Math776: Graph Theory (I)
Fall, 2017
Homework 4 solutions

Select any 5 problems to solve. The total score of this homework is 10 points. You get a bonus point if you solve all 6 problems correctly.

1. [**page 83, #4**] Let X and X' be minimal separators in G such that X meets at least two components of $G - X'$. Show that X' meets at least two components of $G - X$, and X meets all the components of $G - X'$.

Solution: Suppose that X' meets $G - X$ in only one component. Call this component C.

Then $X' \subseteq X \cup C$. So the components of $G - X'$ are components which come from $C - X'$ and a component which contains the rest of G. So, X meets only one component of $G - X'$. This a contradiction. Hence, X' meets at least two components of $G - X$.

Then, it follows from symmetry that X meets every component of $G - X'$.

2. [**page 83, #10**] Let e be an edge in a 3-connected graph $G \neq K_4$. Show that either $G \dot{-} e$ or G/e is again 3-connected.

Solution: Let $e = xy$ be an edge in a 3-connected graph $G \neq K_4$. We want to show that either $G \dot{-} e$ or G/e is 3-connected. Suppose not, so neither of these graphs are 3-connected. Then each of these new graphs has a set of at most two vertices that disconnects it. First we look at G/e. If neither of the vertices are the compressed ends of e then these vertices would disconnect G, a contradiction. Let the other vertex in the separator be called z. Since $\{xy, z\}$ is a separator of G/e, $\{x, y, z\}$ will be a separator of G. And this set is a minimal separator in G so each of these connects to every component of $G - \{x, y, z\}$.

Now we look at $G \dot{-} e$. Neither x nor y can be in the separator or they would be part of a 2-separator of G, so let $\{u, v\}$ be a separator of $G \dot{-} e$. Now we consider where u and v live in $G - \{x, y, z\}$. If they are in the same component, then there is at least one component containing neither u nor v. Since x and y have edges to this connected component there must be a xy path that does not use e or go through u or v. Thus x and y are in the same component in $G - \{u, v\}$. This would mean that $\{u, v\}$ separates G, a contradiction, since removing e doesn't affect anything. Hence u and v are in different components of $G - \{x, y, z\}$ and there are only these two components.

Let a, b, c be the ends of the edges to x, y, z respectively, in the component containing u. Since G is 3-connected there is an ab path that does not go through x or u. If such a path doesn't go through v then we can travel from x to a to this path to b then to y. This would mean that x and y are in the same component of $G - \{u, v\}$. This is a contradiction as argued above. Thus every such path goes through v. So there is some path from a to v that doesn't go through x, b, or u. This path must go through either y or z first to get to the other side. Going through y would place x and y in the same component of $G - \{u, v\}$, again a contradiction. Thus there is a path from a to z that doesn't go through x, v, or u. So a and z are in the same component of $G - \{u, v\}$, and thus x is as well since it is a neighbor of a.

Now we know that y and z are in different components of $G - \{u, v\}$ since x is in the same one as z. Since G is 3-connected there is a bc path that does not go through y or u. Every such path must go through v or else y and z would be in the same component of $G - \{u, v\}$. Thus there is a path from b to v that does not go through y, c or u. So this path must go through x or z first. But if it goes through x then x and y would be in the same component of $G - \{u, v\}$. And if it goes through z then y and z would be in the same component, also a contradiction. Therefore G must not be 3-connected and we have shown that there must be a contradiction so either $G \dot{-} e$ or G/e is 3-connected.

3. **[page 84, #18**] Let $k \geq 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

Solution:

Let $k \geq 2$ and let G be a k-connected graph with $|G| \geq 2k$. As G is k-connected, it is connected, and as $\delta(G) \geq \kappa(G) \geq k \geq 2$, it has no leaves, so it is not a tree, so it has a cycle.

Let C be a largest cycle in G. First, as $\delta(G) \geq \kappa(G) \geq k$ and G has a cycle, $|C| \geq k + 1$. Assume for the sake of contradiction that $|C| < 2k$. Then there is a $v \in G \backslash C$. Let $A = N(v)$ and $B = V(C)$. as $\delta(G) \geq \kappa(G) \geq k, |A| \geq k$. Furthermore, any set X of size less than k cannot separate A and B as that would disconnect v and some $c \in C$, contradiction that G is k-connected. Thus the size of a minimum separator is at least k, and by Menger's theorem, there are at least k disjoint AB paths.

By the pigeon-hole principle (with vertices in A as pigeons and edges in C as holes), there are $a, a' \in A$ and $c_1, c_2 \in C$ such that $c_1, c_2 \in E(G)$ there are distinct $a - c_1$ and $a' - c_2$ paths P_a and $P_{a'}$. (Note that these paths may be of length on if a vertex of C is adjacent to v.) Let P be the c_1c_2 path in C of size at least two, Then

$$C' = vP_a \overset{o}{P} P_{a'}v$$

has size at least one larger than C, contradicting the maximality of C.

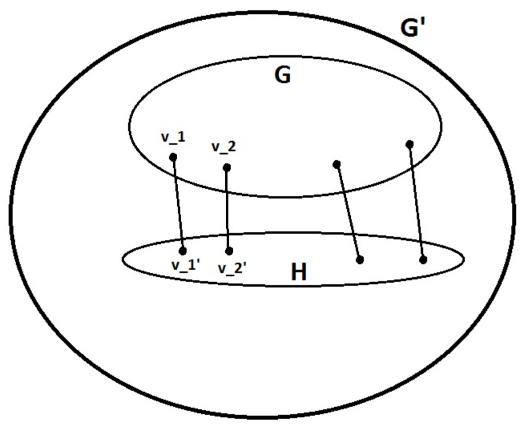
We conclude $|C| \geq 2k$.

4. [page 84, #19] Let $k \geq 2$. Show that in a k-connected graph any k vertices lie on a common cycle.

Solution: Let G be a k-connected graph, and let $v_1, \dots, v_k \in V(G)$. Let C be a cycle containing as many of these specified vertices as possible, without loss of generality say $v_1, \dots, v_l$, and suppose that $l < k$. Then there exists a v_{l+1} outside of C, and by Menger's Theorem, the minimum number of vertices not equal to v_{l+1} separating v_{l+1} from C is equal to the maximum number of independent $N(v_{l+1})$-C paths. Hence, since G is k-connected, there are at least k paths from v_{l+1} to C, independent save for v_{l+1} as the initial vertex. However, these paths must meet C in between each of the vertices $v_1, \dots, v_l$ with no two paths meeting in the same portion of the cycle $v_i C v_{i+1}$, or else there exists a larger cycle containing v_{l+1}. On the other hand, if no such cycle exists, then there are at least k elements from $v_1, \dots, v_k$ in C (since there are k paths meeting C in this way), a contradiction.

5. [page 84, #24] Derive Tutte's 1-factor theorem from Mader's theorem.

Solution: Let $G = (V, E)$ be a graph. For each vertex $v \in V(G)$, add a new vertex v', and connect v to v'. Call this new graph G', and let $H = \{v'\}$. We have the following diagram:



Assume $q_G(S) \leq |S|$ for all $S \subseteq V(G)$. We want to show that G contains a 1-factor.

Notice, there are $\frac{|G|}{2}$ many independent H-paths by construction. So, we have $M_{G'}(H) \leq \frac{|G|}{2}$. Observe, if $M_{G'}(H) = \frac{|G|}{2} = \frac{|G'|}{4}$, then G has a 1-factor. So, we need to show

$$\frac{|G|}{2} \leq M_{G'}(H) = |S| + \sum_{C_i \in C_F} \lfloor \frac{1}{2} |\delta C| \rfloor$$

for all $S \subseteq V(G - H)$ and $F \subseteq E(G - S) - E(H)$, where C_F is the set of connected components of F.

Suppose we have r components of $G-H$. We then have $|G| = |S|+|C_1|+\ldots+|C_r|$. So,

$$\begin{aligned}
|S| + \lfloor\frac{1}{2}C_1\rfloor + \ldots + \lfloor\frac{1}{2}C_r\rfloor &= |S| + \frac{1}{2}|C_1| + \ldots + \frac{1}{2}|C_r| - \frac{1}{2}q_G(S) \\
&= \frac{|G|}{2} + \underbrace{\frac{|S|}{2} - \frac{1}{2}q_G(S)}_{\geq 0} \qquad \text{since } q_G(S) \leq |S| \\
&\geq \frac{|G|}{2}
\end{aligned}$$

Therefore, $M_{G'}(H) = \frac{|G|}{2}$, hence, we G has a 1-factor.

6. [page 84, #26] For every $k \in \mathbb{N}$ find an $l = l(k)$, as large as possible, such that not every l-connected graph is k-linked.

Solution: We want to find a function $l(k)$ as large as possible such that an $l(k)$ connected graph does not have to be k-linked. We will let $l(k) = 3k-3$ and show that a graph can be 3k-3 connected but not k-linked. Pick $s_1, s_2, ..., s_k$ and $t_1, t_2, ..., t_k$. Then add $k-1$ vertices to finish the vertex set of G. We will not have edges between s_i and t_i for any i. However, every other possible edge will be included in the edge set of G. Thus to get from any vertex to any other vertex we can go through any of the other 3k-3 vertices. Thus G is 3k-3 connected since there are this many independent paths between any two vertices. But G is not k-linked since any link must include one of the vertices from the k-1 set. Thus there cannot be a complete set of k links so $l(k) = 3k-3$ works.