Math776: Graph Theory (I)
Fall, 2017
Homework 3 solution

1. **[page 31, #39**] Prove Gallai's theorem that the edge set of any graph G can be written as a disjoint union $E(G) = C \cup D$ with $C \in \mathcal{C}(G)$ and $D \in \mathcal{C}^*(G)$.

   Proof: Let G be an arbitrary graph. Suppose for $|G| < n$ there is a partition of G such that $G[V_1]$ and $G[V_2]$ both have even degree and $D = \{ab \in E(G) | a \in V_1, b \in V_2\}$. Consider G with $|G| = n$. If $deg(v)$ is even for all $v \in V(G)$, then we are done as $G = D$ and $C = \emptyset$. So, let $v \in V(G)$ such that $deg(v)$ is odd. Construct $G' = G \backslash \{v\}$ by adding the edge ab if $ab \notin E(G)$ and deleting the edge ab if $ab \in E(G)$ for $a, b \in N(v)$. We note that constructing the edges of G' in this way preserves the parity of the vertices in $N(v)$. By the induction hypothesis, there is a partition of G' such that $G'[V_1]$ and $G'[V_2]$ both have even degree. As $deg(v)$ is odd, there are an odd number of neighbors of v in one of V_1 or V_2 and an even number in the other. WLOG, suppose there are an odd number of neighbors of v in V_1. Then we add v to V_2 and reconstruct the edges between v and its neighbors and delete the edges that were added. Then $G[V_1]$ and $G[V_2]$ both have even degree. Thus, $E(G) = C \cup D$.

2. **[page 54, #11**] Let G be a bipartite graph with bipartition $\{A, B\}$. Assume that $\delta(G) \geq 1$, and that $d(a) \geq d(b)$ for every edge ab with $a \in A$. Show that G contains a matching of A.

   Solution: Assume G has a minimal set S such that S does not satisfy the marriage condition. In other words $|N(S)| < |S|$. Remove one vertex of S, call it S'. Since S was minimal we are now guaranteed a matching in S'. Also the $|N(S')| = |N(S)|$ or else we would have had a matching in S. The edges from S' to $N(S') = \sum_{a_i \in S'} d(a_i) = \sum_{b_j \in N(S')} d(b_j)$. So for each $a_i \exists b_j$ such that $d(a_i) = d(b_j)$. When we add our one vertex back into S', it will be connected to one of the vertices in $N(S')$. This will disrupt the equality above and $\exists$ a b_j such that $d(a_i) < d(b_j)$.

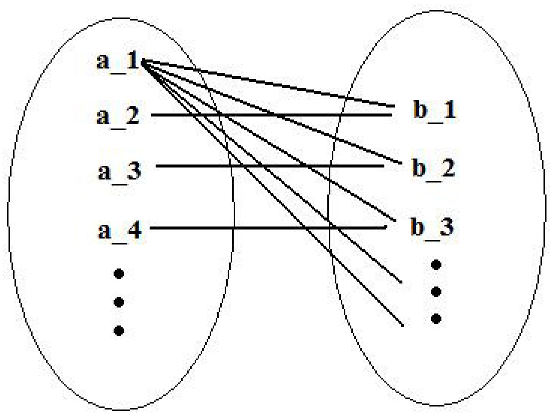
   So there is no minimal set that violates the marriage condition. So all subsets have the marriage condition. So we have a matching on A. $\square$

3. **[page 55, #5**] Derive the marriage theorem from König's theorem.

   Solution: The König's theorem says that in a bipartite graph G, $max|M| = min|K|$. where M is a matching, and K is a vertex cover of edges. We use this theorem to prove the Hall' theorem which says that G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$. We use contradiction method, given a graph G, it satisfies the Hall condition, but has no matching of A, then according to König's theorem, the size if cover U is less than size of A, i.e. $|U| < |A|$. We say $U = A^{'} + B^{'}$ with $A^{'} \subseteq A$ and $B^{'} \subseteq B$. Then $|A^{'}| + |B^{'}| = |U| < |A|$. Hence $|B^{'}| < |A| - |A^{'}| = |A \setminus A^{'}|$. Since U is a cover of edges, there is no edge between $A \setminus A^{'}$ and $\setminus AB^{'}$, so, $|N(A \setminus A^{'})| \leq |B^{'}| < |A \setminus A^{'}|$, which contradicts to the hall condition when set $S = A \setminus A^{'}$.

4. [**page 55, #8** Find an infinite counterexample to the statement of the marriage theorem.

Proof: Let G be a bipartite graph with partition classes A and B. Enumerate the vertices in each of these partition classes. Then, for a fixed index i, let a_i be adjacent to b_{i-1} where $i \geq 2$, and let a_1 be adjacent to every vertex in B (see picture below). Notice, when the size of A and B are infinite, any subset of the vertices in A has at least as many neighbors as the size of the subset. Therefore, $|N(S)| \geq |S|$ for all $S \subseteq A$. Notice, this is not the case when the sizes of A and B are finite. However, for $i \geq 2$, a_i has only one edge incident, and thus, this edge must be used when considering a matching for A. Therefore, choosing each of these edges, one must then find an independent edge that is incident to a_1, which is not possible. Hence, no matching of A exists.



5. [**page 55, #9**] Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \in \mathbb{N}$. Show that there are disjoint subsets $D_k \subset A_k$, with $|D_k| = d_k$ for all $k \leq n$ if and only if

$$|\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$$

for all $I \subset \{1, \ldots, n\}$.

Solution: We construct a bipartite graph. In the left hand partition L, we place d_i copies of a vertex labeled D_i for $i = 1, \ldots, n$, and into the right hand partition R we place each of the distinct elements of $\cup_{i=1}^{n} A_i$. For $i = 1, \ldots, n$, make each copy of D_i adjacent to each $a \in A_i$.

If there exist disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$ for all $k \leq n$, then this is equivalent to there being a matching in the bipartite graph we constructed. Hence, by the Marriage Theorem, $|\cup_{i\in I} A_i| = |N(D_i\big|_{i\in I})| \geq \sum_{i\in I} |D_i| = \sum_{i\in I} d_i$ for all $I \subseteq \{1, \ldots, n\}$.

On the other hand, if $|\cup_{i\in I} A_i| \geq \sum_{i\in I} d_i$ for all $I \subseteq \{1, \ldots, n\}$, then we have $|N(D_i\big|_{i\in I})| = |\cup_{i\in I} A_i| \geq \sum_{i\in I} d_i = \sum_{i\in I} |D_i|$ for all $I \subseteq \{1, \ldots, n\}$. Furthermore, if we take an $S \subset L$ without the full number of copies of some D_i, then $|S| < \sum_{i\in I} |D_i| \leq |N(D_i\big|_{i\in I})| = |N(S)|$. Therefore, by the Marriage Theorem, there exists a matching, and we have constructed disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$ for all $k \leq n$.

2. [**page 55, #14**] Show that all stable matchings of a given graph cover the same vertices. (In particular, they have the same size.)

Solution: Let M, M' be two stable matchings of G. For a contradiction, suppose $\exists\ v_0 \in M' \setminus M$. Then v_0 has a neighbor v_1 with $v_0v_1 \in M'$. Note that v_1 must be matched in M, otherwise we may add v_0v_1 to M to get a larger stable matching, a contradiction. Since v_1 is matched in M, then v_1 has a neighbor v_2 with $v_1v_2 \in M$. We have that $v_0v_1v_2$ is a path with edges alternately in M' and M. Continue in this manner to get a full path $P = v_0v_1 \cdots v_n$ (for some $n \in \mathbb{N}$) and consider v_{n-1}. We have the preferences $v_{n-2} <_{v_{n-1}} v_n$ in M, but $v_n <_{v_{n-1}} v_{n-2}$ in M', a contradiction.

Thus, such a v_0 cannot exist, so M and M' must cover the same vertices.