i$ is an independent set. Thus, we can pair them one by one to get a $k-1$ independent set whose union is $V(G)$. Thus G is $(k-1)$-colorable. Contradiction!

5. **[page 140, #24**] For every k, find a 2-chromatic graph whose choice number is at least k.

Solution: We claim that the complete bipartite graph K_{k,k^k} has choice number at least k.

Consider the following coloring assignment. Divide a total of k^2 colors into k color classes $C_1, C_2, \ldots, C_k$ evenly. Assign each vertex v_i on the left with a color set C_i. Assign each possible k-tuple in $C_1 \times C_2 \times \cdots \times C_k$ to one of vertex on the right side. If this graph is k-choosable, then there is a selection of colors $(c_1, c_2, \ldots, c_k)$ for the k vertices on the left. Since there is a vertex v on the right assigned with the exact colors $\{c_1, c_2, \ldots, c_k\}$. There is no way to select a proper color for v.

6. **[page 140, #13**] Prove that the choice number of K_2^r is r. (Here K_2^r is the complete r-partite graph with each part of size 2.)

Solution: We will prove by induction that $\mathrm{ch}(K_2^r) \leq r$. For $r = 1$, K_2^1 is the graph of two vertices and no edges which has choice number 1 since no two vertices are adjacent. This satisfies the condition.

Assume $r \geq 2$. Consider K_2^r with r parts $V_1, V_2, \ldots, V_r$, each of size 2. There are two cases:

Case 1: There is an $V_i = \{u_i, v_i\}$ such that the lists of colors at u_i and v_i has a common color c. Now delete the two vertices in V_i from G, and delete color c from the lists of all vertices but u_i and v_i. The remaining graph forms a K_2^{r-1}, and each vertex has a list of colors with size at least $r-1$. By inductive hypothesis, K_2^{r-1} is $(r-1)$-choosable. We find select a color from each list to form a proper coloring of K_2^{r-1}. Now extend this coloring to G by assigning both u_i and v_i the color c.

Case 2: For each $1 \leq i \leq r$, the color lists of colors of two vertices in V_i has NO common color. Now consider the first part $V_1 = \{u_1, v_1\}$. Now pick any color c_1 in the list of u_1. The color c_1 can be in at most $r-1$ lists of other vertices. Since there are r colors available for the vertex v_1, there is a color c_2 in the v_1's list such that c_2 is not in the list of any other vertex, whose list contains color c_1. Now color vertex u_1 by c_1 and vertex v_1 by c_2. Delete c_1 and c_2 from the lists of the rest of vertices. Since no c_1 and c_2 are in the same list, there are at least $r-1$ remaining colors in each list of K_2^{r-1}. By inductive hypothesis, we can select one color of each list to form a proper coloring of $G - V_1$, thus, a proper coloring of G.

This shows that K_2^r is r-choosable. To show that K_2^r is not $(r-1)$-choosable, note the each vertex in K_2^r has $r-1$ neighbors. Therefore, if every vertex has the same $r-1$ choices of colors, there would be no proper coloring of K_2^r. Therefore, K_2^r is r-choosable, but not $(r-1)$-choosable. Therefore, K_2^r has choice number r.