Math776: Graph Theory (I)
Fall, 2017
Homework 2 solution

1. **[page 31, #20**] Show that a graph is 2-edge-connected if and only if it has a *strongly-connected* orientation, one in which every vertex can be reached from every other vertex by a directed path.

   Solution:

   $\Rightarrow$: Let G be a 2-edge-connected graph. Since G is 2-edge-connected, there is a cycle C in G. We can orient the edges in the same direction so that C is strongly connected. Suppose that the edges oriented forms a subgraph H and it is a strongly-connected orientation.

   If H is not an induced subgraph of G, orient any missing edges arbitrarily. Without loss of generality, we can assume H is an induced subgraph of G.

   If $H \neq G$, pick a vertex $x \notin H$. Since G is 2-edge-connected, we have two edge-disjoint paths P_1 and P_2 from x to some vertices in H. We can orient the edges in P_1 toward x and the edges in P_2 leaving x. Then repeat this process until $H = G$.

   ($\Leftarrow$) Let H be a strongly connected orientation of G. Suppose that G has a bridge whose endpoints are x and y. Then there is only one directed edge between x and y, and so H is not strongly connected.

2. **[page 31, #21**] Find a short inductive proof for the existence of normal spanning trees in finite connected graphs.

   Solution: The graph G of one vertex has a normal spanning tree by definition, as does P_2. Let any graph G of size $|G| = n - 1$ have a normal spanning tree. Then consider another vertex which may be connected to an arbitrary number of vertices. Call this vertex u. Choose one of the vertices $v \in N(u)$. Let u be the parent of v. Let all other vertices in $N(u)$ be directly above v in the tree. We will now reform the tree in the following way to make it normal. Consider all vertices which are in $N(u) \setminus v$. We will let all vertices in $N(N(u) \setminus v)$ vertices become parents of the vertices in $N(u) \setminus v$ unless they are already on uPr in T. We will continue to do this process, until there are no more vertices which are in the neighborhoods which are not on uPr in T. This tree is normal, because certainly for u, all the vertices in can be related to are either on uPr or above it in the tree T. It suffices to show that all the other vertices above u in T still have the normality condition. All of these vertices must either be connected to something above it, in which case we still have the normality condition. Or if they are connected to something below it, say w is connected to something lower, x in T. Then by our construction of T, x must be on wPr. Therefore, the normality condition holds for all the vertices which have been moved in the tree.

3. [page 31, #24] Show that every automorphism of a tree fixes a vertex or an edge.

Proof: Let T be a tree. If $|V(T)| = 1$ then clearly the identity automorphism must fix the vertex. If $|V(G)| = 2$, there are two automorphisms - the identity and the map that switches the two vertices. In the latter case, the edge is fixed. Now suppose for all T with $|V(T)| \leq n$, every automorphism fixes either an edge or a vertex. Let $|V(T)| = n + 1$ and let $\{v_1, \ldots, v_k\} \in V(T)$ be the set of vertices with degree one. We note that this set is nonempty as T is a tree. Let ϕ be an automorphism of T. Then, $\phi(v_i) \in \{v_1, \ldots, v_k\}$ for $1 \leq i \leq k$ as automorphisms preserve degrees. Consider $T' = T \backslash \{v_1, \ldots, v_k\}$. Then ϕ is also an automorphism of T', T' is also a tree, and $|V(T')| < n+1$. By the induction hypothesis, ϕ fixes an edge or a vertex of T'. Thus, ϕ fixes an edge or a vertex of T.

4. [page 32, #27] Prove or disprove that a graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertices.

Solution: $\Rightarrow$: Let G be a bipartite graph, and consider two adjacent vertices $x, y \in G$. We may assume G is connected since a graph is bipartite if and only if each of its connected components is bipartite. We may partition G into two subgraphs X and Y such that $E(X) = E(Y) = \emptyset$. Without loss of generality, assume $x \in X$ and $y \in Y$. Let $z_x \in X$ and $z_y \in Y$. Since there are no edges within X or Y, $d(x, z_x)$ is even, $d(y, z_x)$ is odd, $d(x, z_y)$ is odd, and $d(y, z_y)$ is even. Hence, no two adjacent vertices share distance to any other vertex.

$\Leftarrow$: Suppose that G is not bipartite. By *Proposition 1.6.1*, G contains an odd cycle. Let C be an odd cycle of the smallest order. Observe that C must be geodesic in the sense:

$$d_C(u, v) = d_G(u, v)$$

for any vertex u and v in C. Otherwise, we can construct another odd cycle with smaller order!

Say $|C| = 2k + 1$ and C has vertices $v_0, v_1, \ldots, v_{2k+1}$. Let $x = v_0$, $y = v_{2k+1}$, and $z = v_k$. We have $d_C(x, y) = 1$ and $d_C(x, z) = d_C(y, z) = k$. Since C is geodesic, we have

$$d_G(x, y) = 1, \text{ and } d_G(x, z) = d_G(y, z) = k.$$

5. [page 32, #28] Find a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of average degree at least $f(k)$ has a bipartite subgraph of minimum degree at least k.

Solution: Let $f(k) = 4k$. The graph of G with an average degree of $4k$ will have a subgraph H with minimum degree $2k$. Now take the maximal bipartite graph in H with the maximal number of edges, call it H_B.
<ins>Claim</ins>: H_B will have minimum degree $\geq k$.

Assume not. Then $\exists v \in H_B$ such that $d(v) < k$. That means that v must have lost over half of its neighbors from $H \rightarrow H_B$. So v is on the same 'side' as over half of its neighbors from H in H_B. But by moving v to the other 'side' of the partition, v will be able to connect to those vertices and H'_B will have more edges than $H_B \rightarrow\leftarrow$ because H_B was said to be maximal with respect to edges. So H_B must have minimal degree $\geq k$.

6. [**page 32, #30**] Prove or Disprove that every connected graph contains a walk that traverses each of its edges exactly once in each direction.

Proof: Let G be an connected graph. For each edge uv of G, we replace it by two arcs uv and vu. Call the resulting directed graph D. It is clearly D is strongly connected and for each vertex v $d^+_H(v) = d^-_H(v)$. By Euler's theorem, there exists an Euler circuit of H, which traverses each of its edges in G exactly once in each direction.